Computing Schrödinger propagators on Type-2 Turing machines

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Abstract

We study Turing computability of the solution operators of the initial-value problems for the linear Schrödinger equation \( u_t = i\Delta u + \phi \) and the nonlinear Schrödinger equation of the form \( iu_t = -\Delta u + mu + |u|^2 u \). We prove that the solution operators are computable if the initial data are Sobolev functions but noncomputable in the linear case if the initial data are \( L^p \)-functions and \( p \neq 2 \). The computations are performed on Type-2 Turing machines.

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1. Introduction

Most physicists believe that the future behavior of processes that are described by well-established theories can be computed with arbitrary precision, at least in principle, from sufficiently precisely given initial conditions, where the computations can be performed on digital computers, hence on Turing machines. Nevertheless, the Pour-El/Richards’ paradox [9] (a three-dimensional wave with computable amplitude at time 0 and noncomputable amplitude at time 1) gave cause for speculations that it might be possible to design “wave computers” beating the Turing machine. As a consequence, Church’s Thesis had to be revised. However, a careful analysis of wave propagation [14] has shown that, in physically reasonable settings, wave propagation is computable, and it seems very unlikely that such wave computers can be built. Of course, this result does not at all settle the question whether the common belief holds true. The question whether it is always possible to compute physical processes modeled by differential equations...
remains largely open, in particular, for nonlinear problems. Examples of such problems are the Navier–Stokes equation, the complex of problems associated with Feigenbaum’s constant, and the Schrödinger equation. These problems are of classical importance.

Since different nonlinear problems generally have little in common with each other, nonlinear problems may have to be dealt with on a case-by-case basis. In this paper, we study computability of solution operators of the Schrödinger equation in the context of the Turing machine-based computability theory of real functions. We will prove that the Schrödinger propagator is Turing computable in physically relevant settings. Physically, the initial-value problem for the linear Schrödinger equation

\[ u_t = i\Delta u + \phi, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3, \ i = \sqrt{-1}, \ u(0, x) = f(x) \]  (1)

describes the movement of a particle. The square of the modulus, \(|u(t, x)|^2\), of its solution \(u\) is the probability density for finding the particle at time \(t\) and place \(x\). The probability interpretation requires that

\[ \int_{\mathbb{R}^3} |u(t, x)|^2 \, dx = 1, \ \forall t \geq 0 \]

for physically relevant solutions. Thus, to study existence and regularity of the initial-value problem (1) requires \(L^2\)-norm based Sobolev spaces \(H^s(\mathbb{R}^d)\). Indeed, it is known classically that problem (1) is well-posed in \(H^3(\mathbb{R}^d)\)-settings. In this paper, we will further show that the solution operator of problem (1) is Turing computable in such settings. Namely, we will construct a Turing machine that computes arbitrarily accurate approximations to the solution \(u\) from approximations to the initial data \(f\) and the forcing term \(\phi\). This result provides another piece of evidence to support the common belief.

We also prove in Section 5 that the solution operator of the initial-value problem for the following nonlinear Schrödinger equation

\[ i \frac{du}{dt} = -\frac{d^2u}{dx^2} + mu + |u|^2u, \quad m, t, x \in \mathbb{R}, \ u(0, x) = \varphi(x) \]  (2)

is computable in the \(H^1(\mathbb{R})\)-setting. There is a rich mathematical theory for the Schrödinger equation of the form

\[ i \frac{du}{dt} = (-\Delta + m)u + |u|^2u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \]

where \(m\) is a real constant [2,3,11]. The equation is widely used in several domains of applied physics. For example, it can be considered as the classical approximation to the field equation for a quantum mechanical nonrelativistic many body system with a two body \(\delta\)-function interaction. It is also used in the Landau–Ginsburg theory of superconductivity.

As for the computational model, we use Type-2 theory of effectivity (TTE for short), developed by Weihrauch and others [12,4,5,7,9,6]. TTE is a Turing machine-based theory of real computing. It offers a uniform language to study computability properties of real functions and function spaces in a realistic way, in particular, it allows simple and realistic definitions of computability on a variety of spaces occurring in analysis.

The plan of the paper is as follows. Section 2 summarizes basic definitions and facts to be used in later sections, which are devoted to study computability properties of solution operators of the Schrödinger equations. Sections 3 and 4 discuss the initial-value problems for the
linear Schrödinger equations, while Section 5 studies the initial-value problem for a nonlinear Schrödinger equation.

2. Preliminaries

In this section we review some basic definitions from TTE. For details the reader is referred to the textbook [13]. TTE is a representation based approach to computable analysis. Its basic idea is to represent infinite objects, for example, real numbers or functions of real variables, by infinite strings over some appropriate finite alphabet $\Sigma$ containing numbers 0 and 1. A representation of a set $X$ is simply a surjective map $\delta : \subseteq \Sigma^\omega \to X$, where $\Sigma^\omega$ is the set of infinite sequences over $\Sigma$ with the product topology (also called the Cantor topology) and “$\subseteq$” indicates that the map might be partial. If $\delta(p) = x$, then $p$ is called a $\delta$-name (or $\delta$-code) of $x$. The pair $(X, \delta)$ is called a represented space. Through a representation computations on $X$ can be defined by means of computations on $\Sigma^\omega$, which are explicitly executable on Turing machines: A function $\psi : \subseteq \Sigma^\omega \to \Sigma^\omega$ is called computable if there exists a Turing machine that computes and transforms each sequence $p$, written on the input tape, into the corresponding sequence $\psi(p)$, written on the one-way output tape. A Turing machine allowing infinite inputs and outputs is formally called a Type-2 Turing machine. For convenience, Type-2 Turing machines are also called Turing machines. A fundamental fact regarding the Turing computability on $\Sigma^\omega$ is that if $\psi$ is computable, then it is continuous with respect to the Cantor topology. Based on Turing computability on $\Sigma^\omega$, a notion of computable functions on represented spaces can now be introduced naturally.

**Definition 1 (Computable function).** Let $(X, \delta)$ and $(Y, \delta')$ be represented spaces. A function $f : \subseteq X \to Y$ is called $(\delta, \delta')$-computable if there exists a computable function $\psi : \subseteq \Sigma^\omega \to \Sigma^\omega$ such that $f \circ \delta(p) = \delta' \circ \psi(p)$ for all $p \in \text{dom}(f \circ \delta)$, or if the following diagram commutes:

$$
\begin{array}{ccc}
\Sigma^w & \xrightarrow{\psi} & \Sigma^w \\
\downarrow{\delta} & & \downarrow{\delta'} \\
X & \xrightarrow{f} & Y
\end{array}
$$

If $\psi$ is merely a continuous function, then $f$ is called $(\delta, \delta')$-continuous. As a fundamental fact, $f$ is $(\delta, \delta')$-continuous if it is $(\delta, \delta')$-computable. When $X$ and $Y$ are topological $T_0$-spaces with countable bases, $f$ is continuous if and only if $f$ is $(\delta, \delta')$-continuous, provided that $\delta$ and $\delta'$ are admissible representations. For details on admissible representations, the reader is referred to [10,13].

For any represented spaces $(X, \delta)$ and $(Y, \delta')$ with admissible $\delta$ and $\delta'$, there is a canonical admissible representation $[\delta \to \delta']$ of $C(X; Y)$, the set of all continuous functions from $X$ to $Y$ [10,13]. This function space representation $[\delta \to \delta']$ admits evaluation and type conversion.

**Lemma 2 (Evaluation and type conversion).**

1. (Evaluation): The evaluation function $(f, x) \mapsto f(x)$ is $(\{\delta \rightarrow \delta'\}, \delta, \delta')$-computable.
2. (Type conversion): Let $\delta_i : \subseteq \Sigma^\omega \to X_i$ be a representation of the set $X_i$, $0 \leq i \leq k$. Let $f : X_1 \times \cdots \times X_k \to X_0$ and define $F(x_1, \ldots, x_{k-1})(x_k) := f(x_1, \ldots, x_k)$. Then $f$ is
(δ₁, ..., δₖ, δ₀)-computable (-continuous), iff \( F \) is (δ₁, ..., δₖ₋₁, [δₖ → δ₀])-computable (-continuous).

Although a set can be represented possibly in different ways, there are certain canonical representations for computable metric spaces. One such representation is called Cauchy representation that is defined below. Let \( \mathbb{N} \) denote the set \( \{0, 1, 2, \ldots \} \), and let \( v_{\mathbb{N}} : \Sigma^\omega \rightarrow \mathbb{N}, v_{\mathbb{N}}(1^n0^\omega) = n \), be a representation of \( \mathbb{N} \).

**Definition 3** (Computable metric space and Cauchy representation).

1. A computable metric space is a tuple \( \mathbf{M} = (M, d, D, x) \) such that \( (M, d) \) is a metric space, \( D \) a countable dense subset of \( M \), and \( x : \mathbb{N} \rightarrow D \) a numbering of \( D \) satisfying the following property:

\[
\{(a, b, u, v) \in \mathbb{N}^4 \mid v_{\mathbb{Q}}(a) < d(x(u), x(v)) < v_{\mathbb{Q}}(b)\} \text{ is r.e.,}
\]

where \( v_{\mathbb{Q}} : \mathbb{N} \rightarrow \mathbb{Q} \) is some standard numbering of \( \mathbb{Q} \), the set of rational numbers.

2. The Cauchy representation \( \delta_M : \subseteq \Sigma^\omega \rightarrow M \) associated with a computable metric space \( \mathbf{M} \) is defined as follows: For any \( p \in \Sigma^\omega \), \( \delta_M(p) = x \) if and only if there are numbers \( p_0, p_1, \ldots, p_k \) such that \( p = 0^{p_0}10^{p_1}1 \ldots \) and \( d(x, x(p_k)) \leq 2^{-k} \) for all \( k \in \mathbb{N} \).

Thus, from the definition, a Cauchy name of an element \( x \) in a computable metric space is a coded sequence over a chosen countable dense set that converges to \( x \) rapidly. If \( (X, \delta) \) and \( (Y, \delta') \) are two Cauchy represented spaces, then a function \( f : X \rightarrow Y \) is \((\delta, \delta')\)-computable if there is a Turing machine that transforms a sequence of approximations to \( x \) to a sequence of approximations to \( f(x) \). In this sense, algorithms in TTE are “approximating” algorithms. The Cauchy representation is admissible.

An example of a computable metric space is \((\mathbb{R}, d_{\mathbb{R}}, \mathbb{Q}, v_{\mathbb{Q}})\), where \( \mathbb{R} \) is the set of real numbers, \( d_{\mathbb{R}}(x, y) = |x - y| \). The Cauchy representation associated with \((\mathbb{R}, d_{\mathbb{R}}, \mathbb{Q}, v_{\mathbb{Q}})\) is denoted by \( \rho \). For any \( x \in \mathbb{R} \) and \( p = 0^{p_0}10^{p_1}1 \ldots \in \Sigma^\omega \), \( \rho(p) = x \) if the sequence \( \{v_{\mathbb{Q}}(p_k)\} \) of rational numbers converges to \( x \) rapidly. For \( d \)-dimensional Euclidean space \( \mathbb{R}^d \), we use \( \rho^d \) to denote the corresponding Cauchy representation.

In the remaining of this section, we review several function spaces and some Cauchy representations associated with these spaces. These Cauchy representations will be used in later sections to lay down our main results. Let \( C^\infty(\mathbb{R}) \) denote the set of infinitely differentiable functions defined on \( \mathbb{R} \).

**Definition 4** (Function spaces).

1. (Schwartz space): The Schwartz space \( S(\mathbb{R}) \) is defined as follows: \( S(\mathbb{R}) = \{ \phi \in C^\infty(\mathbb{R}) : \forall \alpha, \beta \in \mathbb{N}, \sup_{x} |x^\alpha \phi^{(\beta)}(x)| < \infty \} \) with metric

\[
d_S(\phi, \varphi) = \sum_{\alpha, \beta = 0}^{\infty} 2^{-\alpha, \beta} \frac{\|\phi - \varphi\|_{\alpha, \beta}}{1 + \|\phi - \varphi\|_{\alpha, \beta}}, \quad \forall \phi, \varphi \in S(\mathbb{R}),
\]

where \( \|\phi\|_{\alpha, \beta} := \sup_{x} |x^\alpha \phi^{(\beta)}(x)| \), \( \phi^{(\beta)} \) is the \( \beta \)th derivative of \( \phi \), and \( \langle , \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) is some standard pairing function.
(2) \((L^p\text{-space}):\) For any \(1 \leq p < \infty\), the space \(L^p(\mathbb{R}^d)\) is the set of all measurable complex valued functions \(f\) such that \(\int_{\mathbb{R}^d} |f(x)|^p dx < \infty\) with the norm \(\|f\|_{L^p} = \left\{\int_{\mathbb{R}^d} |f(x)|^p dx\right\}^{1/p}.

(3) \((\text{Sobolev space}):\) For any \(s \in \mathbb{R}\), the Sobolev space \(H^s(\mathbb{R}^d)\) is the set of all functions \(f \in L^2(\mathbb{R}^d)\) such that

\[
\|f\|_{H^s} = \left(\int_{\mathbb{R}^d} (1 + |x|^2)^{s/2} |f(x)|^2 dx\right)^{1/2},
\]

where \(\|f(x)|^2 \leq \sum_{|\alpha| = k} \|D^\alpha f\|_{L^2}^2\) for all \(k \geq 0\) is a non-negative integer, the norm \(\|\cdot\|_{H^k}\) is equivalent to the norm \(\|\cdot\|_{H^k}^p\) defined as

\[
\|f\|_{H^k}^p = \left(\|f\|_{L^2}^{2} + \|f'\|_{L^2}^{2} + \cdots + \|f^{(k)}\|_{L^2}^{2}\right)^{1/2},
\]

where \(f^{(k)}\) is a multi index of order \(|\alpha| = \alpha_1 + \cdots + \alpha_d\) is called a multi index of order \(|\alpha| = \alpha_1 + \cdots + \alpha_d\), each component \(\alpha_i\) is a nonnegative integer. \(D^\alpha f\) is defined as

\[
D^\alpha f = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d} f.
\]

To introduce a Cauchy representation on a computable metric space, we need to identify a countable dense set \(D\). Our choice for \(D\) in \(L^p(\mathbb{R}^d)\) is the set of all rational complex valued finite step functions defined on \(\mathbb{R}^d\) as follows:

\[
D = \left\{ \sum_{i=0}^{k} c_i \cdot I_{r_i,s_i} : k \in \mathbb{N}, c_i \text{ rational complex}, r^i, s^i \in \mathbb{Q}^d, r^i < s^i \right\},
\]

where \(r^i = (r^i_1, r^i_2, \ldots, r^i_d), s^i = (s^i_1, s^i_2, \ldots, s^i_d), r^i < s^i\) if and only if \(r_j < s_j\) for all \(1 \leq j \leq d\), \(r^i, s^i \in \mathbb{Q}^d\) and \(I_{r^i,s^i}(x) = 1\) if \(x \in (r^i, s^i)\) and \(I_{r^i,s^i}(x) = 0\) otherwise. The set \(D\) is dense in \(L^p(\mathbb{R}^d)\). Let \(v_D : \mathbb{N} \rightarrow D\) be some standard numbering of \(D\). Then for any computable \(p, (L^p(\mathbb{R}^d), \|\cdot\|_{L^p}, D, v_D)\) is a computable metric space.

**Definition 5 (Representing \(L^p\)-functions).** Let \(\delta_{L^p}\) be the Cauchy representation associated with this computable metric space. Namely, \(p = 0^{p_0}10^{p_1}1 \ldots\) is a \(\delta_{L^p}\)-name of an \(L^p\)-function \(f\) if \(\|f - v_D(p_k)\|_{L^p} \leq 2^{-k}\) for all \(k \in \mathbb{N}\).

Roughly speaking, a \(\delta_{L^p}\)-name of an \(L^p\)-function \(f\) is a (coded) sequence of rational finite step functions convergent to \(f\) rapidly in \(L^p\)-norm. Since Sobolev \(H^s\)-functions are weighted \(L^2\)-functions, it is natural to represent Sobolev functions in terms of \(\delta_{L^2}\).

**Definition 6 (Representing Sobolev functions).** For any \(s \in \mathbb{R}\), define a representation \(\delta_{H^s} : \Sigma^0 \rightarrow H^s(\mathbb{R}^d)\) by

\[
\delta_{H^s}(q) := T_s^{-1} \circ \delta_{L^2}(q).
\]

Thus a sequence \(q \in \Sigma^0\) is a \(\delta_{H^s}\)-name of \(f \in H^s(\mathbb{R}^d)\) if it is a \(\delta_{L^2}\)-name of the weighted Fourier transform \(T_s(f)\) of \(f\). By the definitions above, computations on \(L^2(\mathbb{R}^d)\) are carried out by Turing machines on coded sequences of rational finite step functions, and computations on \(H^s(\mathbb{R}^d)\) are simply reduced to computations on \(L^2(\mathbb{R}^d)\).

Since the representations \(\rho\) of \(\mathbb{R}\) and \(\delta_{L^p}\) of \(L^p(\mathbb{R}^d)\) are admissible, the representation \([\rho \rightarrow \delta_{L^p}]\) induced by \(\rho\) and \(\delta_{L^p}\) gives rise to a representation of \(C(\mathbb{R}; L^p(\mathbb{R}^d))\), the set of all continuous functions from \(\mathbb{R}\) to \(L^p(\mathbb{R}^d)\) with the compact-open topology. Also it can be proved that the
representation $T_s^{-1} \circ [\rho \rightarrow \delta_{L^2}]$ is the same as the representation $[\rho \rightarrow \delta_{H^s}]$ for the space $C(\mathbb{R}; H^s(\mathbb{R}^d))$, the set of all continuous functions defined on $\mathbb{R}$ with values in $H^s(\mathbb{R}^d)$.

**Proposition 7.** $[\rho \rightarrow \delta_{H^s}] = T_s^{-1} \circ [\rho \rightarrow \delta_{L^2}]$.

We omit the proof for it can be derived directly from Definition 6 and the definition of $[\delta \rightarrow \delta']$ (see, for example, Definition 3.3.13 in [13]). This proposition allows us to reduce computations on $C(\mathbb{R}; H^s(\mathbb{R}^d))$ to computations on $C(\mathbb{R}; L^2(\mathbb{R}^d))$.

For the Schwartz space $\mathcal{S}(\mathbb{R})$ we choose the set of “smoothly truncated polynomials with complex rational coefficients” as a desired countable dense set. This set is defined as follows: Let $\gamma_n(x) = 3 \cdot 2^n + 1 \cdot 3 \cdot 2^{n+1}$ be a sequence of mollifying functions, where

$$
\gamma(x) = \begin{cases} 
az e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\
0 & \text{otherwise}
\end{cases}
$$

$a$ is a constant such that the integral $\int_{\mathbb{R}} \gamma(x) \, dx = 1$. Let $\mathcal{P}$ be the set of complex-valued polynomials defined on $\mathbb{R}$ with complex rational coefficients and $J_n^k$ be the characteristic function of the closed interval $\{x \in \mathbb{R} \mid |x| \leq k - \frac{1}{2^n}\}$, $k, n \in \mathbb{N}$. For any $k \in \mathbb{N}$, $k \geq 1$, let $\mathcal{P}_k = \{\gamma_n \ast (P \cdot J_n^k) : n \in \mathbb{N}, P \in \mathcal{P}\}$, where $f * g$ is the convolution of $f$ and $g$. It is well-known in analysis that every function in $\mathcal{P}_k$ is $C^\infty$ with support contained in $\{x \in \mathbb{R} \mid |x| \leq k\}$. Let $\mathcal{P}^* = \bigcup_{k=1}^{\infty} \mathcal{P}_k$ and $\nu_{\mathcal{P}}$ be some standard numbering of $\mathcal{P}^*$. Then $(\mathcal{S}(\mathbb{R}), d_S, \mathcal{P}^*, \nu_{\mathcal{P}})$ is a computable metric space (see, for example, [16]).

**Definition 8 (Representing Schwartz functions).** Let $\delta_S$ be the Cauchy representation associated with the computable metric space $(\mathcal{S}(\mathbb{R}), d_S, \mathcal{P}^*, \nu_{\mathcal{P}})$. Namely, $q = (q_0, q_1, q_2, \ldots)$ is a $\delta_S$-name of a Schwartz function $f$ if $\nu_{\mathcal{P}}(q_k)$ is a smoothly truncated rational polynomial and $d_S(f, \nu_{\mathcal{S}}(q_k)) \leq 2^{-k}$ for all $k \in \mathbb{N}$.

The following two lemmas address computability of basic arithmetics, integration, and Fourier transform on Schwartz space $\mathcal{S}(\mathbb{R})$ ([16, Lemma 5.7], [15, Lemma 3.7]).

**Lemma 9.**

1. On $\mathcal{S}(\mathbb{R})$, the function $(a, \psi) \mapsto a\psi$ is $(\rho, \delta_S, \delta_S)$-computable; the absolute evaluation $(\psi, t) \mapsto |\psi(t)|$ is $(\delta_S, \rho, \rho)$-computable; the addition $(\phi, \psi) \mapsto \phi + \psi$ and the multiplication $(\phi, \psi) \mapsto \phi \cdot \psi$ are $(\delta_S, \delta_S, \delta_S)$-computable.

2. The function $(\psi, t) \mapsto E_m(t) \cdot \psi, E_m(t)(\xi) := e^{-\xi^2 t - i m t}$ is $(\delta_S, \rho, \delta_S)$-computable for computable $m \in \mathbb{R}$.

3. The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}), \phi \mapsto (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i \xi x} \phi(x) \, dx$, and the inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}), \phi \mapsto (2\pi)^{-1/2} \int_{\mathbb{R}} e^{i \xi x} \phi(\xi) \, d\xi$, are both $(\delta_S, \delta_S)$-computable.

**Lemma 10.** The function $H : C(\mathbb{R}; \mathcal{S}(\mathbb{R})) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{S}(\mathbb{R}), H(u, a, b) = \int_a^b u(t) \, dt$, is $(\rho \rightarrow \delta_S, \rho, \rho, \delta_S)$-computable.

In other words, Lemma 10 says that there is a Turing machine which computes (function) approximations to $\int_a^b u(t) \, dt$ with arbitrary precision from approximations to $u$, $a$, and $b$. 
For integrating a function \( f \in C(\mathbb{R}; L^2(\mathbb{R}^d)) \) we need its modulus of continuity. In the remaining of this section, we show how to construct an algorithm that computes a modulus of uniform continuity for \( f \in C(\mathbb{R}; L^2(\mathbb{R}^d)) \) on the interval \([0, 1]\). This algorithm will be used in the proof of Theorem 14. A modulus of uniform continuity of \( f : \mathbb{R} \to L^2(\mathbb{R}^d) \) on \([0, 1]\) is a function \( \mu : \mathbb{N} \to \mathbb{N} \) such that \( \| f(x) - f(y) \|_{L^2} \leq 2^{-\mu(n)} \) whenever \( |x - y| \leq 2^{-\mu(n)} \) for any \( x, y \in [0, 1] \). For the construction another representation of \( C(\mathbb{R}; L^2(\mathbb{R}^d)) \) other than \( [\rho \to \delta_{L^2}] \) is needed. We recall that the topology on \( C(\mathbb{R}; L^2(\mathbb{R}^d)) \) is the compact-open topology.

Consider the set of all bounded rational closed intervals in \( \mathbb{R} \).

**Definition 11.** Define the compact-open representation \( \delta_{co} \) of \( C(\mathbb{R}; L^2(\mathbb{R}^d)) \) as follows: for any \( h \in C(\mathbb{R}; L^2(\mathbb{R}^d)) \) and \( q \in \Sigma^\omega \), \( \delta_{co}(q) = h \) iff there are numbers \( q_0, q_1, \ldots \in \text{dom}(\nu_U) \) such that \( q = 0^{q_0} 1^{q_1} 0^{q_2} \ldots \) and

\[
\{ V \in U \mid h \in V \} = \{ \nu_U(q_i) \mid i \in \mathbb{N} \}.
\]

In other words, \( \delta_{co}(q) = h \) if \( p \) is (encodes) a list of all pairs \((I, B)\) such that \( h[I] \subseteq B \). The two representations, \([\rho \to \delta_{L^2}]\) and \( \delta_{co} \), of \( C(\mathbb{R}; L^2(\mathbb{R}^d)) \) are equivalent. We recall that two representations \( \delta \) and \( \delta' \) of a set \( X \) are called equivalent (denoted as \( \delta \equiv \delta' \)) if the identity map \( id : X \to X \) is \((\delta, \delta')\)-computable as well as \((\delta', \delta)\)-computable. The proof of Lemma 12 is omitted for it is similar to that of Theorem 6.1.7 in [13].

**Lemma 12.** \( \delta_{co} \equiv [\rho \to \delta_{L^2}] \).

The following lemma provides an algorithm that computes a modulus function on \([0, 1]\) for any function \([\rho \to \delta_{L^2}](p) \in C(\mathbb{R}; L^2(\mathbb{R}^d)) \) from its given \([\rho \to \delta_{L^2}]-name \) \( p \).

**Lemma 13.** There is a computable function \( d : \Sigma^\omega \to \Sigma^\omega \) such that \( d(p) = 0^{\mu(0)} 1^{\mu(1)} 0^{\mu(2)} 1 \ldots \) for some modulus \( \mu \) of uniform continuity of \([\rho \to \delta_{L^2}](p) \) on \([0, 1]\) for all \( p \in \text{dom}(\{[\rho \to \delta_{L^2}]\}) \).

**Proof.** By Lemma 12 it suffices to consider \( \delta_{co} \) instead of \([\rho \to \delta_{L^2}]\). Assume that \( \delta_{co}(p) = h \). Consider \( n \in \mathbb{N} \). Since \( p \) is a list of all pairs \((I, B)\) such that \( h[I] \subseteq B \), there is, for every \( t \in [0, 1] \), some pair \((I_t, B_t)\) in the list such that \( B_t \) has a radius \( < 2^{-n-1} \) and \( t \in I_t/3 \), where \( [a; b]/3 := (a + (b - a)/3); b - (b - a)/3) \). Finitely many intervals \( I_t/3, \ldots, I_j/3 \) cover \([0, 1]\). Choose \( \mu(n) \) such that \( 2^{-\mu(n)} \) is less than the length of each of these \( I_t/3 \). Then \( \mu \) is a modulus of uniform continuity. By systematic search, some Type-2 machine can compute an encoded modulus from \( p \). \( \square \)
3. The linear Schrödinger equation

We now formulate and prove our first theorem. For a more rigorous formulation we replace $u$ by $\bar{u}$ in (1) and define $u(t)(x) := \bar{u}(t, x)$.

**Theorem 14.** For any computable $s \in \mathbb{R}$, consider the following initial-value problem for the inhomogeneous linear Schrödinger equation:

$$
\begin{aligned}
\bar{u}_t(t, x) &= i \Delta \bar{u}(t, x) + \phi(t)(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d, \ \phi \in C(\mathbb{R}; H^s(\mathbb{R}^d)), \\
u(0) &= f, \quad f \in H^s(\mathbb{R}^d).
\end{aligned}
$$

(4)

Then the solution operator $S : C(\mathbb{R}; H^s(\mathbb{R}^d)) \times H^s(\mathbb{R}^d) \times \mathbb{R} \to H^s(\mathbb{R}^d)$, $(f, f, t) \mapsto u(t)$, is $([\rho \to \delta_{H^s}], \delta_{H^s}, \rho, \delta_{H^s})$-computable.

**Proof.** For any given initial condition $f \in H^s(\mathbb{R}^d)$, continuous forcing $\phi : \mathbb{R} \to H^s(\mathbb{R}^d)$ and time $t$, the solution $u(t), u(t) \in H^s(\mathbb{R}^d)$, of problem (4) is given explicitly by the following solution formula

$$\mathcal{F}(u(t))(\xi) = E(t)(\xi) \cdot \mathcal{F}(f)(\xi) + \int_0^t E(t - \tau)(\xi) \cdot \mathcal{F}(\phi(\tau))(\xi) \, d\tau,$$

(5)

where $E$ is the map: $\mathbb{R} \to C(\mathbb{R}^d; \mathbb{C})$, $E(t)(\xi) = e^{-it|\xi|^2}$, and $\mathcal{F}(f)$ is the Fourier transform of $f$. Eq. (5) can be derived from (4) by a standard technique of ODE (ordinary differential equation): Taking the Fourier transform of (4) with respect to the variable $x$ yields $\hat{\bar{u}}_t = -i \hat{\Delta} \hat{u} + \hat{\phi}$ and $\hat{u}(0) = \hat{f}$, where $\hat{u} = \mathcal{F}(u)$. Notice that $\xi^2 = \xi_1^2 + \cdots + \xi_d^2 = |\xi|^2$. Assume that $u$ is the solution of problem (4). Let $w(s) = e^{-i\xi^2(t-s)} \hat{u}$. Then $dw/ds = e^{-i\xi^2(t-s)} \hat{\phi}$ (in the sense of weak derivative). Integrating from 0 to $t$ yields (5). Thus if (4) has a solution, this solution is given by (5).

Next multiplying (5) by $(1 + |\xi|^2)^{s/2}$ yields the equation

$$(1 + |\xi|^2)^{s/2} \cdot \mathcal{F}(u(t))(\xi) = E(t)(\xi) \cdot (1 + |\xi|^2)^{s/2} \cdot \mathcal{F}(f)(\xi)$$

$$+ \int_0^t E(t - \tau)(\xi) \cdot (1 + |\xi|^2)^{s/2} \cdot \mathcal{F}(\phi(\tau))(\xi) \, d\tau.$$ 

In terms of $T_s$ from Definition 4, the above integral equation can be rewritten in the following form:

$$T_s(u(t)) = E(t) \cdot T_s(f) + \int_0^t E(t - \tau) \cdot T_s(\phi(\tau)) \, d\tau$$

with $T_s(u(t)), T_s(f) \in L^2(\mathbb{R}^d)$ and $T_s(\phi) \in C(\mathbb{R}; L^2(\mathbb{R}^d))$. Thus, to show that $u(t)$ can be computed from $f, \phi$, and $t$, it is enough to prove that

$$(g, \psi, t) \mapsto E(t) \cdot g + \int_0^t E(t - \tau) \cdot \psi(\tau) \, d\tau$$

(6)

is $(\delta_{L^2}, [\rho \to \delta_{L^2}], \rho, \delta_{L^2})$-computable. The following parts constitute this proof.
(a) It is easy to see that the function \((t, \xi) \mapsto e^{-i|\xi|^2} \) is \((\rho, \rho^d, \rho^2)\)-computable. Then by Lemma 2 (type conversion), the function \(E : \mathbb{R} \to C(\mathbb{R}^d; \mathbb{C}), E(t)(\xi) = e^{-i|\xi|^2} \), is \((\rho, [\rho^d \to \rho^2])\)-computable.

(b) By generalizing the proof of Lemma 4.3 in [14] from dimension 3 to dimension \(d\), it can be proved that addition \((f, g) \mapsto f + g\) is \((\delta_{L^2}, \delta_{L^2}, \delta_{L^2})\)-computable and multiplication \((f, g, K) \mapsto fg\), where \(f \in C(\mathbb{R}^d; \mathbb{C})\) satisfying \(\sup_{x \in \mathbb{R}} |f(x)| < K\) and \(g \in L^2(\mathbb{R}^d)\) (i.e., \(fg\) is the multiplication of the bounded continuous function \(f\) and the \(L^2\)-function \(g\)) is \((\rho^d \to \rho^2], \delta_{L^2}, \rho, \delta_{L^2})\) computable.

(c) We conclude from (a) and (b) that \((g, t) \mapsto E(t) \cdot g\) is \((\delta_{L^2}, \rho, \delta_{L^2})\)-computable and the map \(h, h(t, \psi)(\tau) = E(t - \tau) \cdot \psi(\tau)\), is \((\rho, [\rho \to \delta_{L^2}], [\rho \to \delta_{L^2}])\)-computable.

(d) Finally what is left to show is that the integration

\[
(a, b, h) \mapsto \int_a^b h(\tau) \, d\tau, \quad a, b \in \mathbb{R}, \quad h \in C(\mathbb{R}; L^2(\mathbb{R}^d))
\]

is \((\rho, \rho, [\rho \to \delta_{L^2}], \delta_{L^2})\)-computable. To this end, let us first consider the special case when \(a = 0\) and \(b = 1\). The integral can be defined by the limit of Riemann sums:

\[
\int_0^1 h(\tau) \, d\tau = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^k h\left(\frac{i}{k}\right).
\]

For obtaining an approximate sum with error \(\leq 2^{-n-1}\) (which can be used as the \(n\)th term in a Cauchy name), choose \(k = 2^{\mu(n+2)}\), where \(\mu\) is a modulus of uniform continuity of \(h\) on the interval \([0, 1]\) and compute \(h\left(\frac{i}{k}\right)\) with precision \(2^{-n-2}\). This computation is possible by Lemma 2 (evaluation) and the fact that the modulus \(\mu\) can be computed from any given \([\rho \to \delta_{L^2}]\)-name of \(h\) (Lemma 12). The general case \(a, b\) can be reduced to the special one (cf. [13, proof of Theorem 6.4.1.2]). \(\square\)

Corollary 15. The solution operator \((\phi, f) \mapsto u\) of the initial-value problem

\[
\begin{aligned}
&\bar{u}_t(t, x) = i\Delta u(t, x) + \phi(t)(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d, \ \phi \in C(\mathbb{R}; H^s(\mathbb{R}^d)), \\
u(0) = f, \quad f \in H^s(\mathbb{R}^d)
\end{aligned}
\]

is \(([\rho \to \delta_{H^s}], \delta_{H^s}, [\rho \to \delta_{H^s}])\)-computable.

Proof. Apply Lemma 2 (type conversion) to Theorem 14. \(\square\)

Corollary 16. The solution operator \((f, t) \mapsto u(t)\) of the initial-value problem for the homogeneous linear Schrödinger equation

\[
u_t = i\Delta u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d, \ u(0) = f, \quad f \in H^s(\mathbb{R}^d) \quad (7)
\]

is \((\delta_{H^s}, \rho, \delta_{H^s})\)-computable.

4. The Schrödinger propagator on \(L^p\)-spaces

Since \(H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)\) and the Fourier transform on \(L^2(\mathbb{R}^d)\) is \((\delta_{L^2}, \delta_{L^2})\)-computable [8], Corollary 16 implies that the solution operator \((f, t) \mapsto u(t)\) of the initial-value problem (7) for the homogeneous linear Schrödinger equation is \((\delta_{L^2}, \rho, \delta_{L^2})\)-computable. Is it still possible
to compute the solution operator when the initial function $f$ is $L^p$ with $p \neq 2$? The following theorem answers this question negatively.

**Theorem 17.** For any computable real number $t \neq 0$, the solution operator $S(t) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ of problem (7), $f \mapsto u(t)$, is $(\delta_{L^p}, \delta_{L^p})$-computable if and only if $p = 2$.

**Proof.** We consider the case when $p < 2$. The same argument applies to the situation where $p > 2$. For any $t \neq 0$, we compute the solution of the Schrödinger equation $u_t = i\Delta u$ with the initial value $g(x) = f(x)/\|f\|_{L^p}$, where $f(x) = e^{-a|x|^2/2}$ and $a > 0$ is a constant. We note that $g$ is infinitely differentiable, $g \in L^p(\mathbb{R}^d)$, and $\|g\|_{L^p} = 1$. Since

$$S(t)g = F^{-1}(e^{-it|\xi|^2} \hat{f})/\|f\|_{L^p} = \left(\frac{1}{a} - d/2(1/a + 2it)^{-d/2}e^{-|x|^2/(2(1/a + 2it))}\right)/\|f\|_{L^p},$$

the $L^p$-norm of $S(t)g$ can be computed explicitly as follows:

$$\|S(t)g\|_{L^p}^p = \int_{\mathbb{R}^d} \left|a^{-d/2}(1/a + 2it)^{-d/2}e^{-|x|^2/(2(1/a + 2it))}\right|^p dx / \|f\|_{L^p}^p$$

$$= \frac{|1 + 2ita|^{-pd/2}}{\|f\|_{L^p}^p} \int_{\mathbb{R}^d} e^{-a|x|^2/(2(1 + 4a^2t^2))} e^{(a^2t^2|x|^2)/(1 + 4a^2t^2)} |x|^p dx$$

$$= \frac{|1 + 2ita|^{-pd/2}}{\|f\|_{L^p}^p} \int_{\mathbb{R}^d} e^{-pa|x|^2/(2(1 + 4a^2t^2))} dx$$

$$= \frac{|1 + 2ita|^{-pd/2}}{\|f\|_{L^p}^p} \left(\frac{1 + 4a^2t^2}{ap}\right)^{d/2} (2\pi)^{d/2}$$

$$= \frac{|1 + 2ita|^{-pd/2}}{\|f\|_{L^p}^p} \left(\frac{2\pi}{ap}\right)^{d/2}$$

$$= |1 + 2ita|^{-pd/2} \left(1 + 4t^2a^2\right)^{d/2} \to \infty$$

as $a \to +\infty$. Hence

$$\sup_{\phi \in L^p(\mathbb{R}^d), \|\phi\|_{L^p} = 1} \|S(t)\phi\|_{L^p} = \infty.$$

This shows that the operator $S(t)$ is unbounded on $L^p(\mathbb{R}^d)$. Since $S(t)$ is a linear operator, the unboundedness implies that $S(t)$ is not continuous. We recall that $\delta_{L^p}$ is admissible, therefore, any operator $F : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is continuous if and only if it is $(\delta_{L^p}, \delta_{L^p})$-continuous. Since $S(t)$ is not continuous, it is not $(\delta_{L^p}, \delta_{L^p})$-continuous, hence it is not $(\delta_{L^p}, \delta_{L^p})$-computable. 

Let $(X, \delta)$ and $(Y, \delta')$ be two represented spaces. By definition, if $f$ is $(\delta, \delta')$-computable, then it preserves computability, that is, it maps every $\delta$-computable element in $X$ to a $\delta'$-computable element in $Y$. The converse however might not be true. Theorem 17 shows that $S(t)$ is not $(\delta_{L^p}, \delta_{L^p})$-computable for any $t \neq 0$ and $p \neq 2$. Does $S(t)$ preserve computability? The answer is again negative.
Corollary 18. For any $t \neq 0$ and any computable $p \neq 2$, the solution operator $S(t) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ of problem (7) does not preserve computability.

Proof. As is known classically, $S(t)$ is a closed linear operator. By the proof of Theorem 17, $S(t)$ is unbounded. According to Pour-El and Richards’ First Main Theorem [9], a closed linear map $F : X \to Y$ between Banach spaces must be bounded if it maps every computable element in $X$ to a computable element in $Y$. Since $S(t)$ is unbounded, it must map some $\delta_{L^p}$-computable initial $L^p(\mathbb{R}^d)$ function $f$ to a solution $S(t)f \in L^p(\mathbb{R}^d)$ that is not $\delta_{L^p}$-computable. □

5. A nonlinear Schrödinger propagator

In this section we prove our second main theorem. As before we define $u(t)(x) := u(t, x)$.

Theorem 19. The solution operator $S : H^1(\mathbb{R}) \to C(\mathbb{R}; H^1(\mathbb{R}))$, $\varphi \mapsto u$, of the initial-value problem for the nonlinear Schrödinger equation

\[
\frac{d\tilde{u}}{dt} = -\frac{d^2\tilde{u}}{dx^2} + m\tilde{u} + |\tilde{u}|^2\tilde{u}, \quad u(0) = \varphi,
\]

where $m$ is a computable real number, is $(\delta_{H^1}, [\rho \to \delta_{H^1}])$-computable.

In words, this is to say that there exists a Turing machine that computes approximations (with arbitrary precision) to the solution $u$ from approximations to the initial data $\varphi$. The proof requires another representation $\tilde{\delta}_{H^s}$ of $H^s(\mathbb{R})$ that is equivalent to the representation $\delta_{H^s}$. The new representation $\tilde{\delta}_{H^s}$ makes use of the fact that the set of Schwartz functions is dense in $H^s(\mathbb{R})$.

Definition 20. The representation $\tilde{\delta}_{H^s} : \Sigma^\omega \to H^s(\mathbb{R})$ is defined as follows: for any $f \in H^s(\mathbb{R})$ and any infinite tuple $p = (p_0, p_1, p_2, \ldots) \in \Sigma^\omega$,

\[
\tilde{\delta}_{H^s}(p) = f \iff p_i \in \text{dom}(\delta_{\tilde{S}}) \quad \text{and} \quad \|\tilde{S}(p_i) - f\|_{H^s} \leq 1/2^i.
\]

Thus, a $\tilde{\delta}_{H^s}$-name of a Sobolev function $f$ is a sequence (of names) of Schwartz functions that converges to $f$ rapidly in $H^s$-norm. Here, an infinite tuple $(p_0, p_1, \ldots)$ is defined as follows: $(p_0, p_1, p_2, \ldots)(\langle k, n \rangle) = p_k(n)$. Recall that $\langle , , \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is some standard paring function.

The proof of the following lemma can be found in [15].

Lemma 21. $\delta_{H^s} \equiv \tilde{\delta}_{H^s}$, if $s \in \mathbb{R}$ is computable.

To prepare for the proof of Theorem 19, we first present and prove several lemmas. Physically, if $u$ is a solution of the Schrödinger equation (8), then its $L^2$-norm and the energy are conserved, which in turn implies the following estimates:

Lemma 22. If $u : \mathbb{R} \to H^1(\mathbb{R})$ is a solution of (8), then for any $t, t', t_0 \in \mathbb{R},$

\[
\|u(t)\|_{H^1} \leq f(\|u(t_0)\|_{H^1}),
\]

\[
\|U(t-t', u(t'))\|_{H^1} \leq f(\|u(t_0)\|_{H^1}),
\]

(9) (10)
where \( f(x) = x \cdot (1 + 4\sqrt{2}x^2)^{1/2} \) and \( U : \mathbb{R} \times S(\mathbb{R}) \to S(\mathbb{R}) \) is the so-called free evolution defined by
\[
U(t, \psi)(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{i\xi x - i|\xi|^2 t - i\xi t} \mathcal{F}(\psi)(\xi) \, d\xi.
\]
(11)

In particular, for any \( \phi \in H^1(\mathbb{R}) \) and any \( t \in \mathbb{R} \), \( \|U(t, \phi)\|_{H^1} \leq f(\|\phi\|_{H^1}) \).

**Proof.** See Appendix. \( \square \)

In the following we fix an initial condition \( \varphi \in H^1(\mathbb{R}) \) of problem (8)
\[
i \frac{d\bar{u}}{dt} = -\frac{d^2 \bar{u}}{dx^2} + m\bar{u} + |\bar{u}|^2 \bar{u}, \quad u(0) = \varphi.
\]
Classically, the solution of the initial-value problem (8) can be constructed, in terms of \( t_0 \) and \( u(t_0) \), locally in a neighborhood of \( t_0 \) by making use of the contraction-mapping principle, starting from \( t_0 = 0 \). Assume that \( u(t) \) has been constructed over the time interval \([0, t_0]\). We show in the following how to extend the construction in a neighborhood of \( t = t_0 \). In order to formally describe this construction, let us consider the following problem with the value \( w(t_0) \) given:
\[
i \frac{dw}{dt} = -\frac{d^2 w}{dx^2} + mw + |w|^2 w, \quad w(t_0) = \psi, \quad \psi \in H^1(\mathbb{R}).
\]
(12)
The same technique used to derive (5) from (4) can be applied to recast problem (12) in the form of the following integral equation
\[
w(t) = U(t - t_0, \psi) - i \int_{t_0}^t U(t - \tau, |w(\tau)|^2 w(\tau)) \, d\tau.
\]
(13)
We note that if \( \psi = u(t_0) \) in (12), then by the uniqueness of the solution of (8), the solution of (13) is also the solution of (8).

Associated with the above integral equation we define two maps \( A \) and \( G \). For \( t_0 \in \mathbb{R} \), the map \( G(t_0) : C(\mathbb{R}; H^1(\mathbb{R})) \to C(\mathbb{R}; H^1(\mathbb{R})) \) is defined by
\[
G(t_0)(\psi)(t) = -i \int_{t_0}^t U(t - \tau, |\psi(\tau)|^2 \psi(\tau)) \, d\tau,
\]
(14)
and for \( t_0 \in \mathbb{R} \) and \( \psi \in H^1(\mathbb{R}) \), the map \( A(t_0, \psi) : C(\mathbb{R}; H^1(\mathbb{R})) \to C(\mathbb{R}; H^1(\mathbb{R})) \) is defined by
\[
A(t_0, \psi)(\psi)(t) = U(t - t_0, \psi) + G(t_0)(\psi)(t).
\]
(15)
The following lemma shows that both \( A \) and \( G \) are contraction mappings in a neighborhood of \( t_0 \) for “not too big” \( \psi \) compared with \( \|\varphi\|_{H^1} \).

**Lemma 23.** Let \( r_\varphi \) be a rational number such that \( r_\varphi - 1 \leq \|\varphi\|_{H^1} \leq r_\varphi \), let \( R_\varphi \) be the least integer upper bound of \( f(r_\varphi) + 1 \) and let \( I := [t_0, t_0 + T_\varphi] \), where \( T_\varphi = 1/(32R_\varphi^2) \). Then, for any \( \psi \in H^1(\mathbb{R}) \) such that \( \|\psi\|_{H^1} \leq R_\varphi \) and any \( v_1, v_2 \in \{ v \in C(\mathbb{R}; H^1(\mathbb{R})) : \|v\|_I \leq \frac{4}{3} R_\varphi \} \),
\[
\|A(t_0, \psi)(v_1) - A(t_0, \psi)(v_2)\|_I = \|G(t_0)(v_1) - G(t_0)(v_2)\|_I \leq \frac{1}{2} \|v_1 - v_2\|_I,
\]
(16)
where \( \|v\|_I := \|v\|_{C(I; H^1(\mathbb{R}))} = \sup_{t \in I} \|v(t)\|_{H^1} \) for \( v \in C(\mathbb{R}; H^1(\mathbb{R})) \).
Proof. By Fact 30 in Appendix and the Minkowski inequality \((\int_{\mathbb{R}} |f + g|^2 dx)^{1/2} \leq (\int_{\mathbb{R}} |f|^2 dx)^{1/2} + (\int_{\mathbb{R}} |g|^2 dx)^{1/2}\), it follows that for any \(u, v \in H^1(\mathbb{R})\), \(\|uv\|_{H^1} \leq \sqrt{\frac{3}{2} \|u\|_{H^1} \|v\|_{H^1}}\). Let \(C_0 = \sqrt{\frac{3}{2}}\).

Then for any \(v_1, v_2 \in C(\mathbb{R}; H^1(\mathbb{R}))\) satisfying \(\|v_1\|_I \leq \frac{1}{4} R_\varphi\) and \(\|v_2\|_I \leq \frac{1}{4} R_\varphi\), we have

\[
\|G(t_0)(v_1) - G(t_0)(v_2)\|_I \\
= \sup_{t \in I} \left\| \int_{t_0}^t U(t - \tau, \varphi) v_1(\tau) - v_2(\tau) d\tau \right\|_{H^1} \\
\leq T_\varphi \sup_{t \in I, \tau \leq \tau \leq t} \|U(t - \tau, \varphi) v_1(\tau) - v_2(\tau)\|_{H^1} \\
= T_\varphi \sup_{\tau \in I} \|v_1(\tau) - v_2(\tau)\|_{H^1} \\
= T_\varphi \|v_1 - v_2\|_I,
\]

where \(\bar{v}\) is the complex conjugate of \(v\). Since

\[
v_1^2 \bar{v}_1 - v_2^2 \bar{v}_2 = v_1^2 \bar{v}_1 - v_2^2 \bar{v}_1 + v_2^2 \bar{v}_1 - v_2^2 \bar{v}_2 \\
= (v_1^2 - v_2^2) \bar{v}_1 + v_2^2 (\bar{v}_1 - \bar{v}_2) \\
= (v_1 + v_2) \bar{v}_1 (v_1 - v_2) + v_2^2 (\bar{v}_1 - \bar{v}_2),
\]

it follows that

\[
\|v_1^2 \bar{v}_1 - v_2^2 \bar{v}_2\|_I \leq C_0 \left( (\|v_1\|_I + \|v_2\|_I) \|\bar{v}_1\|_I \|v_1 - v_2\|_I + \|v_2\|^2 \|\bar{v}_1 - \bar{v}_2\|_I \right) \\
\leq C_0 \left( \frac{32}{9} R_\varphi^2 + \frac{16}{9} R_\varphi^2 \right) \|v_1 - v_2\|_I = \frac{48}{9} C_0 R_\varphi^2 \|v_1 - v_2\|_I
\]

Thus,

\[
\|G(t_0)(v_1) - G(t_0)(v_2)\|_I \leq T_\varphi \frac{48}{9} C_0 R_\varphi^2 \|v_1 - v_2\|_I \\
= \frac{1}{32 R_\varphi^2} \cdot \frac{48}{9} \cdot \frac{5}{2} R_\varphi^2 \|v_1 - v_2\|_I \\
= \frac{5}{12} \|v_1 - v_2\|_I < \frac{1}{2} \|v_1 - v_2\|_I.
\]

We observe that the constant \(T_\varphi\) defined in Lemma 23 depends only on the size of the initial value \(\varphi\) of problem (8). By Eqs. (13) and (16), the fixed point of the contraction \(A(t_0, \psi)\) is the solution of (13) over the time interval \(I\) satisfying \(w(t_0) = \psi\). Thus, we can compute the solution with the initial data from \(H^1(\mathbb{R})\) as long as we can compute \(A\) on \(H^1(\mathbb{R})\). The following lemma shows that the restriction of \(A\) on the Schwartz space \(S(\mathbb{R})\), a subset of \(H^1(\mathbb{R})\), is computable. This restriction will also be denoted as \(A\).

**Lemma 24.** The restriction of the operator \(A\) to \(S(\mathbb{R}): \mathbb{R} \times S(\mathbb{R}) \times C(\mathbb{R}; S(\mathbb{R})) \rightarrow C(\mathbb{R}; S(\mathbb{R}))\), \(t_0, \psi, v \mapsto A(t_0, \psi)(v)\), is \((\rho, \delta_S, [\rho \rightarrow \delta_S], [\rho \rightarrow \delta_S])\)-computable.

**Proof.** By Lemmas 9 and 10, the function \((t_0, \psi, v, t) \mapsto A(t_0, \psi)(v)(t)\) is \((\rho, \delta_S, [\rho \rightarrow \delta_S], \rho, \delta_S)\)-computable. Applying Lemma 2, \(t_0, \psi, v \mapsto A(t_0, \psi)(v)\) is then \((\rho, \delta_S, [\rho \rightarrow \delta_S], [\rho \rightarrow \delta_S])\)-computable. \(\square\)
Corollary 25. The map
\[ F_1 : (t_0, \psi, n) \mapsto (A(t_0, \psi))^n(0), \]
where \( (A(t_0, \psi))^n(v) = A(t_0, \psi)((A(t_0, \psi))^{n-1}(v)) \) is the \( n \)th iteration of \( A(t_0, \psi) \), is \((\rho, \delta, v_{[\rho]}, [\rho \to \delta])\)-computable.

Proof. This is true because \( F_1 \) is a primitive recursion of the computable operator \( A \) (Lemma 24). \( \square \)

Since \( S(\mathbb{R}) \) is dense in \( H^1(\mathbb{R}) \) (see Lemma 21), it is possible to approximate \( \psi \in H^1(\mathbb{R}) \) by a sequence \( \psi_k \) of \( S(\mathbb{R}) \)-functions such that \( \| \psi_k - \psi \|_{H^1} \leq 2^{-k} \). The next lemma presents an algorithm to choose approximations among \( A(t_0, \psi_k)^n(0) \) that converge to the fixed point of \( A(t_0, \psi) \) rapidly.

Lemma 26. There are two computable functions \( g_1, g_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that the following holds: for \( \psi = u(t_0) \in H^1(\mathbb{R}) \) and any sequence \( \{\psi_k\} \subset S(\mathbb{R}) \) satisfying \( \| \psi - \psi_k \|_{H^1} \leq 2^{-k} \) for \( k = 0, 1, \ldots \),
\[ \| v_{\psi} - (A(t_0, \psi g_2(R_\psi, j))^{g_1(R_\psi, j)}(0) \|_I \leq 2^{-j}, \]
where \( v_{\psi} \) is the fixed point of \( A(t_0, \psi) \) in \( \{ v \in C(I; H^1(\mathbb{R})) : \| v \|_I \leq \frac{4}{3} R_\psi \} \) with \( I = [t_0, t_0 + T_\psi] \). In particular, for any \( t \in I \),
\[ \| v_{\psi}(t) - (A(t_0, \psi g_2(R_\psi, j))^{g_1(R_\psi, j)}(0)(t) \|_{H^1(I)} \leq 2^{-j}. \]

Proof. First we prove that for any \( \phi \in H^1(\mathbb{R}) \) satisfying \( \| \phi \|_{H^1} \leq R_\phi \), \( \| (A(t_0, \phi))^n(0) \|_I \leq \frac{4}{3} R_\phi \) for all \( n \in \mathbb{N} \). For \( n = 1 \), \( \| A(t_0, \phi)(0) \|_I = \| \phi \|_{H^1} \leq R_\phi \) by assumption. Assume that \( \| (A(t_0, \phi))^n(0) \|_I \leq \frac{4}{3} R_\phi \) for all \( n \in \mathbb{N} \). Denote \( (A(t_0, \phi))^n(0) \) by \( v \). Then
\[ \| (A(t_0, \phi))^{n+1}(0) \|_I = \| A(t_0, \phi)(v) \|_I \leq \| \phi \|_{H^1} + T_\phi \| v^2 \|_I \]
\[ \leq R_\phi + T_\phi C_0^2 \| v \|_I^3 \leq R_\phi + \frac{1}{32 R_\phi^2} \cdot \frac{5}{2} \cdot \frac{64}{27} R_\phi^3 \]
\[ \leq R_\phi + \frac{5}{27} R_\phi < \frac{4}{3} R_\phi. \]
Thus, \( \| (A(t_0, \phi))^n(0) \|_I \leq \frac{4}{3} R_\phi \) for all \( n \in \mathbb{N} \), provided \( \| \phi \|_{H^1} \leq R_\phi \).

Since \( \psi = u(t_0) \), it follows from (9) that \( \| \psi \|_{H^1} \leq f(\| u(0) \|_{H^1}) = f(\| \phi \|_{H^1}) \leq R_\phi \). Then, by Lemma 23, \( A(t_0, \psi) \) is a contraction on \( \{ v \in C(I; H^1(\mathbb{R})) : \| v \|_I \leq \frac{4}{3} R_\phi \} \) and therefore has a fixed point \( v_{\psi} \).

Let \( v^n := (A(t_0, \psi^n))^n(0) \). Then \( \| v^n \|_I \leq \frac{4}{3} R_\phi \) for all \( n \in \mathbb{N} \). Let \( v^n := (A(t_0, \psi^n))^n(0) \). Since \( \| \psi - \psi_k \|_{H^1} \leq 2^{-k} \), \( \| \psi_k \|_{H^1} \leq \| \psi \|_{H^1} + \| \psi - \psi_k \|_{H^1} \leq f(\| u(0) \|_{H^1}) + 2^{-k} \leq f(\| u(0) \|_{H^1}) + 1 \leq R_\phi \) and consequently \( \| v^n \|_I \leq \frac{4}{3} R_\phi \) for all \( n \in \mathbb{N} \).

We observe that
\[ \| v_{\psi} - v^n \|_I \leq \| v_{\psi} - v^n \|_I + \| v^n - v^n \|_I. \]
Also we recall that for any function $h$ on a Banach space with $\|h(x) - h(y)\| \leq \frac{1}{2} \|x - y\|$, the fixed point $x_h$ of $h$ satisfies $\|x_h - h^n(0)\| \leq \|h(0)\| \cdot 2^{-n+1}$. Thus, by applying (10) we obtain
\[
\|v^n - v^n_k\|_I \leq 2^{-n+1} \|A(t_0, \psi)(0)\|_I \leq 2^{-n+1} \|U(t - t_0, \psi)\|_I
\leq 2^{-n+1} f(\|\varphi\|_{H^1}) \leq 2^{-n+1} R_\varphi.
\]
Next we show by induction that $\|v^n - v^n_k\|_I \leq n \cdot f(2^{-k})$. This is true for $n = 0$. Applying the special case of (10) with $\phi = \psi - \psi_k$ and Lemma 23, we obtain
\[
\|v^{n+1} - v_k^{n+1}\|_I = \|A(t_0, \psi)(v^n) - A(t_0, \psi_k)(v_k^n)\|_I
\leq \|U(t - t_0, \psi) - U(t - t_0, \psi_k)\|_I + \|G(t_0, v^n) - G(t_0, v_k^n)\|_I
\leq \|U(t - t_0, \psi - \psi_k)\|_I + \frac{1}{2} \|v^n - v_k^n\|_I
\leq f(\|\psi - \psi_k\|_{H^1}) + \frac{1}{2} n \cdot f(2^{-k})
\leq f(2^{-k}) + n \cdot f(2^{-k})
\leq (n + 1) \cdot f(2^{-k}).
\]
Now define
\[
\begin{align*}
g_1(R_\varphi, j) &:= \mu n \left[2^{-n+1} R_\varphi \leq 2^{-j-1} \right], \\
g_2(R_\varphi, j) &:= \mu k \left[g_1(R_\varphi, j) \cdot f(2^{-k}) \leq 2^{-j-1} \right].
\end{align*}
\]
The statement of the lemma follows from (17). \(\square\)

We remark that both Lemmas 23 and 26 hold true on the interval $[t_0 - T_\varphi, t_0]$. We also recall that $v_\psi$ is the solution of (13) satisfying $\psi = u(t_0)$; namely, $v_\psi(t) = w(t)$ for any $t \in I$.

A byproduct of Lemma 26 is that $\{(A(t_0, \psi_{g_2(R_\varphi, j)}))(0)\}$ is a “$\tilde{\delta}_{H^1}$-name” of $w(t)$ for all $t \in I$ because $(A(t_0, \psi_{g_2(R_\varphi, j)}))(0) \in H^1$ are Schwartz functions and $\|v_\psi(t) - (A(t_0, \psi_{g_2(R_\varphi, j)}))(0)\|_{H^1} \leq 2^{-j}$. Moreover, since $g_1$ and $g_2$ are computable, this $\tilde{\delta}_{H^1}$-name of $w(t)$ is computable from $t_0, \psi$, and $t$. More precisely

**Corollary 27.** The maps
\[
F_+ : (t_0, \psi, t) \mapsto w(t), \; t \in [t_0, t_0 + T_\varphi],
\]
and
\[
F_- : (t_0, \psi, t) \mapsto w(t), \; t \in [t_0 - T_\varphi, t_0]
\]
are $(\rho, \tilde{\delta}_{H^1}, \rho, \tilde{\delta}_{H^1})$-computable, where $w(t)$ is the solution of (13) with $\psi = u(t_0)$.

Since $\psi = u(t_0)$, $w(t)$ is also the solution of (8) over the time interval $[t_0, t_0 + T_\varphi]$. Thus the solution of (8) is extended from $[0, t_0]$ to $[t_0, t_0 + T_\varphi]$.

Now we are ready to lay down the proof of Theorem 19. We need to show how to compute the solution $u(t)$ of the initial-value problem (8) from the initial condition $\varphi \in H^1(\mathbb{R})$ and arbitrary time $t \in \mathbb{R}$. The proof consists of the following parts.
(a) since \( \varphi \mapsto \| \varphi \|_{H^1} \) is \((\delta_{H^1}, \rho)\)-computable, we can compute a rational number \( r_\varphi \) such that
\[
    r_\varphi - 1 \leq \| \varphi \|_{H^1} \leq r_\varphi
\]
and then compute \( R_\varphi = \mu n[ f(r_\varphi + 1) \leq n] \) and \( T_\varphi = 1/(32 R_\varphi^2) \). Here \( f(x) = x \cdot (1 + 4\sqrt{2}x^2)^{1/2} \) is defined in Lemma 22.

(b) Next we compute the solution \( u(z \cdot T_\varphi) \) at times \( z \cdot T_\varphi \) for integers \( z \in \mathbb{Z} \). Define
\[
    H_+(\varphi, 0) := H-(\varphi, 0) := \varphi
\]
and
\[
    H_+(\varphi, n + 1) := F_+(n \cdot T_\varphi, H_+(\varphi, n), (n + 1) \cdot T_\varphi), \tag{18}
    H_-(\varphi, n + 1) := F_-(n \cdot T_\varphi, H_-(\varphi, n), (n + 1) \cdot T_\varphi). \tag{19}
\]
By Corollary 27, \( H_+(\varphi, n) = u(n \cdot T_\varphi) \) and \( H_-(\varphi, n) = u(-n \cdot T_\varphi) \), and in addition, \( u(n \cdot T_\varphi) \) and \( u(-n \cdot T_\varphi) \) are computable from \( n \) and \( \varphi \) because both \( H_+ \) and \( H_- \) are primitive recursions of either the function \( F_+ \) or the function \( F_- \), which are computable according to Corollary 27.

(c) Finally we show how to compute \( u(t) \) from \( \varphi \) and \( t \). We begin by computing an integer \( z \in \mathbb{Z} \) from \( t \) and \( T_\varphi \) such that
\[
    zT_\varphi \leq t \leq (z + 1)T_\varphi.
\]
Then we compute \( u(z \cdot T_\varphi) \), and further compute \( F_+(z \cdot T_\varphi, u(z \cdot T_\varphi), t) \). By Corollary 27, \( F_+(z \cdot T_\varphi, u(z \cdot T_\varphi), t) = u(t) \). The proof is complete. \( \square \)

Appendix A.

In this section we prove Lemma 22. We first prove the following proposition. Let \( \| \cdot \| \) denote the \( L^2 \)-norm and \( \| \cdot \|_\infty \) the \( L^\infty \)-norm.

**Proposition 28.** For any \( v_1, v_2 \in H^1(\mathbb{R}) \), if \( \| v_1 \| = \| v_2 \| \) and \( E(v_1) = E(v_2) \), then \( \| v_2 \|_{H^1} \leq (1 + \| v_1 \|_{H^1}^2)^{1/2} \| v_1 \|_{H^1} \), where \( E(v) = \| v' \|^2 + \int_{\mathbb{R}} V(v) \, dx \) is the energy function and \( V(z) = z^2 z^2/2 \) is a real valued function.

The following facts are needed for proving the proposition.

**Fact 29 (The Sobolev inequality).** For any \( v \in H^1(\mathbb{R}) \),
\[
    \| v \|_\infty \leq \frac{\sqrt{2}}{2} \left( \| v \| \cdot \| v' \| \right)^{1/2}.
\]

**Proof.** See [1, Theorem 1, p. 167]. \( \square \)

**Fact 30.** For any \( v \in H^1(\mathbb{R}) \), \( \| v \|_\infty \leq \frac{\sqrt{2}}{2} \| v \|_{H^1} \).

**Proof.** Since \( \| v \| \cdot \| v' \| \leq \frac{1}{2} \left( \| v \|^2 + \| v' \|^2 \right) \), we have
\[
    \| v \|_\infty \leq \frac{\sqrt{2}}{2} \left( \| v \| \cdot \| v' \| \right)^{1/2} \leq (1/2)^{1/2} \left( \| v \|^2 + \| v' \|^2 \right)^{1/2} = \frac{\sqrt{2}}{2} \| v \|_{H^1}. \quad \square
\]
Fact 31. For any \( v \in H^1(\mathbb{R}) \), \( \|v\|^2_{H^1} \leq \|v\|^2 + E(v) \).

Proof. Since \( E(v) = \|v'\|^2 + \int_{\mathbb{R}} V(v(x)) \, dx \), \( V(v) = (v^2 \cdot \bar{v}^2)/2 \), we obtain that \( E(v) \geq \|v'\|^2 \), which in turn implies that
\[
\|v\|^2_{H^1} = \|v\|^2 + \|v'\|^2 \leq \|v\|^2 + E(v). \quad \square
\]

Fact 32. For any \( v \in H^1(\mathbb{R}) \), \( E(v) \leq \|v'\|^2 + \|v\|_{H^1}^4 \).

Proof. Since
\[
\int_{\mathbb{R}} V(v(x)) \, dx = \int_{\mathbb{R}} \frac{|v(x)|^4}{2} \, dx = \frac{1}{2} \int_{\mathbb{R}} |v(x)|^2 \cdot |v(x)|^2 \, dx \\
\leq \|v\|_{\infty}^2 \int_{\mathbb{R}} |v(x)|^2 \, dx = \|v\|_{\infty}^2 \cdot \|v\|^2 \\
\leq \left( \frac{\sqrt{2}}{2} \right)^2 \cdot \|v\|_{H^1}^2 \cdot \|v\|_{H^1}^2 \leq \frac{1}{2} \|v\|_{H^1}^2 \cdot \|v\|_{H^1}^2 \leq \|v\|_{H^1}^4,
\]
\[
E(v) = \|v'\|^2 + \int_{\mathbb{R}} V(v(x)) \, dx \leq \|v'\|^2 + \|v\|_{H^1}^4. \quad \square
\]

Proof (Proposition 28).
\[
\|v_2\|_{H^1}^2 \leq \|v_2\|^2 + E(v_2) \quad \text{(Fact 31)}
\]
\[
= \|v_1\|^2 + E(v_1) \quad \text{(Assumption)}
\]
\[
\leq \|v_1\|^2 + \|v'_1\|^2 + \|v_1\|_{H^1}^4 \quad \text{(Fact 32)}
\]
\[
= \|v_1\|_{H^1}^2 + \|v_1\|_{H^1}^4
\]
\[
= (1 + \|v_1\|_{H^1}^2)\|v_1\|_{H^1}^2. \quad \square
\]

Next we prove Lemma 22. Two more facts are needed.

Fact 33. If \( u \) is the solution of \( iu_t = -\Delta u + mu + |u|^2 u \), \( u(0) = u_0 \), then for any \( s, t \in \mathbb{R} \), \( \|u(t)\| = \|u(s)\| \) and \( E(u(t)) = E(u(s)) \).

Proof. See Proposition 3.1 of [2]. \( \square \)

Fact 34. If \( u \) is the solution of \( iu_t = -\Delta u + mu + |u|^2 u \), \( t, x \in \mathbb{R} \) with \( u(0) = u_0 \), then for any \( s, t \) and \( t_0 \in \mathbb{R} \),
\[
\|U(t-s, u(s))\|_{H^1} \leq (1 + \|u(t_0)\|_{H^1}^2)^{1/2}\|u(t_0)\|_{H^1}. \]

Proof. First, by Fact 33, \( \|u(s)\| = \|u(t_0)\| \) and \( E(u(s)) = E(u(t_0)) \). Secondly, from the definition of \( U(t, \psi) \) (defined in Lemma 22), it follows that the Fourier transform, \( \hat{U}(t, \psi) \), of \( U(t, \psi) \)
is equal to $e^{-i\xi^2 t - imt} \hat{\psi}$. Then
\[
\|U(t-s, u(s))\|_{H^1} = \|(1 + |\xi|^2)^{1/2} \hat{U}(t-s, u(s))\|_{L^2}
\]
\[
= \|(1 + |\xi|^2)^{1/2} e^{-i\xi^2 (t-s) - im(t-s)} \hat{u}(s)\|_{L^2}
\]
\[
= \|e^{-i\xi^2 (t-s) - im(t-s)} (1 + |\xi|^2)^{1/2} \hat{u}(s)\|_{L^2}
\]
\[
= \|(1 + |\xi|^2)^{1/2} \hat{u}(s)\|_{L^2} = \|u(s)\|_{H^1}.
\]
By applying Proposition 28 to $u(s)$ and $u(t_0)$, we obtain the desired inequality. □

**Proof (Lemma 22).** First it follows from Fact 33 and Proposition 28 that for any $t, t', t_0 \in \mathbb{R}$,
\[
\|u(t)\|_{H^1} \leq (1 + \|u(t_0)\|_{H^1}^2)\|u(t_0)\|_{H^1} \leq (1 + 4\sqrt{2}\|u(t_0)\|_{H^1}^2)\|u(t_0)\|_{H^1}.
\]
Then Fact 34 leads to the following inequality
\[
\|U(t-t', u(t'))\|_{H^1} \leq (1 + \|u(t_0)\|_{H^1}^2)\|u(t_0)\|_{H^1}
\]
\[
\leq (1 + 4\sqrt{2}\|u(t_0)\|_{H^1}^2)\|u(t_0)\|_{H^1}. \quad \square
\]

**References**