The Price of Anarchy of Large Walrasian Auctions*

Richard Cole and Yixin Tao
Courant Institute, NYU
September 3, 2015

Abstract

As is well known, many classes of markets have efficient equilibria, but this depends on agents being non-strategic, i.e. that they declare their true demands when offered goods at particular prices, or in other words, that they are price-takers. An important question is how much the equilibria degrade in the face of strategic behavior, i.e. what is the Price of Anarchy (PoA) of the market viewed as a mechanism?

Often, the PoA bounds are modest constants such as $\frac{4}{3}$ or 2. Nonetheless, in practice a guarantee that no more than 25% or 50% of the economic value is lost may be unappealing. This paper asks whether significantly better bounds are possible under plausible assumptions. In particular, we look at how these worst case guarantees improve in large markets, i.e. when there are many copies of each item and many agents. We show that if there is some uncertainty about the numbers of copies of each good, then the PoA tends to 1 as the market sizes increases, and we also note that some such assumption is unavoidable.

1 Introduction

When is there no gain to participants in a game from strategizing? One answer applies when players in a game have no prior knowledge; then a game that is strategy proof ensures that that truthful actions are a best choice for each player. However, in many settings there is no strategy proof mechanism. Also, even if there is a strategy proof mechanism, with knowledge in hand, other equilibria are possible, for example, the “bullying” Nash Equilibrium as illustrated by the following example: there is one item for sale using a second price auction, the low-value bidder bids an amount at least equal to the value of the high-value bidder, who bids zero; the resulting equilibrium achieves arbitrarily small social welfare compared to the optimal outcome.

To make the notion of gain meaningful one needs to specify what the game or mechanism is seeking to optimize. Social welfare and revenue are common targets. For the above example, the social welfare achieved in the bullying equilibria can be arbitrarily far from the optimum. However, for many classes of games, over the past fifteen years, bounds on the gains from strategizing, a.k.a. the Price of Anarchy (PoA), have been obtained, with much progress coming thanks to the invention of the smoothness methodology \cite{7, 17, 21}; many of the resulting bounds have been shown to be tight. Often these bounds are modest constants, such as $\frac{4}{3}$ \cite{18} or 2 \cite{20}, etc., but rarely are there provably no losses from strategizing, i.e. a PoA of 1.

This paper investigates when bounds close to 1 might be possible. In particular, we study Walrasian auctions for large markets. Walrasian auction are used in market settings where there are goods for sale and agents, called bidders, who want to buy these goods. Each agent has

*This work was supported in part by NSF Award CCF-1217989.
varying preferences for different subsets of the goods, preferences that are represented by valuation functions. The goal of the auction is to identify equilibrium prices; these are prices at which all the goods sell, and each bidder receives a favorite (utility maximizing) collection of goods, where each bidder’s utility is quasi-linear: the difference of its valuation for the goods and their cost at the given prices. Such prices, along with an associated allocation of goods, are said to form a Walrasian equilibrium.

Walrasian equilibria are known to exist when each bidder’s demand satisfies the gross substitutes property [10], but this is the only substantial class of settings in which they are known to exist. In these settings, the equilibria form a lattice, with the prices at the top of the lattice being those that result from a Dutch auction, while those at the bottom result from an English auction.

Babaioff et al. [2] analyzed the PoA of the games induced by Walrasian mechanisms, i.e. the prices were computed by a method, such as the English or Dutch auction, that yields equilibrium prices when these exist. Note that the mechanism can be applied even when Walrasian equilibria do not exist, though the resulting outcome cannot be a Walrasian equilibrium. But even when Walrasian equilibria exist, because bidders may strategize, in general the outcome will be a Nash equilibrium rather than a Walrasian one. Among other results, Babaioff et al. showed an upper bound of 4 for any Walrasian mechanism when there was no overbidding and the bids and valuations satisfied the gross substitutes property. They also obtained lower bounds on the PoA that were greater than 1, even when overbidding was not allowed, which excludes bullying equilibria; e.g. the English auction has a PoA of at least 2.

Babaioff et al. also noted that the prices computed by double auctions, widely used in financial settings, are essentially computing a price that clears the market and maximizes trade; one example they mention is the computation of the opening prices on the New York Stock Exchange, and another is the adjustment of prices of copper and gold in the London market.

By a large market, we intend a market in which there are many copies of each good, and in addition the demand set of each bidder is small. The intuition is that then each bidder will have a small influence and hence strategic behavior will have only a small effect on outcomes. In fact, this need not be so. For example, the bullying equilibrium persists: simply increase the numbers of items and bidders for each type to \( n \), and have the buyers of each type follow the same strategy as before.

What allows this bullying behavior to be effective is the precise match between the number of items and the number of low-value bidders. The need for this exact match also arises in the lower-bound examples in [2] (as with the bullying equilibrium, it suffices to pump up the examples by a factor of \( n \)). To remove these equilibria that demonstrate PoA values larger than 1, it suffices introduce some uncertainty regarding the numbers of items and/or bidders. Indeed, in a large setting it would seem unlikely that such numbers would be known precisely. We will create this uncertainty by using distributions to determine the number of copies of each good. In contrast, prior work on non-large markets eliminated the potentially unbounded PoA of the bullying equilibrium by bounding overbidding \([3, 6, 8, 21]\).

Our main result is that the PoA of the Walrasian mechanism tends to 1 as the market size grows. This result assumes that valuations are bounded regardless of the size of the market. We specify this more precisely when we state our results in Section 3. This bound applies to both Nash and Bayes-Nash equilibria; as it is proved by means of a smoothness argument, it extends to mixed Nash and coarse correlated equilibria, and outcomes of no-regret learning.

We also investigate a more restricted setting in which we can drop this assumption. This setting is a one-good market, again with many copies of the good, and with unit-demand bidders. Here we consider the first and second price auctions, and prove that their PoAs both tend to 1 as the market size grows. These results are achieved by a direct argument and so apply only to Nash
and Bayes-Nash equilibria. Interestingly, for the second price auction, we show that the simple rearrangement of the bullying equilibrium is essentially the only inefficiency: the bid ordering and the value ordering of bidders differ only by swaps of adjacent bidders, at least among those bidders who have non-zero probability of receiving an item.

1.1 Related Work

The most closely related work is due to Feldman et al. [7]. They also consider large settings and show that for several market settings when using simple, non-Walrasian mechanisms, the PoA tends to 1 as the market size grows to \( \infty \). Their results are derived from a new type of smoothness argument. Depending on the result, they require either uncertainty in the number of goods or the number of bidders. In contrast, our main result uses a previously known smoothness technique. They also show that for traffic routing problems, the PoA of the atomic case tends to that of the non-atomic case as the number of units of traffic grows to \( \infty \).

The idea of uncertainty in the number of agents or items first arose in the Economics literature. Myerson used it in the context of voting games [14], and Swinkels in the context of auctions [19]. Later, uncertainty in the number of agents was used with the Strategy Proof in the Large concept [1].

As already noted, Babaioff et al. gave bounds on the PoA of Walrasian equilibria. Another approach is to bound the gains to individual agents, called the incentive ratio; Chen et al. showed these values were bounded by small constants in Fisher market settings [5, 4].

Achieving good outcomes in the large was first looked at in the context of exchange economies by Roberts and Postlewaite [16], which they modeled as a replica economy, the \( n \)-fold duplication of a base economy, showing that individual utility gains from strategizing tend to zero as the economy grows. Subsequently, Jackson and Manelli showed that with some regularity assumptions, the equilibrium allocations converge to the competitive equilibrium [11]. Kalai studied the notion of extensive robustness for large games [12], and Kalai and Shmaya investigated large repeated games using the notion of compressed equilibria [13]. Pai et al. studied repeated games and the use of differential privacy as a measure of largeness [15]. In a different direction, Gradwohl and Reingold investigated fault tolerance in large games for \( \lambda \)-continuous and anonymous games [9].

2 Preliminaries

Definition 2.1. A market \( M \) comprises a set of \( N \) bidders \( B_1, B_2, \ldots, B_N \), and a set of \( l \) goods \( G \), with \( n_j \) copies of good \( j \), for \( 1 \leq j \leq l \). We write \( \bar{n} = (n_1,n_2,\ldots,n_l) \), where \( n_j \) denotes the number of copies of good \( j \), and we call it the multiplicity vector. We also write \( \bar{n} = (n_{-j}, n_j) \), where \( n_{-j} \) is the vector denoting the number of copies of goods other than good \( j \). We refer to an instance of a good as an item. For an allocation \( x_i \) to \( B_i \), which is a subset of the available goods, we write \( x_i = (x_i^1, x_i^2, \ldots, x_i^l) \) where \( x_i^j \) denotes the number of copies of good \( j \) in the allocation \( x_i \). There is a set of prices \( p = (p_1, p_2, \ldots, p_l) \), one per good; we also write \( p = (p_j, p_{-j}) \). Each bidder \( B_i \) has a valuation function \( v_i : X \rightarrow \mathbb{R}_+ \), where \( X \) is the set of possible assignments, and a quasi-linear utility function \( u_i(x) = v_i(x) - x \cdot p \).

A Walrasian equilibrium is a collection of prices \( p \) and an allocation \( x_i \) to each bidder \( B_i \) such that (i) the goods are fully allocated but not over-allocated, i.e. for all \( j \), \( \sum_i x_i^j = n_j \), and (ii) each bidder receives a utility maximizing allocation at prices \( p \), i.e. \( u_i(x_i) = v_i(x_i) - x_i \cdot p = \max_x [v_i(x) - x \cdot p] \).

In a Walrasian mechanism for market \( M \) each bidder produces a bid function \( b_i : X \rightarrow \mathbb{R}_+ \). We write \( \bar{b} = (b_1, b_2, \ldots, b_N) \) and \( \bar{b} = (b_i, b_{-i}) \). The mechanism computes prices and allocations as if the bids were the valuations.
Given the bidders and their bids, $p(\vec{n}; \vec{b})$ denotes the prices produced by the Walrasian mechanism at hand when there are $\vec{n}$ copies of goods and $\vec{b}$ is the bidding profile. Also, $p_j(\vec{n}; \vec{b})$ denotes the price of good $j$ and $p(\vec{n}; \vec{b}) = (p_j(\vec{n}; \vec{b}), p_{-j}(\vec{n}; \vec{b}))$. Finally, we let both $x_i(\vec{n}; \vec{b})$ and $x_i(\vec{n}; b_i, b_{-i})$ denote the allocation to $B_i$ provided by the mechanism.

**Definition 2.2.** A valuation or bid function satisfies the gross substitutes property if, for each utility maximizing allocation $x$ at prices $p = (\vec{n}; p_j, p_{-j})$, at prices $(\vec{n}; q_j, p_{-j})$ such that $q_j > p_j$, there is a utility maximizing allocation $y$ with $y_{-j} \geq x_{-j}$ (i.e. $y_k \geq x_k$ for $k \neq j$).

A Bayes-Nash equilibrium is an outcome with no expected gain from an individual deviation:

$$\forall b_{-i} : \mathbb{E}_{v_i, b_{-i}}[u_i(x_i(\vec{n}; b_i, b_{-i}), p((b_i, b_{-i})))] \geq \mathbb{E}_{v_i, b_{-i}}[u_i(x_i(\vec{n}; b'_i, b_{-i}), p((b'_i, b_{-i})))].$$

The social welfare $SW(x)$ of an allocation $x$ is the sum of the individual valuations: $SW(x) = \sum_i v_i(x_i)$. We also write $SW(OPT)$ for the (expected) optimal social welfare, the maximum (expected) achievable social welfare, and $SW(NE)$ for the smallest social welfare achievable at a Bayes-Nash equilibrium.

Finally, the Price of Anarchy is the worst case ratio of $SW(OPT)$ to $SW(NE)$ over all instances in the class of games at hand, which in this context comprise markets $M_N$ of $N$ buyers:

$$PoA = \max_{M_N} \frac{SW(OPT)}{SW(NE)}.$$

**Definition 2.3.** A large market is a sequence of markets $M_1, M_2, \ldots, M_N, \ldots$, where $N$ denotes the number of bidders. It satisfies the following two properties.

i. The demand of every bidder is for at most $k$ items. Formally, if allocated a set of more than $k$ items, the bidder will obtain equal utility with a subset of size $k$.

ii. Let $F(n_j, j, N|n_{-j})$ denote the probability that there are exactly $n_j$ copies of good $j$ when given $n_{-j}$ copies of other goods, and let $F(j, N) = \max_{n_j, n_{-j}} F(n_j, j, N|n_{-j})$. Then, for all $n$ and $j$, $\lim_{N \to \infty} F(j, N) = 0$.

It will be convenient to write $m_j = \mu(n_j)$ for the expected number of copies of good $j$.

As a running example, to illustrate the application of the large market definition, we will use a binomial distribution with $2m_j$ potential copies of good $j$, each with a probability $\frac{1}{2}$ of being present. For this binomial distribution, $F(j, N) \leq \frac{1}{\sqrt{2m_j}}$.

3 Our Results

We begin by looking at a simple setting: a one good, many-copies market with unit-demand bidders. We analyze both the first and second price auctions, meaning that the prices are set as follows: in the first price auction all copies are sold at the lowest bid made by a winning bidder, and in the second price auction all copies are sold at the highest bid made by a losing bidder, and in both cases the winners are the highest bidders. Further, in the second price auction, any tie-breaking rule suffices, while the first price auction uses either a fixed tie-breaking ordering of the bidders, or a uniform random ordering.

As there is just one good, we drop the index $j$: we set $m = m_1$ and let $F(n, N)$ denote the probability that there are exactly $n$ copies of the good and $F(N) = \max_n F(n, N)$. We need one mild assumption about $F$, namely that once $F(n, N)$ becomes zero, it remains at zero as $n$ increases.
Assumption 3.1. For each value of $N$, there is an $m' > m$, such that $F(n, N) > 0$ for $1 \leq n \leq m'$ and $F(n, N) = 0$ for $n > m'$. Finally, $F(0) \geq 0$.

Theorem 3.1. If Assumption 3.1 holds, then for the second price auction,

$$\text{SW(NE)} \geq \left(1 - \frac{F(N)}{1 - F(N)}\right) \cdot \text{SW(OPT)}.$$ 

For specificity, we assume that ties are resolved by uniform random selection of winners; we make the same assumption for the first price auction.

If $F$ is the previously described binomial distribution, Theorem 3.1 implies $\text{SW(NE)} \geq \left(1 - \frac{1}{\sqrt{2m-1}}\right) \cdot \text{SW(OPT)}$.

Theorem 3.2. If Assumption 3.1 holds, then for the first price auction,

$$\text{SW(NE)} \geq \left(1 - \frac{16F(N)}{1 - 2F(N)}\right) \cdot \text{SW(OPT)}.$$ 

Again, if $F$ is the binomial distribution in question, then $\text{SW(NE)} \geq \left(1 - O\left(\frac{1}{\sqrt{2m}}\right)\right) \cdot \text{SW(OPT)}$.

We now turn to our main result. Here we need two assumptions about large markets; similar assumptions were made for the large market results in [7].

Assumption 3.2. Bounded Expected Valuation For each bidder and each item, her expected value for this single item is less than a constant $\zeta$:

$$\max_s E[v_i(s)] \leq \zeta.$$ 

Assumption 3.3. Market Welfare The optimal social welfare grows linearly with the number of bidders: $\text{SW(OPT)} \geq \rho N$, for some constant $\rho > 0$.

We can achieve Assumption 3.3 by making following assumptions.

Assumption 3.4. Market Size Let $\mu(n_j)$ be the expected number of copies of good $j$, for $1 \leq j \leq l$, and let $\Gamma(n_j)$ be its standard deviation. The assumption is that for each $j$, $\mu(n_j) = \Theta(N)$ and $\Gamma(n_j) \leq (1 - \lambda)\mu(n_j)$ for some constant $\lambda > 0$. Let $\alpha > 0$ be such that $\mu(n_j) \geq \alpha N$ for all $j$ and sufficiently large $N$.

Assumption 3.5. Value Lower Bound There is a parameter $\rho' > 0$ such that for any bidder, its largest expected value for one item is at least $\rho'$:

$$\rho' \leq \max_s E[v_i(s)].$$ 

Lemma 3.1. Let $\rho = \lambda^2 \alpha^2 \frac{2\lambda + \lambda^2}{(1 + \lambda)^2} \rho'$. If Assumptions 3.4 and 3.5 hold, then $\text{SW(OPT)} \geq \rho N$.

Theorem 3.3. In a large market which satisfies Assumptions 3.2 and 3.3,

$$\text{SW(NE)} \geq \left(1 - \frac{3 \cdot k \cdot (k + 1) \cdot \zeta \cdot l}{\rho} \cdot Y \cdot [\log_2 \frac{1}{Y^2}]\right) \cdot \text{SW(OPT)},$$

where $Y = \sum_{1 \leq j \leq l} F(j, N) \lfloor 2l(k + 2)^2(k + 1 + l)^{k+1} \rfloor$. 

5
3.1 Proof Sketches

Second Price The key observation is that the orderings of the top bidders by value and by bid are very similar. Specifically, let $m'$ be the largest number of items which can occur with non-zero probability, and let $v'_{(i)}$ denote the $i$th largest value. Then given an ordering of the top $m'$ bidders by their bids, to obtain an ordering by value it suffices to swap adjacent bidders; further, the resulting ordering includes the top $m' - 1$ bidders by value. It follows that $\text{SW(OPT)} - \text{SW(NE)}$ is bounded by $\mathbb{E}[\sum_{i=1}^{m'-1} F(N)(v'_{(i)} - v'_{(i+1)}) + F(N)v'_{(m')} \leq F(N)v'_{(1)} \leq F(N)\frac{\text{SW(OPT)}}{1 - F(N)}$. The result now follows.

First Price Using the fact that $v_i \geq b_i$ if there is a non-zero probability of winning with a bid of $b_i$, by a direct calculation we show that $\text{SW(OPT)} - \text{SW(NE)}$ is at most

$$\int_{v, b} \sum_{i} \sum_{n=\text{rank}_i(v, b_{-i})+1}^{\text{rank}_i(b_i, b_{-i})} (v_i - b_{(n-1)}) \cdot \Pr[\#\text{items} = n] \, dv \, db$$

$$+ 2 \int_{v, b} \sum_{n=2}^{+\infty} b_{(n-1)} \cdot \Pr[\#\text{items} = n] \, dv \, db + \int_{v, b} \sum_{i} v_i \cdot \Pr\left[\#\text{items} = \text{rank}_i(v_i, v_{-i})\right] \, dv \, db,$$

where $b_{(n_i)}$ is the $n$th bid in the bid ordering. Then because $(b_i, b_{-i})$ is a Nash equilibrium we deduce this is bounded by

$$\int_{v, b} \sum_{i} v_i \cdot \Pr\left[\#\text{items} = \text{rank}_i(b_i, b_{-i})\right] \, dv \, db$$

$$+ 2 \int_{v, b} \sum_{n=2}^{+\infty} b_{(n-1)} \cdot \Pr[\#\text{items} = n] \, dv \, db + \int_{v, b} \sum_{i} v_i \cdot \Pr\left[\#\text{items} = \text{rank}_i(v_i, v_{-i})\right] \, dv \, db.$$

Finally, we show that each of the above three terms is bounded by $O(F(N) \cdot \text{SW(OPT)})$.

Walrasian Equilibrium The key idea is to define $(k, \epsilon, U)$-good and bad multiplicity vectors $\vec{n}$, wr.t. bids $\vec{b}$. By counting their number, we will show that the fraction of $(k + 1, \epsilon, U)$-bad vectors is $O(F(N))$. On the other hand, if the vector is $(k + 1, \epsilon, U)$-good, we will show that a bidder can cause the prices, when they are all bounded by $U$, to vary by at most $(k + 1)\epsilon$. Essentially, a vector $\vec{n}$ is $(k, \epsilon, U)$-good if changing the supplies by at most $k$ items causes prices $p_j \leq 1$ to change in total by at most $k \epsilon$. Then, using the fact that the equilibrium is Walrasian, we can show that for $(k + 1, \epsilon, U)$-good vectors $\vec{n}$,

$$u_i(x_i(v_i, b_{-i})) \geq v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\vec{n}; (b_i, b_{-i})) - k(k + 1)\epsilon.$$

On summing over $i$ and taking expectations, we can then deduce that

$$\sum_i \mathbb{E}\left[u_i(x_i(v_i, b_{-i}))\right] \geq \text{SW(OPT)} - R(b_i, b_{-i}) - N \cdot k \cdot (k + 1) \cdot \epsilon \cdot \zeta - O(N \cdot F(N) \cdot k \cdot \zeta \cdot l),$$

where $R(b_i, b_{-i})$ denotes the revenue with bidding profile $(b_i, b_{-i})$. By Lemma 3.1, $\text{SW(OPT)} = \Theta(N)$. We can now apply the smooth technique for Bayesian settings [21] to obtain our result.
4 Second Price Auction

We will show that almost surely the value ordering is very similar to the order of bids, as stated in the following lemma. We let \( b_1 \geq b_2 \geq \ldots \geq b_n \) denote the bids in sorted order, and let \( v_i \) be the value corresponding to bid \( b_i \), for \( 1 \leq i \leq n \). Also, we let \( v_1 \geq v_2 \geq \ldots \geq v_n \) denote the values in sorted order.

Recall that \( m' \) denotes the maximum number of copies of the good that might be present. The following lemma shows that the ordering of the top \( m' - 1 \) values is very similar to the ordering of the top \( m' \) bids.

**Lemma 4.1.** Consider the list of (bid, value) tuples. We will consider two sortings of this list: \( L_V \) is a sort by value, and \( L_B \) a sort by bids. Almost surely, to obtain the top \( m' - 1 \) tuples in \( L_V \) it suffices to take the top \( m' \) tuples in \( L_B \) and then swap some disjoint pairs of adjacent tuples.

Note that this captures the already described bullying Equilibrium with one item and two bidders.

With Lemma 4.1 in hand it is straightforward to bound the loss in social welfare.

**Theorem 4.1.**

\[
SW(NE) \geq \left(1 - \frac{F(N)}{1-F(N)}\right) \cdot SW(OPT).
\]

**Proof.** By Lemma 4.1, for each of the top \( m' - 1 \) bidders in the value ordering, almost surely the only possible rearrangement from the bid ordering to the value ordering is a swap with an adjacent neighbor. Thus the expected reduction in the contribution to the social welfare, compared to \( SW(OPT) \) is at most:

\[
E \left[ \sum_{i=1}^{m'-1} (v_i - v_{i+1}) F(N) + v_m F(N) \right] \leq E \left[ v_1 F(N) \right].
\]

Now \( SW(OPT) \geq E \left[ (1 - F(N)) v_1 \right] \). Consequently,

\[
SW(NE) \geq SW(OPT) \left(1 - \frac{F(N)}{1-F(N)}\right).
\]

It remains to show Lemma 4.1. The next lemma provides the key observation. We let \( \text{rank}_k(x, b_{-k}) \) denote the rank of bid \( x \) w.r.t. the bid ordering \( b_{-k} \), where \( \text{rank}_k \) means the rank given by the auction, which may depend on \( k \)’s identity.

**Lemma 4.2.** Suppose \( b_i \) is an optimal strategy for bidder \( i \) with value \( v_i \).

If \( b_i \neq v_i \), then \( \Pr \left[ \exists b_j (j \neq i) \left( (b_j \text{ is strictly between } v_i \text{ and } b_i) \land \text{rank}_j(b_j, b_{-j}) \leq m' \right) \right] = 0. \)

If \( b_i > v_i \), then \( \Pr \left[ \exists b_j (j \neq i) \left( (b_j \text{ equals } b_i) \land (\text{rank}_i(b_i, b_{-i}) \leq \text{rank}_j(b_j, b_{-j}) \leq m') \right) \right] = 0. \)

If \( b_i < v_i \), then \( \Pr \left[ \exists b_j (j \neq i) \left( (b_j \text{ equals } b_i) \land (\text{rank}_j(b_j, b_{-j}) \leq \min\{m', \text{rank}_i(b_i, b_{-i})\}) \right) \right] = 0. \)
Proof. Suppose not, then \( b_i \) is strictly dominated by strategy \( v_i \).

We begin by showing that the top \( m' \) tuples in \( L_B \) are almost in sorted order by value, the only rearrangements needed being swaps of adjacent tuples. (This does not prove Lemma 4.1 because possibly the top \( m' \) tuples in \( L_B \) do not include all of the top \( m' - 1 \) values.)

**Lemma 4.3.** Let \( h \leq m' \). Then, almost surely, \( v(h) \leq v(g) \) for \( g \leq h - 2 \), and \( v(h) \geq v(k) \) for \( m' \geq k \geq h + 2 \).

Proof. We can classify the top \( m' \) bids as higher than, equal to, or less than their values. For the high bids, \( b_{(x_1)} \geq b_{(x_2)} \geq \ldots \geq b_{(x_i)} \) it will be the case that almost surely \( b_{(x_1)} > v_{(x_1)} \geq b_{(x_2)} > v_{(x_2)} \geq \ldots \geq b_{(x_i)} > v_{(x_i)} \), for if \( b_{(x_1)} \geq b_{(x_i)} > v_{(x_i)} \), this contradicts Lemma 4.2. Similarly for the low bids, \( v_{(y_1)} > b_{(y_1)} \geq v_{(y_2)} > b_{(y_2)} \geq \ldots \geq v_{(y_i)} > b_{(y_i)} \), and for the remaining bids \( v_{(1)} = b_{(1)} \geq v_{(2)} = b_{(2)} \geq \ldots \geq v_{(m'_k)} = b_{(m'_k)} \).

Almost surely, it cannot be that \( v_{(x_a)} < v_{(z_a)} = b_{(z_a)} \leq b_{(x_a)} \), for this contradicts Lemma 4.2. Thus either \( v_{(x_a)} < b_{(x_a)} = b_{(z_a)} \) or \( v_{(z_a)} = b_{(x_a)} \leq v_{(x_a)} < b_{(x_a)} \). Similarly, either \( b_{(y_a)} \leq v_{(y_a)} = b_{(z_a)} \) or \( v_{(z_a)} = b_{(x_a)} \leq b_{(y_a)} < v_{(y_a)} \). However, \( b_{(y_a)} \leq v_{(x_a)} < v_{(y_a)} \leq b_{(x_a)} \) is possible (any other interleaving of these pairs contradicts Lemma 4.2). But almost surely, two such interleavings cannot occur: i.e., \( \{b_{(y_1)}, b_{(y_2)}\} \leq v_{(x_0)} \leq \{v_{(y_1)}, v_{(y_2)}\} \leq b_{(x_0)} \) cannot occur as this contradicts the ordering of low bids. Likewise, the analogous situation with a pair of high bids is also almost surely impossible.

In sum, for each bid among the top \( m' \) bids, among these bids there is at most one w.r.t. which it is out of order in the value ordering.

**Corollary 4.1.** Almost surely, \( v_{(m'_1)} \leq v_{(k)} \) for any \( k \leq m' - 3 \).

We finish by considering which of the top \( m' \) bidders can have a value below the highest \( m' \) values, if any.

**Lemma 4.4.** For \( r > m' \), almost surely \( v_r \leq v_{(m'_1)}, v_{(m'_2)} \).

Proof. Suppose that \( v_r > v_{(m')} \). Necessarily \( b_r \leq b_{(m')} \). To avoid contradicting Lemma 4.2, almost surely \( v_r \leq b_{(m')} \), and so \( v_{(m')} < v_r \leq b_{(m')} \). In a similar way, \( b_{(m')} \leq v_{(m')} \).

Therefore, \( b_r \leq v_{(m')} \). Almost surely this does not occur by Lemma 4.2 and hence \( v_r \leq v_{(m'-1)} \). The same argument applies to \( v_{(m'-2)} \),

**Proof of Lemma 4.3** By Corollary 4.1 and Lemma 4.4 for \( r > m' \) and \( k \leq m' - 3 \), almost surely \( v_r \leq v_{(m'-1)}, v_{(m'-2)} \) and \( v_{(m'-1)} \leq v_{(k)} \). Thus the only value \( v_r \) outside the top \( m' \) values that can be among the top \( m' \) bids will have rank \( m' \) among the bids. Thus the top \( m' - 1 \) values all lie among the top \( m' \) bids. Finally, Lemma 4.3 ensures that the only possible rearrangements are disjoint swaps of adjacent bids.

**5 First Price Auction**

Our goal is to show that \( \text{SW} (\text{NE}) \geq (1 - \epsilon) \text{OPT} \), where \( \epsilon \) is a suitable function of \( m \), the number of copies of the item.

As for the second price auction, \( \text{rank}_k (x, b_{-i}) \) denotes the rank that would be returned by the mechanism to the \( k \)th bidder with a bid of \( x \) given that the other bidders bid \( b_{-i} \). Recall that this may depend on bidder \( k \)’s identity.
Theorem 5.1.

\[ \text{SW(NE)} \geq \left( 1 - \frac{16F(N)}{1 - 2F(N)} \right) \text{SW(OPT)}. \]

Let \( v_i \) be a random variable denoting the \( i \)th largest value, and let \( b_i \) be a random variable denoting the \( i \)th largest bid. We first have a lower bound on \( \text{SW(NE)} \). \( \text{rank}_i(\text{bid profile}) \) denotes the rank of bidder \( i \) in the bid profile, which could be \( \vec{b}, \vec{v} \) or some other profile.

Lemma 5.1.

\[
\text{SW(NE)} \geq \int_{v,b} \sum_i v_i \cdot \text{Pr}[\text{\#items} \geq \max\{\text{rank}_i(b_i, b_{-i}), \text{rank}_i(v_i, v_{-i})\}] + \sum_{n=\text{rank}_i(v_i, v_{-i})+1}^{\text{rank}_i(b_i, b_{-i})} b_{(n-1)} \cdot \text{Pr}[\text{\#items} = n] \ dv \ db \\
- 2 \int_{v,b} \sum_{n=2}^{+\infty} b_{(n-1)} \cdot \text{Pr}[\text{\#items} = n] \ dv \ db.
\]

Lemma 5.1 is based on the following lemma.

Lemma 5.2.

\[
\int_{v,b} \sum_i v_i \cdot \sum_{n=\text{rank}_i(v_i, v_{-i})}^{\text{rank}_i(v_i, v_{-i})-1} \text{Pr}[\text{\#items} = n] \ dv \ db \\
\geq \int_{v,b} \sum_i \sum_{n=\text{rank}_i(v_i, v_{-i})}^{\text{rank}_i(b_i, b_{-i})} b_{(n-1)} \cdot \text{Pr}[\text{\#items} = n] \ dv \ db \\
- 2 \int_{v,b} \sum_{n=2}^{+\infty} b_{(n-1)} \cdot \text{Pr}[\text{\#items} = n] \ dv \ db.
\]

Observation 5.1. If \( \text{Pr}[\text{bidder } i \text{ wins by bidding } b_i] > 0 \), then \( v_i \geq b_i \).

Proof. Otherwise a bid of \( b_i \) would be strictly dominated by a bid of \( v_i \). \( \square \)
Proof of Lemma 5.2. See below for justifications.

\[
\int_{v,b} \sum_{i} \sum_{n=\text{rank}_i(b_i, b_{-i})}^{\text{rank}_i(v_i, v_{-i})-1} \Pr[\text{#items} = n] \, dv \, db \\
= \int_{v,b} \sum_{n=1}^{+\infty} \Pr[\text{#items} = n] \cdot \sum_{i} v_i \cdot 1 \left[ \text{rank}_i(v_i, v_{-i}) - 1 \geq n \land \text{rank}_i(b_i, b_{-i}) \leq n \right] \, dv \, db \\
\geq \int_{v,b} \sum_{n=2}^{+\infty} \Pr[\text{#items} = n] \cdot \sum_{i} v_i \cdot 1 \left[ \text{rank}_i(v_i, v_{-i}) - 1 \geq n \land \text{rank}_i(b_i, b_{-i}) < n \right] \, dv \, db \\
\geq \int_{v,b} \sum_{n=2}^{+\infty} \Pr[\text{#items} = n] \cdot \sum_{i} b_{(n-1)} \cdot 1 \left[ \text{rank}_i(v_i, v_{-i}) - 1 \geq n \land \text{rank}_i(b_i, b_{-i}) \leq n \right] \, dv \, db \\
\geq \int_{v,b} \sum_{n=2}^{+\infty} \Pr[\text{#items} = n] \cdot \sum_{i} b_{(n-1)} \cdot 1 \left[ \text{rank}_i(v_i, v_{-i}) - 1 \geq n \land \text{rank}_i(b_i, b_{-i}) < n \right] \, dv \, db \\
\geq \sum_{n=1}^{+\infty} \sum_{i} b_{(n-1)} \cdot \Pr[\text{#items} = n] \, dv \, db \\
- \sum_{n=2}^{+\infty} \Pr[\text{#items} = n] \, dv \, db \\
\geq \int_{v,b} \sum_{i} \sum_{n=\text{rank}_i(b_i, b_{-i})}^{\text{rank}_i(v_i, v_{-i})-1} b_{(n-1)} \cdot \Pr[\text{#items} = n] \, dv \, db \\
- \sum_{n=2}^{+\infty} \Pr[\text{#items} = n] \, dv \, db \\
\geq \int_{v,b} \sum_{i} \sum_{n=\text{rank}_i(v_i, v_{-i})+1}^{\text{rank}_i(b_i, b_{-i})} b_{(n-1)} \cdot \Pr[\text{#items} = n] \, dv \, db \\
- 2 \sum_{n=2}^{+\infty} b_{(n-1)} \cdot \Pr[\text{#items} = n] \, dv \, db.
\]

For the second inequality, note that if \( \text{rank}_i(b_i, b_{-i}) \leq n - 1 \), then by Observation 5.1, \( v_i \geq b_i \), and also \( b_i \geq b_{(n-1)} \); so \( v_i \geq b_{(n-1)} \).

For the fourth inequality, it suffices to note that for every \( i \) with \( \text{rank}_i(v_i, v_{-i}) \geq n + 1 \) and \( \text{rank}_i(b_i, b_{-i}) \leq n \), there must be a \( j \) with \( \text{rank}_j(v_j, v_{-j}) < n + 1 \) and \( \text{rank}_j(b_j, b_{-j}) > n \). \( \square \)
Proof of Lemma 5.1

$$\text{SW(NE)} = \int_{v,b} \sum_i v_i \cdot \Pr \left[ \#\text{items} \geq \max \{ \text{rank}_i(b_i, b_{i-1}), \text{rank}_i(v_i, v_{i-1}) \} \right]$$

$$+ \sum_i v_i \cdot \sum_{n=\text{rank}_i(b_i, b_{i-1})}^{\text{rank}_i(v_i, v_{i-1})-1} \Pr[\#\text{items} = n] \, dv \, db$$

$$\geq \int_{v,b} \sum_i v_i \cdot \Pr \left[ \#\text{items} \geq \max \{ \text{rank}_i(b_i, b_{i-1}), \text{rank}_i(v_i, v_{i-1}) \} \right] \, dv \, db$$

$$+ \int_{v,b} \sum_i \sum_{n=\text{rank}_i(v_i, v_{i-1})}^{\text{rank}_i(b_i, b_{i-1})+1} b_{(n-1)} \cdot \Pr[\#\text{items} = n] \, dv \, db$$

$$- 2 \int_{v,b} \sum_{n=2}^{+\infty} b_{(n-1)} \cdot \Pr[\#\text{items} = n] \, dv \, db \quad \text{(by Lemma 5.2)}.$$

Lemma 5.3. For any permutation $\sigma(\cdot|v, b)$ of the bidders,

$$\sum_i \int_{v,b} v_i \cdot \Pr \left[ \#\text{items} = \sigma(i|v, b) \right] \, dv \, db \leq \frac{4F(N)}{1 - 2F(N)} \text{SW(OPT)}.$$

Proof. Since for any $x$, $\Pr[\#\text{items} = x] \leq F(N)$, if we pick $\overline{m}$ satisfying $\Pr[\#\text{items} < \overline{m}] < \frac{1}{2} \leq \Pr[\#\text{items} \leq \overline{m}]$, then for any $i \leq \overline{m}$, $\Pr[\#\text{items} \geq i] \geq \frac{1}{2} - F(N)$. Thus,

$$\int_{v,b} \sum_{i=1}^{N} v_i \cdot \Pr \left[ \#\text{items} = \sigma(i|v, b) \right] \, dv \, db$$

$$\leq \int_{v,b} \sum_{i=1}^{N} v_i \cdot \Pr \left[ \#\text{items} = \sigma(i|v, b) \right] \cdot 1 \left[ \sigma(i|v, b) \leq \overline{m} \right] \, dv \, db$$

$$+ \int_{v,b} \sum_{i=1}^{N} v_i \cdot \Pr \left[ \#\text{items} = \sigma(i|v, b) \right] \cdot 1 \left[ \overline{m} + 1 \leq \sigma(i|v, b) \leq m' \right] \, dv \, db$$

$$= \int_{v,b} \sum_{i=1}^{N} v_i \cdot \Pr \left[ \#\text{items} = \sigma([i]|v, b) \right] \cdot 1 \left[ \sigma([i]|v, b) \leq \overline{m} \right] \, dv \, db$$

$$+ \int_{v,b} \sum_{i=1}^{N} v_i \cdot \Pr \left[ \#\text{items} = \sigma([i]|v, b) \right] \cdot 1 \left[ \overline{m} + 1 \leq \sigma([i]|v, b) \leq m' \right] \, dv \, db$$

As

$$\sum_{i=1}^{N} \cdot \Pr \left[ \#\text{items} = \sigma([i]|v, b) \right] \cdot 1 \left[ \overline{m} + 1 \leq \sigma([i]|v, b) \leq m' \right]$$

$$\leq \sum_{i=1}^{N} \Pr \left[ \#\text{items} = \sigma([i]|v, b) \right] \cdot 1 \left[ \sigma([i]|v, b) \leq \overline{m} \right] \leq \overline{m} F(N).$$
and \( \Pr \left[ \# \text{items} = \sigma([i]|v, b) \right] \leq F(N) \),

\[
\int_{v,b} \sum_{i=1}^{N} v[i] \cdot \Pr \left[ \# \text{items} = \sigma([i]|v, b) \right] \cdot 1 \left[ \sigma([i]|v, b) \leq m \right] \, dv \, db \\
+ \int_{v,b} \sum_{i=1}^{N} v[i] \cdot \Pr \left[ \# \text{items} = \sigma([i]|v, b) \right] \cdot 1 \left[ m + 1 \leq \sigma([i]|v, b) \leq m' \right] \, dv \, db \\
\leq \int_{v,b} \sum_{i=1}^{N} v[i] \cdot F(N) \, dv \, db + \int_{v,b} \sum_{i=1}^{m} v[i] \cdot F(N) \, dv \, db.
\]

Therefore,

\[
\int_{v,b} \sum_{i=1}^{N} v[i] \cdot \Pr \left[ \# \text{items} = \sigma(i|v, b) \right] \, dv \, db \leq 2 \cdot \int_{v,b} \sum_{i=1}^{m} v[i] \cdot F(N) \, dv \, db \\
\leq 2 \cdot F(N) \cdot \int_{v,b} \sum_{i=1}^{m} v[i] \cdot \frac{\Pr \left[ \# \text{items} \geq i \right]}{2 - F(N)} \, dv \, db \leq \frac{4F(N)}{1 - 2F(N)}SW(OPT).
\]

Lemma 5.4. If \( \Pr[\#\text{items} = n] > 0 \) and \( \text{rank}_i(v_i, v_{-i}) = n - 1 \) then \( \text{rank}_i(v_i, v_{-i}) \geq \text{rank}_i(v_i, b_{-i}) \).

Proof. For all the bidders with \( \text{rank}_j(b_j, b_{-j}) \leq n \), by Observation 5.1, \( v_j \geq b_j \). If \( \text{rank}_i(v_i, b_{-i}) \geq n \), then \( b_{(1)}, b_{(2)}, \ldots, b_{(n-1)} \geq v_i \), and so \( \text{rank}_i(v_i, v_{-i}) \geq n \). (If there are ties, they can be handled either by a fixed tie-breaking ordering of the bidders, or by using a uniform random ordering.) \( \square \)

Proof of Theorem 3.2:

\[
SW(OPT) = \int_{v,b} \sum_{i} v[i] \cdot \Pr \left[ \# \text{items} \geq \text{rank}_i(v_i, v_{-i}) \right] \, dv \, db.
\]
Substituting from Lemma 5.1 gives:

\( \text{SW(OPT)} - \text{SW(NE)} \)

\[
\leq \int_{v,b} \sum_{i = \text{rank}(b_i, b_{-i})}^{\text{rank}(b_i, b_{-i}) - 1} v_i \cdot \Pr[\# \text{items} = n] \, dv \, db \\
- \int_{v,b} \sum_{i = \text{rank}(b_i, b_{-i})}^{\text{rank}(b_i, b_{-i}) - 1} b_{(n-1)} \cdot \Pr[\# \text{items} = n] \, dv \, db + 2 \int_{v,b} \sum_{n = 2}^{+\infty} b_{(n-1)} \cdot \Pr[\# \text{items} = n] \, dv \, db \\
\leq \int_{v,b} \sum_{i = \text{rank}(b_i, b_{-i})}^{\text{rank}(b_i, b_{-i}) - 1} v_i \cdot \Pr[\# \text{items} = n] \, dv \, db + \int_{v,b} \sum_{i} v_i \cdot \Pr[\# \text{items} = \text{rank}_i(v_i, v_{-i})] \, dv \, db \\
- \int_{v,b} \sum_{i = \text{rank}(b_i, b_{-i})}^{\text{rank}(b_i, b_{-i}) - 1} b_{(n-1)} \cdot \Pr[\# \text{items} = n] \, dv \, db + 2 \int_{v,b} \sum_{n = 2}^{+\infty} b_{(n-1)} \cdot \Pr[\# \text{items} = n] \, dv \, db \\
\leq (a) \int_{v,b} \sum_{i = \text{rank}(b_i, b_{-i})}^{\text{rank}(b_i, b_{-i}) - 1} (v_i - b_{(n-1)}) \cdot \Pr[\# \text{items} = n] \, dv \, db \\
+ 2 \int_{v,b} \sum_{n = 2}^{+\infty} b_{(n-1)} \cdot \Pr[\# \text{items} = n] \, dv \, db + \int_{v,b} \sum_{i} v_i \cdot \Pr[\# \text{items} = \text{rank}_i(v_i, v_{-i})] \, dv \, db \\
\leq (b) \int_{v,b} \sum_{i} v_i \cdot \Pr[\# \text{items} = \text{rank}_i(b_i, b_{-i})] \, dv \, db \\
+ 2 \int_{v,b} \sum_{n = 2}^{+\infty} b_{(n-1)} \cdot \Pr[\# \text{items} = n] \, dv \, db + \int_{v,b} \sum_{i} v_i \cdot \Pr[\# \text{items} = \text{rank}_i(v_i, v_{-i})] \, dv \, db \\
\leq (c) \int_{v,b} \sum_{i} v_i \cdot \Pr[\# \text{items} = \text{rank}_i(b_i, b_{-i})] \, dv \, db \\
+ 2 \int_{v,b} \sum_{n = 2}^{+\infty} v_{[n-1]} \cdot \Pr[\# \text{items} = n] \, dv \, db + \int_{v,b} \sum_{i} v_i \cdot \Pr[\# \text{items} = \text{rank}_i(v_i, v_{-i})] \, dv \, db \\
\leq \frac{16F(N)}{1 - 2F(N)} \text{SW(OPT)}.
\]

Inequality (a) follows directly by Lemma 5.4; inequality (b) follows because \( b_i \) is a Nash equilibrium bid and hence its utility is at least as large as that obtained by bidding \( v_i \); inequality (c) follows by using Observation 5.1 to obtain \( b_{(i)} \leq v_{[i]} \) where \( v_{[i]} \) is the value of the \( (i) \)-th bidder; the last inequality follows by applying Lemma 5.3 to each term in turn. \( \square \)
6 Walrasian Equilibria

Recall that the English Walrasian mechanism can be implemented as an ascending auction. The prices it yields can be computed as follows: \( p_j \) is the price for good \( j \) that occurs at a Walrasian equilibrium when the supply of good \( j \) is increased by one unit. Similarly, the Dutch Walrasian mechanism can be implemented as a descending auction, and the resulting price \( p_j \) is the price for the \( j \)th good when its supply is decreased by one unit.

We will be considering an arbitrary Walrasian mechanism. Necessarily, its prices must lie between those of the Dutch Walrasian and English Walrasian mechanisms. We let \( p^\text{Eng}(\vec{n}; (b_i, b_{-i})) \) denote the price output by the English Walrasian mechanism and \( p^\text{Dut}(\vec{n}; (b_i, b_{-i})) \) be the price output by the Dutch Walrasian mechanism.

We define the distance between two price vectors \( p \) and \( p' \) with respect to \( U \) as follows:

\[
\text{dist}^U(p, p') = \sum_{j=1}^{m} \left| \min\{p_j, U\} - \min\{p'_j, U\} \right|.
\]

Observation 6.1. In the Dutch Walrasian mechanism, if there are zero copies of a good, letting its price be \(+\infty\) will not affect the mechanism outcome.

Observation 6.2. Suppose bidders’ demands satisfy the Gross Substitutes property. In both the English and Dutch Walrasian mechanisms, if \( n_i \geq n'_i \), then \( p(n_i, n_{-i}) \leq p(n'_i, n_{-i}) \), where \( p \leq p' \) means that, for all \( j \), \( p_j \leq p'_j \).

Definition 6.1. Given bidding profile \((b_i, b_{-i})\), \( \vec{n} = (n_j, n_{-j}) \) is \((\epsilon, U)\)-bad for good \( j \), if in the English Walrasian mechanism the distance between the prices is more than \( \epsilon \) when an additional copy of good \( j \) is added to the market:

\[
\text{dist}^U(p^\text{Eng}((n_j, n_{-j}); (b_i, b_{-i})), p^\text{Eng}((n_j + 1, n_{-j}); (b_i, b_{-i}))) > \epsilon.
\]

Let \( \vec{k} = (k, k, \ldots, k) \) and \( \vec{0} = (0, 0, \ldots, 0) \) be \( l \)-vectors.

Definition 6.2. Given bidding profile \( \vec{b} \), \( \vec{n} \) is \((k, \epsilon, U)\)-bad for good \( j \) if there is a vector \( \vec{n}' \) which is \((\epsilon, U)\)-bad for good \( j \), such that \( n'_h \leq n_h \) for all \( h \), and \( \sum_h n_h \leq k + \sum_h n'_h \). \( \vec{n} \) is \((k, \epsilon, U)\)-good if it is not \((k, \epsilon, U)\)-bad.

In Lemmas 6.1 and 6.2 we bound the number of \((\epsilon, U)\)-bad multiplicity vectors, and then in Lemma 6.3 we bound the probability of a \((k, \epsilon, U)\)-bad vector. Following this, in Lemma 6.4 and 6.5 assuming the multiplicity vector is \((k + 1, \epsilon, U)\)-good, we bound the difference between the English Walrasian mechanism prices and those of the Walrasian mechanism at hand. Next, in Lemma 6.6 again for \((k + 1, \epsilon, U)\)-good multiplicity vectors, we relate \( u_i(x_i(v_i, b_{-i})) \) to \( v_i(x_i(v_i, v_{-i})) \) and the prices paid; we then use this to carry out a PoA analysis. For brevity, we sometimes write \( u_i(v_i, b_{-i}) \) instead of \( u_i(x_i(v_i, b_{-i})) \).
Lemma 6.3. In the English Walrasian mechanism with bidding profile \( \vec{b} \), the probability that \( \vec{n} \) is \((k, \epsilon, U)\)-bad for some good or \( \min_j n_j \leq k \) is at most

\[
\sum_{1 \leq j \leq l} F(j, N) \left[ \frac{L}{\epsilon} U(k + 1)^2 (k + l)^{k + 1} \right].
\]

Let \( n_j^i(b_i, b_{-i}) \) denote the number of copies of good \( j \) that bidder \( i \) receives with bidding profile \((b_i, b_{-i})\) and \( n'^i(b_i, b_{-i}) \) denote the corresponding vector. Also, let \( p^{Eng}(\vec{n}; b_{-i}) \) denote the market equilibrium prices when bidder \( i \) is not present.

Lemma 6.4. \( p^{Eng}_j(\vec{n}; b_{-i}) \leq p_j(\vec{n}; (b_i, b_{-i})) \).

Lemma 6.5. If \( \vec{n} \) is \((k + 1, \epsilon, U)\)-good for all goods, and \( n_j > k + 1 \) for all \( j \), then

\[
\forall j \quad \min\{p_j(\vec{n}; (v_i, v_{-i})), U\} \leq \min\{p_j(\vec{n}; (b_i, b_{-i})), U\} + (k + 1)\epsilon.
\]

Lemma 6.6. If \( \vec{n} \) is \((k + 1, \epsilon, U)\)-good with \( U \geq v_i(s) \) for every single item \( s \) and \( n_j > k + 1 \) for all \( j \), then

\[
u_i(v_i, b_{-i}) \geq v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\vec{n}; (b_i, b_{-i})) - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon,
\]

where the sum is over all the items in allocation \( x_i \).

Lemma 6.7. If Assumption 3.3 holds, then \( \mathbb{E}_{v_i} [\max_s \{v_i(s)\}] < l \cdot \zeta \).

Proof. \( \mathbb{E}_{v_i} [\max_s \{v_i(s)\}] \leq \mathbb{E}_{v_i} [\sum_s v_i(s)] \leq \sum_s \mathbb{E}_{v_i} [v_i(s)] \leq l \cdot \zeta. \)

We begin with a slightly weaker version of Theorem 3.3 which demonstrates the main ideas.

Theorem 6.1. In a large market which satisfies Assumptions 3.2 and 3.3,

\[
\text{SW(NE)} \geq \left( 1 - \frac{3k \cdot \zeta \cdot l}{\rho} \sqrt{\frac{(k + 1) \sum_{1 \leq j \leq l} F(j, N) \left[ l(k + 2)^3(k + 1 + l)^{k+1} \right]}{\epsilon}} \right) \text{SW(OPT)}.
\]

Proof. By Lemma 6.6, if \( \vec{n} \) is \((k + 1, \epsilon \cdot \max_s \{v_i(s)\}, \max_s \{v_i(s)\})\)-good and \( n_j > k + 1 \) for all \( j \), then

\[
u_i(v_i, b_{-i}) \geq v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\vec{n}; (b_i, b_{-i})) - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon \cdot \max_s \{v_i(s)\}.
\]

By Lemma 6.3, the probability that \( \vec{n} \) is \((k + 1, \epsilon \cdot \max_s \{v_i(s)\}, \max_s \{v_i(s)\})\)-bad or \( n_j \leq k + 1 \) for some \( j \) is less than

\[
\sum_{1 \leq j \leq l} F(j, N) \left[ \frac{L}{\epsilon} \cdot (k + 2)^3(k + 1 + l)^{k+1} + k + 2 \right],
\]

15
and \[ \mathbb{E}_{\tilde{\pi}}[u_i(v_i, b_{-i})] \]
\[ \geq \mathbb{E}_{\tilde{\pi}} \left[ v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\tilde{n}; (b_i, b_{-i})) - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon \cdot \max_s \{ v_i(s) \} \right] \]
\[ - \sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\epsilon} \cdot (k + 2)^2(k + 1 + l)^{k+1} + k + 2 \right] \cdot k \cdot \max_s \{ v_i(s) \}. \]

Here, the expectation is taken over the randomness on the multiplicities of the goods; the inequality holds since \( u_i(v_i, b_{-i}) \geq 0 \) and \( v_i(x_i(v_i, v_{-i})) \leq k \cdot \max_s \{ v_i(s) \} \).

Taking the expectation over the valuation of agent \( i \) yields

\[ \mathbb{E}_{v_i}[\mathbb{E}_{\tilde{\pi}}[u_i(v_i, b_{-i})]] \geq \mathbb{E}_{v_i} \left[ \mathbb{E}_{\tilde{\pi}} \left[ v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\tilde{n}; (b_i, b_{-i})) \right. \right. \]
\[ \left. \left. - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon \cdot \max_s \{ v_i(s) \} \right] \right. \]
\[ - \sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\epsilon} \cdot (k + 2)^2(k + 1 + l)^{k+1} + k + 2 \right] \cdot k \cdot \max_s \{ v_i(s) \}. \]

By Lemma 6.7, \( \mathbb{E}_{v_i}[\max_s \{ v_i(s) \}] \leq \zeta \). Thus

\[ \mathbb{E}_{v_i}[\mathbb{E}_{\tilde{\pi}}[u_i(v_i, b_{-i})]] \geq \mathbb{E}_{v_i} \left[ \mathbb{E}_{\tilde{\pi}} \left[ v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\tilde{n}; (b_i, b_{-i})) \right. \right. \]
\[ \left. \left. - \zeta \cdot l \cdot k(k + 1)\epsilon \right. \right. \]
\[ - \zeta \cdot l \cdot \sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\epsilon} \cdot (k + 2)^2(k + 1 + l)^{k+1} + k + 2 \right] \]

Let \( R(b) \) denote the revenue when the bidding profile is \( b \). By Assumption 3.3, the optimal welfare \( SW(OPT) > \rho N \). Now, summing over all the bidders yields
\[
\sum_i \mathbb{E}_{\tilde{v}, \tilde{b}}[u_i(v_i, b_{-i})] \geq \sum_i \mathbb{E}_{\tilde{v}, \tilde{b}} \left[ \mathbb{E}_{\tilde{n}} \left[ v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\tilde{n}; (b_i, b_{-i})) \right] \right] \\
- \zeta \cdot l \cdot \sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\epsilon} (k + 2)^2 (k + 1 + l)^{k+1} k + (k + 2) k \right] \cdot N \\
- \zeta \cdot l \cdot k (k + 1) \epsilon \cdot N \\
\geq \left( 1 - \frac{\zeta \cdot l \cdot \sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\epsilon} (k + 2)^2 (k + 1 + l)^{k+1} k + (k + 2) k \right]}{\rho} \right) \text{SW(OPT)} - R(b_i, b_{-i}).
\]

Using the smooth technique for Bayesian settings [21],

\[
\text{SW(NE)} \geq \left( 1 - \frac{\zeta \cdot l \cdot \sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\epsilon} (k + 2)^2 (k + 1 + l)^{k+1} k + (k + 2) k \right]}{\rho} \right) \text{SW(OPT)}.
\]

Now set \( \epsilon = \sqrt{\frac{\sum_{1 \leq j \leq l} F(j, N) \left[ ((k + 2)^2 (k + 1 + l)^{k+1}) / (k + 1) \right]}{l}} \). The claimed bound follows. \( \square \)

**Proof of Theorem 3.3.** By Lemma [3.3] the probability that \( \tilde{n} \) is \((k+1, \frac{\max_s \{v_i(s)\}}{2^{\rho}}, \max_s \{v_i(s)\})\) bad or \( n_j \leq k + 1 \) for some \( j \) is less than

\[
\sum_{1 \leq j \leq l} F(j, N) \left[ \frac{l}{\max_s \{v_i(s)\}} \max_s \{v_i(s)\} (k + 2)^2 (k + 1 + l)^{k+1} k + 2 \right] \\
= \sum_{1 \leq j \leq l} F(j, N) \left[ l \cdot 2^\rho (k + 2)^2 (k + 1 + l)^{k+1} + k + 2 \right].
\]
So, for any integer $c'$,

$$\mathbb{E}_{\bar{\vec{v}}} [u_i(v_i, b_{-i})] \geq \mathbb{E}_{\bar{\vec{v}}} \left[ v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\bar{\vec{v}}; (b_i, b_{-i})) \right. $$

$$- \sum_{c=1}^{c'} \mathbb{I} \left[ \bar{\vec{v}} \text{ is } (k + 1, \frac{\max_s \{v_i(s)\}}{2^c}, \max_s \{v_i(s)\})\text{-bad } \text{ and } (k + 1, \frac{\max_s \{v_i(s)\}}{2^{c-1}}, \max_s \{v_i(s)\})\text{-good} \right]$$

$$\cdot \left| x_i(v_i, v_{-i}) \cap d_i \right| \cdot (k + 1) \frac{\max_s \{v_i(s)\}}{2^{c-1}}$$

$$- \mathbb{I} \left[ \bar{\vec{v}} \text{ is } (k + 1, \frac{\max_s \{v_i(s)\}}{2^{c'}}, \max_s \{v_i(s)\})\text{-good} \right] \cdot \left| x_i(v_i, v_{-i}) \cap d_i \right| \cdot (k + 1) \frac{\max_s \{v_i(s)\}}{2^{c'}}$$

$$\geq \mathbb{E}_{\bar{\vec{v}}} \left[ v_i(x_i(v_i, v_{-i})) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\bar{\vec{v}}; (b_i, b_{-i})) \right]$$

$$- \sum_{c=1}^{c'} \sum_{1 \leq j \leq l} F(j, N) \left[ l \cdot 2^c (k + 2)^2 (k + 1 + l)^{k+1} + k + 2 \right] \cdot k \cdot (k + 1) \frac{\max_s \{v_i(s)\}}{2^{c-1}}$$

$$- k \cdot (k + 1) \frac{\max_s \{v_i(s)\}}{2^{c'}}$$

Summing over all the bidders and integrating w.r.t. $\vec{v}$ and $\vec{b}$ gives

$$\sum_i \mathbb{E}_{\vec{v}, \vec{b}, \bar{\vec{v}}} [u_i(v_i, b_{-i})] \geq \text{SW(OPT)} - R(b_i, b_{-i})$$

$$- N \cdot c' \sum_{1 \leq j \leq l} F(j, N) \left[ 2l (k + 2)^2 (k + 1 + l)^{k+1} + k + 2 \right] \cdot k \cdot (k + 1) \cdot \zeta \cdot l$$

$$- N \cdot k \cdot (k + 1) \frac{1}{2^{c'}} \cdot \zeta \cdot l.$$
Let \( Y = \sum_{1 \leq j \leq l} F(j, N) \left[ 2l(k + 2)^2(k + 1 + l)^{k+1} \right] \). Set \( c' = \left\lceil \log_2 \frac{1}{Y} - \log_2 \log_2 \frac{1}{Y} \right\rceil \); then \( \frac{1}{2^c} \leq Y \log_2 \frac{1}{Y} \). So,

\[
SW(\text{NE}) \geq \left( 1 - \frac{3 \cdot k \cdot (k + 1) \cdot \zeta \cdot l}{\rho} \cdot Y \cdot \left\lceil \log_2 \frac{1}{Y} \right\rceil \right) SW(\text{OPT}).
\]

\[
\square
\]

7 Discussion

We have shown that in large markets under suitable conditions on the valuations and on the uncertainty about the multiplicities of the goods, the PoA tends to 1 as the market size grows.

We mention a few open questions. Can the results for the one good market be shown via a smoothness argument, thereby enabling them to apply to a larger class of equilibria? Can the results be extended to settings where there is no Walrasian equilibrium? What about for Fisher markets, markets of divisible goods with budgeted players, and for Exchange markets? More generally, when does size ameliorate outcomes in games?

References


A Omitted Proofs

A.1 Proofs from Section 3

Proof of Lemma 3.1: Let \( \#\text{items}_j \) denote the number of copies of good \( j \) that are present, and let \( N_j \) denote the number of buyers for which good \( j \) has the largest expected value (breaking ties arbitrarily). By Chebyshev’s Theorem, \( \Pr \left[ \#\text{items}_j > \mathbb{E}[\#\text{items}_j] - t \cdot \Gamma(\#\text{items}_j) \right] \geq 1 - \frac{1}{t^2} \). We set \( t = 1 + \lambda \), where \( \lambda \) is the parameter in Assumption 3.4. Then by Assumption 3.4, \( \Pr \left[ \#\text{items}_j > \lambda^2 \cdot \mathbb{E}[\#\text{items}_j] \right] \geq \frac{2 \lambda^2}{(1+\lambda)^2} \), which implies \( \Pr[\#\text{items}_j > \lambda^2 \alpha N] \geq \frac{2 \lambda^2}{(1+\lambda)^2} \). If at least \( \lambda^2 \alpha N \) copies of good \( j \) are available, then by Assumption 3.5, there is an assignment with valuation at least \( \rho' \cdot \min \{ N_j, \lambda^2 \alpha N \} \). Therefore, the social welfare is at least \( \sum_j \min \{ N_j, \lambda^2 \alpha N \} \) \( \frac{2 \lambda^2}{(1+\lambda)^2} \cdot \rho' \geq \lambda^2 \alpha^2 \frac{2 \lambda^2}{(1+\lambda)^2} \cdot N \cdot \rho' \). \( \square \)

A.2 Proofs from Section 6

Proof of Lemma 6.1: We prove the result by contradiction. Accordingly, let
\[
S = \left\{ n_j \left| \text{dist}^U(p^\text{Eng}((n_j, n_{-j}); \vec{b}), p^\text{Eng}((n_j + 1, n_{-j}); \vec{b})) > \epsilon \right| \right. \}
\]
and suppose that \( |S| > \frac{l}{\epsilon} U \).

The proof uses a new function \( pf(\cdot) : \) \( pf(n_j) = \sum_{q=1}^{t} \min \{ p^\text{Eng}((n_j, n_{-j}); \vec{b}), U \} \).

Then,
\[
\liminf_{n \to \infty} (pf(0) - pf(n)) = \liminf_{n \to \infty} \sum_{h=0}^{n-1} (pf(h) - pf(h+1)) \\
\geq \sum_{n_j \in S} (pf(n_j) - pf(n_j + 1)) > \frac{l}{\epsilon} U \cdot \epsilon = l \cdot U.
\]

(A.1)
The first inequality follows as by Observation 6.2, \( pf(\cdot) \) is a non-increasing function. Further, by construction, \( 0 \leq pf(h) \leq l \cdot U \) for all \( h \), thus \( \liminf_{n \to \infty} (pf(0) - pf(n)) \leq l \cdot U \), contradicting (A.1). \( \square \)

Proof of Lemma 6.2: For \( (n_j, n_{-j}) \) to be \( (k, \epsilon, U) \)-bad for good \( j \) we need an \( (\epsilon, U) \)-bad vector \( \vec{n}' \preceq \vec{n} \) for good \( j \), with \( \sum_{h \neq j} n_h - n'_h = c \) for some \( 0 \leq c \leq k \) and \( n_j - n'_j \leq k - c \). There are \( \binom{l-c+2}{c} \) ways of choosing the \( n'_j \). For each \( n'_j \), by Lemma 6.1, there are at most \( \frac{1}{\epsilon} U \) points that are \( (\epsilon, U) \)-bad for good \( j \). For each \( (\epsilon, U) \)-bad point, there are \( k - c + 1 \) choices for \( n_j \). This gives a total of
\[
\sum_{c=0}^{k} \frac{l}{\epsilon} U (k - c + 1) \binom{l-2+c}{c} \leq \frac{l}{\epsilon} U (k+1)^2 (k+l)^k
\]
\( (k, \epsilon, U) \)-bad vectors. \( \square \)
Proof of Lemma 6.3: Conditioned on the bidding profile being $\vec{b}$,

$$
\sum_{1 \leq j \leq l} \Pr[\text{\vec{n} is (k, \epsilon, U)-bad for good } j \cup (n_j \leq k)]
\leq \sum_{1 \leq j \leq l} \Pr[\text{\vec{n} is (k, \epsilon, U)-bad for good } j] + \Pr[(n_j \leq k)]
\leq \sum_{1 \leq j \leq l} \sum_{\pi_{-j}} \left( \Pr[\text{\vec{n} is (k, \epsilon, U)-bad for good } j | n_{-j} = \pi_{-j}] + \Pr[(n_j \leq k | n_{-j} = \pi_{-j})] \cdot \Pr[n_{-j} = \pi_{-j}] \right)
\leq \sum_{1 \leq j \leq l} \frac{1}{\epsilon} U(k+1)^2(k+l)^k + k+1 \quad \text{(by Lemma 6.2)}.
$$

Proof of Lemma 6.4: Consider the situation with $n'' = \vec{n} - n^i(b_i, b_{-i})$ and suppose that agent $i$ is not present. Then $p_j(\vec{n}; (b_i, b_{-i}))$ is a market equilibrium.

So, \forall j \quad p_j^{Eng}(\vec{n}'; b_{-i}) \leq p_j(\vec{n}; (b_i, b_{-i})).

Since $\vec{n} \geq \vec{n}'$, \forall j \quad p_j^{Eng}(\vec{n}'; b_{-i}) \leq p_j^{Eng}(\vec{n}''; b_{-i}).

The lemma follows on combining these two inequalities.

Proof of Lemma 6.5: Let $d_i \leq n'(v_i, b_{-i})$ be a minimal set with $v_i(d_i) = v_i(n'(v_i, b_{-i}))$. By Definition 2.3(i), $\sum_j d_j \leq k$. First, if $n'_j(v_i, b_{-i}) > d_i$ then $p_j(\vec{n}; (v_i, b_{-i})) = 0$, as the pricing is given by a Walrasian Mechanism.

Consider the scenario with $n''$ copies of goods on offer, where for all $j$, $n''_j = n_j - d_j$ and suppose that bidder $i$ is not present; then $p(\vec{n}; (v_i, b_{-i}))$ is a market equilibrium.

So, $p_j^Dut(\vec{n}''; b_{-i}) = p_j^{Eng}(\vec{n}''; b_{-i})$.

For all $j' \neq j$, let $n'''_j = n''_j$, and let $n''''_j = n''_j - 1$; then $p_j^Dut(\vec{n}''; b_{-i}) \leq p_j^{Eng}(\vec{n}''''; b_{-i})$.

and by Lemma 6.4 $p_j^{Eng}(\vec{n}''''; b_{-i}) \leq p_j^{Eng}(\vec{n}''''; b_{-i})$.

As $\vec{n}$ is $(k+1, \epsilon, U)$-good for all goods, and as $\sum_k \vec{n}_k - \vec{n}'''' \leq k + 1$, we conclude that

$$
\min\{p_j(\vec{n}; (v_i, b_{-i})), U\} \leq \min\{p_j^{Eng}(\vec{n}''''; (b_i, b_{-i})), U\}
\leq \min\{p_j^{Eng}(\vec{n}''''; (b_i, b_{-i})), U\} + (k + 1)\epsilon \leq \min\{p_j(\vec{n}; (v_i, b_{-i})), U\} + (k + 1)\epsilon. \quad \text{(A.2)}
$$

Proof of Lemma 6.6: As we are using a Walrasian mechanism, for any allocation $x'_i$,

$$
v_i(x_i(v_i, b_{-i})) - \sum_{s \in x_i(v_i, b_{-i})} p_s(\vec{n}; (v_i, b_{-i})) \geq v_i(x'_i) - \sum_{s \in x'_i} p_s(\vec{n}; (v_i, b_{-i})). \quad \text{(A.3)}
$$

We let $S$ denote the set of goods whose prices $p_s(\vec{n}; (v_i, b_{-i}))$ are larger than $U$. Then,

$$
u_i(v_i, b_{-i}) = v_i(x_i(v_i, b_{-i})) - \sum_{s \in x_i(v_i, b_{-i})} p_s(\vec{n}; (v_i, b_{-i}))
\geq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in (x_i(v_i, v_{-i}) \cap d_i) \setminus S} p_s(\vec{n}; (v_i, b_{-i})) \quad \text{(by (A.3))} \quad \text{(A.4)}
$$

22
Since $\vec{n}$ is $(k + 1, \epsilon, U)$-good, by Lemma 6.5
\[
\min\{p_s(\vec{n}; (v_i, b_{-i})), U\} \leq \min\{p_s(\vec{n}; (b_i, b_{-i})), U\} + (k + 1)\epsilon.
\]

Therefore, for any $s \not\in S$,
\[
p_s(\vec{n}; (v_i, b_{-i})) \leq \min\{p_s(\vec{n}; (b_i, b_{-i})), U\} + (k + 1)\epsilon
\]
\[
\leq p_s(\vec{n}; (b_i, b_{-i})) + (k + 1)\epsilon.
\]

So,
\[
v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in (x_i(v_i, v_{-i}) \cap d_i) \setminus S} p_s(\vec{n}; (v_i, b_{-i}))
\]
\[
\geq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in (x_i(v_i, v_{-i}) \cap d_i) \setminus S} p_s(\vec{n}; (b_i, b_{-i}))
\]
\[
- |(x_i(v_i, v_{-i}) \cap d_i) \setminus S| \cdot (k + 1)\epsilon.
\]

For any $s \in S$, on applying Lemma 6.5, we obtain $U = \min\{p_s(\vec{n}; (v_i, b_{-i})), U\} \leq \min\{p_s(\vec{n}; (b_i, b_{-i})), U\} + (k + 1)\epsilon$, which implies $p_s(\vec{n}; (b_i, b_{-i}))) + (k + 1)\epsilon \geq U$. Also,
\[
v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) \leq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) \leq |(x_i(v_i, v_{-i}) \cap d_i) \cap S| \cdot U,
\]
where the first inequality follows by the Gross Substitutes assumption, and the second by Gross Substitutes and because by assumption $v_i(s) \leq U$ for all single items $s$. Thus,
\[
v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in (x_i(v_i, v_{-i}) \cap d_i) \setminus S} p_s(\vec{n}; (b_i, b_{-i})) - |(x_i(v_i, v_{-i}) \cap d_i) \setminus S| \cdot (k + 1)\epsilon
\]
\[
\geq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - |(x_i(v_i, v_{-i}) \cap d_i) \cap S| \cdot U
\]
\[
- \sum_{s \in (x_i(v_i, v_{-i}) \cap d_i) \setminus S} p_s(\vec{n}; (b_i, b_{-i})) - |(x_i(v_i, v_{-i}) \cap d_i) \setminus S| \cdot (k + 1)\epsilon
\]
\[
\geq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in x_i(v_i, v_{-i}) \cap d_i} p_s(\vec{n}; (b_i, b_{-i})) - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon
\]
\[
\geq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\vec{n}; (b_i, b_{-i})) - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon.
\]

(A.6)

By (A.4), (A.5) and (A.6),
\[
u_i(v_i, b_{-i}) \geq v_i((x_i(v_i, v_{-i}) \cap d_i) \setminus S) - \sum_{s \in x_i(v_i, v_{-i})} p_s(\vec{n}; (b_i, b_{-i})) - |x_i(v_i, v_{-i}) \cap d_i| \cdot (k + 1)\epsilon.
\]
\[
\square
\]