Controllability analysis of uncertain polytopic systems with time-varying state delay

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Abstract—In this paper reachability/controllability properties for networked systems described by uncertain polytopic linear plants subject to time-varying state delays and input constraints are analyzed. Up to our best acknowledge, previous technical contributions on the matter were not been capable to properly address time-delay occurrences. To this end a new result making it possible to efficiently compute the set of states that can be robustly steered in a finite number of steps, via state feedback control, to a given target set is here proposed. A final simple example is used to show the applicability of the proposed results.

I. INTRODUCTION

Establishing safety properties of hybrid and networked systems is one of the most interesting, but equally challenging, problems for formal methods. With the increased embedding of digital controllers inside physical devices, such as automobiles and aircraft, the need for automated tools and techniques for formal analysis and verification has become more pressing. Exhaustive testing via simulation is, in most cases, neither possible nor practical. Reachability and safety concepts are now in fact being recognized as central problems in designing controllers.

From a methodological point of view, Reachability/Controllability analysis is a relevant research issue in control theory because of its strict relationship with set invariance, set-membership state estimation, constrained control, fault tolerant control schemes etc., see the books [4], [7], papers [3], [17], [19] and references therein. Starting ideas on reachability analysis and guaranteed state estimation can be traced back to the pioneering control literature (see the seminal papers [22], [6], [21], [14]). The common denominator behind these early approaches is that the exact reachability concept is a high computationally demanding task and it is necessary to resort to approximation schemes in order to properly attack the problem. Approximate reachability paradigms are several in literature exploiting essentially, ellipsoidal, polytopic or even zonotopic calculus to obtain guaranteed inner and/or outer estimates of the exact reachable sets/tubes or sets of possible states consistent with acquired information, system dynamics and uncertainty features (see [16], [15], [1] and references therein).

A common feature of these approximate reachability/controllability sets computational approaches is to resort to the so-called predecessor operator algorithm by computing the set of states that can be robustly steered (using an admissible control input) to a given target set in a single step. The predecessor operator is then used in a recursive fashion in order to compute the set of states that can be robustly steered to the given target set in a finite number of steps (see [15] and [2]). As previously stated, the advent of computational improvements and the development of efficient software tools for classes of constrained systems of practical interest, such as hybrid and networked systems have renewed interest in these problems (see [7], [19], [5], [17] and [12]).

To the best of authors’ knowledge, constrained networked systems have received relatively little attention. A networked system can in fact be regarded as a system subject to time-varying delays and such a paradigm is described via Functional Differential Equations (FDE), which differ from Ordinary Differential Equations (ODE) because they do not admit in general a finite dimensional state representation. As a consequence, performance analysis and control design for such systems suffer therefore from unavoidable structural complications and only conservative results and related approximations are achievable, see the survey paper [20] and references therein.

Moving from the previous considerations, the main aim of this paper is to present novel results and related computational schemes for controllability computations using ellipsoidal calculus for input constrained uncertain polytopic time-delay systems. We would like to point out that the proposed extension is not trivial because adequate relaxations are needed even if all the relevant sets are convex and the system is linear. The key point is then to give proper inner approximations of the one-step controllable sets by acting on both actual and delayed states.

The contribution is twofold: first, Delay-Dependent (DD) closed-loop stability is guaranteed for input-constrained time-delay systems. Then, one-step controllable ellipsoidal regions are computed by resorting to ellipsoidal calculus formulas which are based on a series of convex approximations of the original problem obtained by means of set-invariance concepts and of the descriptor approach [11].

Finally, to validate the proposed results a numerical ex-
ample taken from the literature is discussed.

**NOTATION**

Given a set $S \subseteq X \times Y \subseteq \mathbb{R}^n \times \mathbb{R}^m$, the projection of the set $S$ onto $X$ is defined as

$$\text{Proj}_X(S) := \{ x \in X \mid \exists y \in Y \text{ s.t. } (x, y) \in S \}$$

Given a set $S \subseteq \mathbb{R}^n$, $\text{In}(S) \subseteq S$ denotes its inner ellipsoidal approximation.

II. PROBLEM FORMULATION

In what follows we will refer to multi-model discrete-time delayed linear systems

$$x(t+1) = \Phi(\alpha(t))x(t-\tau(t)) + G(\alpha(t))u(t)$$

(1)

where $t \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, $x(t) \in \mathbb{R}^n$ denotes the state plant and $u(t) \in \mathbb{R}^m$ the control input. The time-varying delay satisfies $0 \leq \tau(t) \leq \tau_{up}$, with the upper-bound $\tau_{up}$ assumed to be known and constant. The parameter vector $\alpha(t) \in \mathbb{R}^l$ is assumed to lie in the unit simplex

$$\mathcal{P}_l := \left\{ \alpha \in \mathbb{R}^l : \sum_{i=1}^l \alpha_i = 1, \quad \alpha_i \geq 0 \right\}$$

(2)

and the system matrices $\Phi(\alpha)$ and $G(\alpha)$ belong to

$$\Sigma(\mathcal{P}_l) := \left\{ (\Phi(\alpha), G(\alpha)) = \sum_{i=1}^l \alpha_i(\Phi_i, G_i), \quad \alpha \in \mathcal{P}_l \right\}$$

(3)

where the pairs $(\Phi_i, G_i)$ denote the polytope vertices $\Sigma(\mathcal{P}_l)$, viz. $(\Phi_i, G_i) \in \text{vert} \{ \Sigma(\mathcal{P}_l) \}, \quad \forall i \in l := \{1, 2, \ldots, l\}$. Moreover, the control input is subject to the following saturation constraints

$$u(t) \in \mathcal{U}, \quad \forall t \geq 0, \quad \mathcal{U} := \{ u \in \mathbb{R}^m \mid u^T u \leq \bar{u} \},$$

(4)

with $\bar{u} > 0$ and $\mathcal{U}$ a compact subset of $\mathbb{R}^m$ containing the origin as an interior point. Then, the problem of interest can be stated as follows:

**One-Step Controllable Sets (OCS) Problem** - Consider the time-delay system (1)-(4), determine:

- a robustly positively invariant set $\mathcal{T}_0$ and a state-feedback law of the form

$$u(t) = K_1 x(t) + K_2 x(t-\tau(t))$$

(5)

compatible with (4), such that the closed-loop system is Delay-Dependent (DD) stable $\forall x \in \mathcal{T}_0, \quad \forall \alpha \in \mathcal{P}_l, \quad \forall t \in [0, \tau_{up}]$;

- a sequence $\{ \mathcal{T}_1, \ldots, \mathcal{T}_N \}$ of $N$ arbitrary sets such that, for any possible state $x \in \mathcal{T}_i, \quad i > 0$, there exists a control move $u(t) \in \mathcal{U}$ such that

$$\Phi(\alpha(t))x(t-\tau(t)) + G(\alpha(t))u(t) \in \mathcal{T}_{i-1}, \quad \forall \alpha \in \mathcal{P}_l, \quad \forall t \in [0, \tau_{up}]$$

The feedback gains $K_1, K_2$ can be derived by exploiting a standard technicality in delayed systems (see [11] and references therein).

We start by considering the following auxiliary state, capable to trace all the delayed state informations,

$$y(t) = x(t+1) - x(t)$$

that allows to obtain the following descriptor form of (1)

$$\begin{cases}
    x(t+1) = y(t) + x(t) \\
    0 = -y(t) + \Phi(\alpha(t))x(t-\tau(t)) + G(\alpha(t))K_1 x(t) + K_2 x(t-\tau(t))
\end{cases}$$

(6)

Then, by defining the augmented state

$$\bar{x}(t) = \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix}^T$$

we rewrite (6) as follows

$$E \bar{x}(t+1) = A \bar{x}(t) + B \sum_{j=t-\tau(t)}^{t-1} y(j)$$

(7)

with

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} I \\ \Phi(\alpha) - G(\alpha)K_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \Phi(\alpha) - G(\alpha)K_2 \end{bmatrix}$$

By resorting to the DD Lyapunov-Krasovskii functional

$$V(t) = \bar{x}^T(t) E P_{DD} E \bar{x}(t) + \sum_{m=-\tau_{up}}^{-1} \sum_{j=t-m-1}^{t-1} y^T(j) [R + Q] y(j)$$

(8)

it can be proved that the DD feedback control law (5) asymptotically stabilizes the plant and satisfies the prescribed constraints if the following matrix inequalities in the objective variables $K_i, \quad i = 1, 2$, $P_{DD}, Q, R$ and $\tau_{up} \leq \bar{\tau}$, evaluated over the plant vertices, are satisfied

$$\begin{bmatrix}
    E^T P_{DD} E - S_{DD} & 0 & A^T P_{DD} \\
    0 & \tau_{max}(R + Q) & B^T_{DD} P_{DD} \\
    P_{DD} A_{DD} & P_{DD} B_{DD} P_{DD} & P_{DD}
\end{bmatrix} \geq 0$$

(9)

$$\begin{bmatrix}
\bar{u}^T E^T P_{DD} E & \begin{bmatrix} K_1 \ K_2 \end{bmatrix} \\
\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} & I
\end{bmatrix} \geq 0$$

(10)

where

$$S_{DD} := \text{diag} \{0, \tau_{up} (R + Q)\}$$

Remark 1 - Note that the condition (9) is bilinear in the variables $K_i, \quad i = 1, 2$ and $P_{DD}$, therefore the computation of the triplet $(\mathcal{T}_0, K_1, K_2)$ will be numerically achieved by means of ad-hoc SDP packages, e.g. [13]. □
### III. MAIN RESULT

First, recall that given the plant (1) and assuming a time-delay free scenario it is possible in principle to compute the sets of states $i$-step controllable to $\mathcal{T}$ via the following recursion (see [7]):

\[
\begin{align*}
\mathcal{T}_0 &:= \mathcal{T} \\
\mathcal{T}_i &:= \{ x \in \mathbb{R}^n : \exists u \in \mathcal{U} : \forall \alpha \in \mathcal{P}_i, \\
& \quad \Phi(\alpha)x + G(\alpha)u \in \mathcal{T}_{i-1}, \}
\end{align*}
\]

where $\mathcal{T}_i$ is the set of states that can be steered into $\mathcal{T}_{i-1}$ using a single move with a causal control. By induction we have that $\mathcal{T}_i$ is the set of states that can be steered into $\mathcal{T}$ in at most $i$ control moves.

To recast such an idea into a computable scheme, explicit time-delay dependencies in the model (1) need to be derived. To this end, w.l.o.g. (1) can be re-written as

\[
\begin{align*}
x(t+1) &= \Phi(\alpha(t))x(t) + \Phi(\alpha(t))x(t-\tau(t)) + \\
& \quad + G(\alpha(t))u(t), \quad \forall \tau(t) \in [1, \tau_{up}]
\end{align*}
\]

and by considering the auxiliary state $y(t)$, the following descriptor form results

\[
\begin{bmatrix}
x(t+1) \\
y(t) + x(t) \\
- y(t) + \Phi(\alpha(t))x(t) + \Phi(\alpha(t))x(t-\tau(t)) + \\
+ G(\alpha(t))u(t) - x(t)
\end{bmatrix}
\]

Hence, by noticing that

\[
x(t-\tau(t)) = x(t) - \sum_{j=-\tau(t)}^{t-1} y(j)
\]

and by imposing the worst-case occurrence on the time-varying delay $\tau(t)$, i.e. $\tau(t) = \tau_{up}$ $\forall t$, we have that

\[
\begin{align*}
\tilde{E} \tilde{x}(t+1) &= \tilde{\Phi}(\alpha(t)) \tilde{x}(t) + \tilde{G}(\alpha)u(t) - \\
& \quad - \tilde{G}_y(\alpha(t)) \sum_{j=-\tau_{up}}^{t-1} y(j)
\end{align*}
\]

where $\tilde{x}(t) = [x(t)^T, y(t)^T]^T$, $\tilde{E} = \text{diag}(I, 0)$,

\[
\tilde{\Phi}(\alpha(t)) = \begin{bmatrix} I & I \\ 2\Phi(\alpha(t)) - I & -I \end{bmatrix},
\]

\[
\tilde{G}(\alpha) = \begin{bmatrix} 0 \\ G(\alpha(t)) \end{bmatrix},
\]

\[
\tilde{G}_y(\alpha(t)) = \begin{bmatrix} 0 \\ \Phi(\alpha(t)) \end{bmatrix}
\]

In principle the computation of the set recursions (11) along the system dynamics (14) prescribes that all the auxiliary variables

\[
y(j) = x(j+1) - x(j), \quad j = t - \tau_{up}, \ldots, t - 1
\]

have to be included in the construction of the augmented system. Unfortunately this gives rise to a huge dimension state space description that could be computationally untractable. In order to mitigate such a difficulty it is possible to achieve a less demanding solution at the price of an increasing level of conservativeness. The following proposition provides inner approximations of the exact one-step controllable sets related to the model description (1).

**Theorem 1**: Let $\mathcal{T}_0 \neq \emptyset$ be a given robustly invariant ellipsoidal region complying with the input constraints and $x_{aug} = [x^T y^T z_1^T z_2^T]^T$ the augmented state describing the dynamics (14) with $z_1, z_2 \in \mathcal{R}^n$ accounting for all the cumulative sum vectors $y(j)$, $j = t - \tau_{up}, \ldots, t - 1$. Then, the ellipsoidal sets sequence

\[
\begin{align*}
\mathcal{E}_0 &= \mathcal{T}_0 \\
\mathcal{E}_i &= \text{Proj}_e\{ In[x_{aug} \in \mathcal{R}^{1n} \text{with } z_1, z_2 \in \mathcal{E}_{i-1} : \\
& \exists u \in \mathcal{U} : \forall \alpha \in \mathcal{P}_i], \text{Proj}_e\{ \tilde{\Phi}(\alpha)_{aug} \bar{x}_{aug} + \\
& \quad + \tilde{G}(\alpha)_{aug} u \} \in \mathcal{E}_{i-1} \},
\end{align*}
\]

if non-empty, satisfies $\mathcal{E}_i \subset \mathcal{T}_i$, where $In[\cdot]$ denotes the inner ellipsoidal approximation operator.

**Proof**: Assume that the sequence of one-step controllable sets $\mathcal{T}_i$ for the descriptor form (14) has been computed by resorting to the whole augmented state description $[x^T(t) y^T(t) z_1^T(t-1) \ldots y^T(t-\tau_{up})]^T$. Note that, because the auxiliary variables $y(\cdot)$ are linear combinations of $x(\cdot)$, at each recursion all the initial vectors $x(j)$, $j = t - \tau_{up}, \ldots, t - 1$ must lie in $\mathcal{T}_i$ in order to ensure that one-step evolution of the time-delay map (1) belongs to $\mathcal{T}_{i-1}$. In view of this reasoning, a way to deal with the one-step controllable sets computation is to impose that at each recursion $x(j) \in \mathcal{T}_{i-1}$, $j = t - \tau_{up}, \ldots, t - 1$. The latter is admissible thanks to the nesting property of the one-step controllable set sequence, see [7].

Now, each sample of the initial segments $x(\cdot)$ belongs to the same set $\mathcal{T}_{i-1}$, and a simple way to proceed is to consider two vectors, namely $z_1, z_2 \in \mathcal{R}^n$ during the controllable set sequence computation, which characterize the terms in the one-step differences $y(\cdot)$. Specifically $z_2$ denotes a slack variable representing the first element in the one-step difference whereas $z_1$ is a term characterizing all the time delayed states w.r.t. $z_1$ along all the possible initial segments. Therefore the computed one-step controllable set, say $\mathcal{E}_i$, is an inner approximation of the exact set $\mathcal{T}_i$ and the descriptor form (14) can be approximated as follows:

\[
\begin{align*}
\tilde{E}_{aug} \tilde{x}_{aug}(t+1) &= \tilde{\Phi}_{aug}(\alpha(t)) \tilde{x}_{aug}(t) + \\
& \quad + \tilde{G}_{aug}(\alpha(t))u(t)
\end{align*}
\]

with

\[
\begin{align*}
\tilde{x}_{aug}(t) &= [x^T(t) y^T(t) z_1^T(t) z_2^T(t)]^T, \\
\tilde{E}_{aug} &= \text{diag}\{I, 0, I, I\}
\end{align*}
\]
We have
\[
\Phi_{\text{aug}}(\alpha)(t) = \begin{bmatrix}
I & I & 0 \\
2\Phi(\alpha)(t) - I & -I & -\tau_u\Phi(\alpha)(t) \\
0 & 0 & 0 \\
\tau_u\Phi(\alpha)(t) & I & \Phi(\alpha)(t)
\end{bmatrix},
\]
and the recursions (15) result.

A procedure for the computation of the sets \(\mathcal{E}_i\) can be straightforwardly deduced by referring to [15],[2]. In fact, we have that
\[
\{\bar{x}_{\text{aug}} \text{ with } z_1, z_2 \in \mathcal{E}_{i-1} : \exists u \in \mathcal{U} : \\
\forall \alpha \in \mathcal{P}_i, \text{ Proj}_x \{\Phi_{\text{aug}}(\alpha)\bar{x}_{\text{aug}} + \bar{G}_{\text{aug}}(\alpha)u \in \mathcal{E}_{i-1}\}
= \{\bar{x}_{\text{aug}} \text{ with } z_1, z_2 \in \mathcal{E}_{i-1} : \exists u \in \mathcal{U} : \\
\forall j = 1, \ldots, l, \text{ Proj}_x \{\Phi_{\text{aug},j}\bar{x}_{\text{aug}} + \bar{G}_{\text{aug},j}u \in \mathcal{E}_{i-1}\}
\geq \{\bar{x}_{\text{aug}} \text{ with } z_1, z_2 \in \mathcal{E}_{i-1} : \exists u \in \mathcal{U} : \\
\forall j = 1, \ldots, l, \text{ Proj}_x \{\Phi_{\text{aug},j}\bar{x}_{\text{aug}} + \bar{G}_{\text{aug},j}u \} \in \text{In}[\mathcal{E}_{i-1}]\}
= \text{Proj}_x \{[\bar{x}_{\text{aug}} u] \text{ with } z_1, z_2 \in \mathcal{E}_{i-1} : u \in \mathcal{U} \}
\text{ and } \forall j = 1, \ldots, l, [\bar{x}_{\text{aug}} u] \in \mathcal{E}_{i-1}^{j}\}
\]
where the ellipsoidal set \(\mathcal{E}_{i-1}^{j}\) (in the extended space \([\bar{x}_{\text{aug}} u]\)) is defined as follows
\[
\mathcal{E}_{i-1}^{j} = \{[\bar{x}_{\text{aug}} u] \text{ with } z_1, z_2 \in \mathcal{E}_{i-1} : \\
\text{Proj}_x \{\Phi_{\text{aug},j}\bar{x}_{\text{aug}} + \bar{G}_{\text{aug},j}u \} \in \text{In}[\mathcal{E}_{i-1}]\}
\]
Therefore, each element \(\mathcal{E}_i\) of the ellipsoids sequence can be computed as follows
\[
\text{Proj}_x \{[\bar{x}_{\text{aug}} u] \text{ with } z_1, z_2 \in \mathcal{E}_{i-1} : \\
\forall \alpha \in \mathcal{P}_i, \Phi_{\text{aug}}(\alpha)\bar{x}_{\text{aug}} + \bar{G}_{\text{aug}}(\alpha)u \in \mathcal{E}_{i-1}\}
\supseteq \text{Proj}_x \left[\text{In} \left( \bigcap_{j=1,\ldots,l} \mathcal{E}_{i-1}^{j} \right) \right]
\cap \cap \left( \mathbb{R}^{2n} \times \text{Proj}_{z_1} \{\mathcal{E}_{i-1}^{1}\} \times \text{Proj}_{z_2} \{\mathcal{E}_{i-1}^{2}\} \right) \times \mathcal{E}_{i}^{U}\}
\]
where \(\mathcal{U}\) can be expressed as the intersection of ellipsoidal sets \(\mathcal{U} = \bigcup \mathcal{E}_{i}^{U}\) where the cartesian products \((\mathbb{R}^{2n} \times \text{Proj}_{z_1} \{\mathcal{E}_{i-1}^{1}\} \times \text{Proj}_{z_2} \{\mathcal{E}_{i-1}^{2}\}) \times \mathcal{E}_{i}^{U}\) are degenerate ellipsoidal sets in the extended space, see [15].

IV. ILLUSTRATIVE EXAMPLE

In order to illustrate our results, we consider the following uncertain multi-model system
\[
\Phi(\alpha) = \begin{bmatrix} 1 & 0 \\
0 & 1.01 + \alpha \end{bmatrix}, \quad G(\alpha) = \begin{bmatrix} -0.02 \\
-0.01 + \alpha \end{bmatrix},
\]
with \(|\alpha| \leq 0.008\), \(\tau_u = 15\) and input saturation constraint \(u^2(t) \leq 10, \forall t\).

The DD terminal pairs have been determined by using the Section II results and are below reported:
\[
K_1 = [-4.9854, 0.0324], \quad K_2 = [0.0132, -0.7953],
\]
\[
\mathcal{E}_{DD} = \{x \in \mathbb{R}^{n} \mid x^T Q_{DD} x \leq 1\},
\]
\[
Q_{DD} = \begin{bmatrix} 7.7159 & 0.4011 \\
0.4011 & 0.0329 \end{bmatrix}
\]
A sequence of \(N = 40\) one-step controllable ellipsoids have been determined by using the results of Theorem 1 and by applying the computable expression (22). The numerical results are depicted in Fig. 1.

Fig. 1. One-step controllable ellipsoids sequence - DD Case

In order to compare the performance degradation related to the delay presence, in figure 2 the one-step controllable ellipsoids sequence has been computed in the delay-free case by retain valid the gains \(K_1, K_2\) computed previously and by choosing \(\tau_u = 0\). It can be obviously observed that the delay free regions \(T_i\) are enlarged versions w.r.t. its delay based versions.

Fig. 2. One-step controllable ellipsoids sequence - Delay-Free Case
A better comprehension of the observed phenomenon results when regulated state trajectories in the delay and no-delay case are compared. To this end, we will consider the receding horizon strategy for constrained networked systems proposed in [10] that exploits the sequence of one-step controllable regions. For this simulation, a sequence of \( \{E_i\}_{i=0}^{T_0} \) has been considered. All the relevant results are given in the next Figs. 3-5.

In particular, in Fig. 3 the time delay history \( \tau(t) \) over a 70 time units simulation horizon is represented. By choosing the initial state equal to \( [1, -1]^T \), the control law gains \( K_1 \) and \( K_2 \) as (23) and the terminal robust invariant region as (24), in Figs. 4 and 5 the DD and free-delay regulated state trajectories are depicted respectively.

As observed, the control input appears to be more active in the delay scenario (DD) because the strategy must compensate the occurrence of state time delays. Nonetheless, the regulated performance, even if deteriorated, hold an acceptable level thanking to the control strategy which is based on the Theorem 1 ellipsoidal sequence \( \{T_i\}_{i=0} \).

![Fig. 3. Delay time history \( \tau(t) \)](image)

**V. CONCLUSIONS**

In this paper the main contribution is to provide a procedure for deriving the set of states that can be robustly steered to a give robustly positively invariant region for input-constrained polytopic time-delay systems. This has been achieved by using set-invariance arguments, ellipsoidal calculus and projections. By resorting to a Lyapunov-Krasovskii descriptor approach, it has been shown that the inner approximations of the exact one-step controllable sets can be easily described as ellipsoids.

The numerical example testifies the applicability of the proposed results.

**REFERENCES**


