Existence of solutions of generalized variational inequalities in reflexive Banach spaces

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**Abstract**

In this work, we study the following Generalized Variational Inequality Problem (for short, GVIP): Given a closed convex set $K$ in a reflexive Banach space $E$ with the dual $E^\ast$, a multifunction $T : K \rightarrow 2^{E^\ast}$, and a vector $b \in E^\ast$, find $\bar{x} \in K$ such that there exists $\bar{u} \in T(\bar{x})$ satisfying

$$\langle \bar{u} - b, y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K.$$  \hfill (1.1)

It has been studied by many authors (for example, see [2,7,8,11]).

If $T$ is a single-valued map, then GVIP($T - b, K$) is called the variational inequality problem (for short, VIP($T - b, K$)), considered and studied by Zeng and Yao [11] and references therein. Of course, if $T$ is a single-valued map and $b \equiv 0$, then inequality (1.1) reduces to the classical variational inequality. Recently, Li [5] has obtained some existence results for solutions of variational inequalities in reflexive Banach space. He used so-called generalized metric projection which permits one to draw strong conclusions on the existence of solutions of variational inequalities.

The purpose of this work is to derive existence results for solutions of GVIP($T - b, K$) in reflexive and smooth Banach spaces by using generalized projection and the well-known Fan–KKM Theorem. Our results extend the recent results of [5].

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2. Preliminaries

Throughout the work, unless otherwise specified, $E$ is a reflexive Banach space with its dual space $E^*$. A Banach space $(E, \| \cdot \|)$ is said to be smooth if
\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = 0
\] (2.1)
exists for all $x, y$ belonging to its unit sphere $U = \{ x \in B : \|x\| = 1 \}$. In this case, the norm $\| \cdot \|$ of $B$ is said to be Fréchet differentiable (and $B$ is said to be uniformly smooth) if the limit in (2.1) is attained uniformly for all $x, y \in U$. The Banach space $B$ is called uniformly convex if for every $\epsilon \in (0, 2)$, there exists $\delta(\epsilon) > 0$ such that $\|x - y\| = 1$ and $\|x - y\| \geq 2 \implies \|\frac{x + y}{2}\| \leq 1 - \delta(\epsilon)$. $B$ is called strictly convex if $U$ does not contain line segments, that is, $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|\lambda x + (1 - \lambda)y\| < 1$ for all $\lambda \in (0, 1)$. We recall that if $B$ is uniformly convex, then $B$ is reflexive and strictly convex; see, for example, Proposition 12.1 (a), (b) in [4]. Note that $B$ is uniformly convex if and only if $B^*$ is uniformly smooth. It is well known that $E$ is smooth if and only if $E^*$ is strictly convex.

From the above definitions, it easy to see that the following implications are valid:

- $E$ is uniformly convex $\implies$ $E$ is locally uniformly convex $\implies$ $E$ is strictly convex.

The mapping $J : E \to E^*$ defined by
\[
J(x) = \{ x^* \in E^* : \langle x^*, x \rangle = \|x^*\|^2 = \|x\|^2 \}, \quad \text{for all } x \in E,
\]
is called the duality mapping of $E$. It is known that $J(x) = \partial \phi(x)$, where $\partial \phi(x)$ denotes the subdifferential of $\phi$ at $x$.

We list the following properties of duality mapping $J$ which are useful for the rest of this work.

**Proposition 2.1** ([9, Proposition 32.22]). Let $E$ be a reflexive Banach space and $E^*$ be strictly convex.

(a) The duality mapping $J : E \to E^*$ is single-valued, surjective, bounded.
(b) If $E$ and $E^*$ are locally uniformly convex then $J$ is a homeomorphism, that is, $J$ and $J^{-1}$ are continuous single-valued mappings.

We consider the functional $V : E^* \times E \to \mathbb{R}$ defined as
\[
V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2,
\]
for all $\varphi \in E^*$ and $x \in E$.

It is easy to see that $V(\varphi, x) \geq (\|\varphi\| - \|x\|)^2$. Thus the functional $V : B^* \times B \to \mathbb{R}$ is nonnegative.

It is clear that $V(\varphi, x)$ is continuous and the maps $x \mapsto V(\varphi, x)$ and $\varphi \mapsto V(\varphi, x)$ are convex. We recall that $(\|\varphi\| - \|x\|)^2 \leq V(\varphi, x) \leq (\|\varphi\| + \|x\|)^2$.

We remark that the main Lyapunov functional $V$ was first introduced by Alber [1] and its properties were studied there. By using this functional $V$, Alber defined a generalized projection operator on uniformly convex and uniformly smooth Banach spaces which is further extended by Li [6] on reflexive Banach spaces.

**Definition 2.1** (See, Definition 6.2 in [1] and Definition 1.1 in [6]). Let $E$ be a reflexive Banach space with its dual $E^*$ and $K$ be a nonempty, closed and convex subset of $E$. The operator $\pi_K : E^* \to 2^E$ defined by
\[
\pi_K(\varphi) = \{ x \in K : V(\varphi, x) = \inf_{y \in K} V(\varphi, y) \},
\]
is said to be a generalized projection operator. For each $\varphi \in E^*$, the set $\pi_K(\varphi)$ is called the generalized metric projection of $\varphi$ on $K$.

We mention the following useful properties of the operator $\pi_K$.

**Proposition 2.2** ([6]). Let $E$ be a reflexive Banach space with its dual space $E^*$ and $K$ be a nonempty, closed and convex subset of $E$, then the following properties hold:

(a) The operator $\pi_K : E^* \to 2^E$ is single-valued if and only if $E$ is strictly convex.
(b) If $E$ is smooth, then for any given $\varphi \in E^*$, $x \in \pi_K(\varphi)$ if and only if $\langle \varphi - J(x), x - y \rangle \geq 0$, for all $y \in K$.
(c) If $E$ is strictly convex, then the generalized projection operator $\pi_K : E^* \to K$ is continuous.

**Theorem 2.1** ([3, Theorem 2.11]). In every reflexive Banach space, an equivalent norm can be introduced so that $E$ and $E^*$ are locally uniformly convex and thus also strictly convex with respect to the new norm on $E$ and $E^*$.

In view of Theorem 2.1 we can assume for the rest of this work that the norm $\| \cdot \|$ of the reflexive Banach space $E$ is such that $E$ and $E^*$ are locally uniformly convex. In this case the generalized metric projection operator $\pi_K$ is single-valued and continuous. Also the duality mapping $J$ is single-valued and continuous.
We will use the following celebrated Fan–KKM Theorem to establish the main existence results in the next section.

**Definition 2.2.** Let \( K \) be a nonempty subset of a linear space \( X \). A multivalued map \( G : K \to 2^X \) is said to be a KKM map if for any finite subset \( \{y_1, y_2, \ldots, y_n\} \) of \( K \), we have

\[
\text{co}\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{i=1}^{n} G(y_i),
\]

where \( \text{co}\{y_1, y_2, \ldots, y_n\} \) denotes the convex hull of \( \{y_1, y_2, \ldots, y_n\} \).

**Theorem 2.2** (Fan–KKM Theorem). Let \( K \) be a nonempty convex subset of a Hausdorff topological vector space \( X \) and let \( G : K \to 2^X \) be a KKM map with closed values. If there exists a point \( y_0 \in K \) such that \( G(y_0) \) is a compact subset of \( K \), then \( \bigcap_{y \in K} G(y) \neq \emptyset \).

In the sequel we will need the following Sion Minimax Theorem (see, [10, Theorem 9.D]).

**Theorem 2.3.** Let \( A \) and \( B \) be convex subsets of some real topological vector spaces with \( B \) compact, and let \( p : A \times B \to \mathbb{R} \). If \( p(. , b) \) is lower semicontinuous and quasiconvex on \( A \) for all \( b \in B \), and if \( p(a, .) \) is upper semicontinuous and quasiconcave on \( B \) for all \( a \in A \), then

\[
\inf_{a \in A} \max_{b \in B} p(a, b) = \max_{b \in B} \inf_{a \in A} p(a, b).
\]

3. Existence results

We establish a relationship between GVIP\((T - b, K)\) and the generalised metric projection.

**Proposition 3.1.** Let \( E \) be a reflexive and smooth Banach space, \( K \) a closed convex set in \( E \) and \( b \in E^* \). Assume that \( \alpha > 0 \) and \( T : K \to 2^E \) is a multivalued map with nonempty values. Then \( \bar{x} \in K \) is a solution of GVI \((T - b, K)\) if and only if there exists \( \bar{u} \in T(\bar{x}) \) such that \( \bar{x} = \pi_{\bar{x}}[J(\bar{x}) - \alpha(\bar{u} - b)] \).

**Proof.** Let \( \bar{x} \) be a solution of GVIP\((T - b, K)\). Then there exists \( \bar{u} \in T(\bar{x}) \) such that for all \( y \in K \),

\[
\langle \bar{u} - f, y - \bar{x} \rangle \geq 0,
\]

\[
\Leftrightarrow \langle \alpha(\bar{u} - f), y - \bar{x} \rangle \geq 0,
\]

\[
\Leftrightarrow \langle -\alpha(\bar{u} - f), \bar{x} - y \rangle \geq 0,
\]

\[
\Leftrightarrow \langle j(\bar{x}) - \alpha(\bar{u} - f) - j(\bar{x}), \bar{x} - y \rangle \geq 0,
\]

\[
\Leftrightarrow \bar{x} = \pi_{\bar{x}}[J(\bar{x}) - \alpha(\bar{u} - f)] \quad \text{(by virtue of Proposition 2.2)}.
\]

This completes the proof. \( \Box \)

We remark that when \( T \) is single-valued, **Proposition 3.1** reduces to Theorem 8.1 of [1]. The following theorem is one of the main results of this work which is motivated by Theorem 2.1 in [5].

**Theorem 3.1.** Let \( E \) be a reflexive and smooth Banach space, \( K \) be a closed convex set in \( E \) and \( b \in E^* \). Let us have \( \alpha > 0 \) and \( T : K \to 2^E \) be a multivalued map such that for each \( x \in K \), \( T(x) \) is weakly * compact and convex. Assume that for each \( y \in K \), the set

\[
G(y) := \{x \in K : \inf_{u \in T(x)} \langle J(x) - \alpha(u - b), 2(y - x) \rangle + \|x\|^2 \leq \|y\|^2 \}
\]

is closed and there exists \( y_0 \in K \) such that \( G(y_0) \) is compact. Then GVIP\((T - b, K)\) has a solution.

**Proof.** First we show that the multivalued \( G(\cdot) \) is a KKM mapping.

(a) For each \( y \in K \), we have \( y \in G(y) \). Hence \( G(y) \) is a nonempty set in \( K \).

(a) For any finite set \( \{y_1, y_2, \ldots, y_n\} \subseteq K \), we claim that

\[
\text{co}\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{j=1}^{n} G(y_j),
\]

Take any \( y \in \text{co}\{y_1, y_2, \ldots, y_n\} \). Then \( y = \sum_{j=1}^{n} \lambda_j y_j \), where \( \lambda_j \in [0, 1] \) and \( \sum_{j=1}^{n} \lambda_j = 1 \). We have the following estimation:

\[
\sum_{j=1}^{n} \inf_{u \in T(x)} \langle J(y) - \alpha(u - b), 2\lambda_j(y_j - y) \rangle \leq \inf_{u \in T(y)} \langle J(y) - \alpha(u - b), 2 \sum_{j=1}^{n} \lambda_j(y_j - y) \rangle = 0.
\]
Hence
\[\sum_{j=1}^{n} \inf_{y \in T(x)} \left( (j(y) - \alpha(u - b), 2\lambda_j(y - y)) + \lambda_j y^2 \right) \leq \|y\|^2 \leq \sum_{j=1}^{n} \lambda_j y^2.\]

This implies that
\[\sum_{j=1}^{n} \inf_{y \in T(x)} \left( (j(y) - \alpha(u - b), 2\lambda_j(y - y)) + \lambda_j y^2 - \lambda_j y^2 \right) \leq 0.

Therefore there exists \(j > 0\) such that
\[\inf_{y \in T(x)} \left( (j(y) - \alpha(u - b), 2\lambda_j(y - y)) + \lambda_j y^2 - \lambda_j y^2 \right) \leq 0.

Consequently,
\[\inf_{y \in T(x)} \left( (j(y) - \alpha(u - b), 2(y - y)) + \|y\|^2 \leq \|y\|^2.\]

It follows that \(y \in G(y) \subseteq \bigcup_{n=1}^{\infty} G(y)\) and so \(\co\{y_1, y_2, \ldots, y_n\} \subseteq \bigcup_{n=1}^{\infty} G(y)\).

\((a_4)\) The set \(G(y)\) is closed and there exists \(y_0 \in K\) such that \(G(y_0)\) is compact. Thus all conditions of Theorem 2.2 are fulfilled.

By Theorem 2.2, there exists \(\bar{x} \in K\) such that \(\bar{x} \in G(y)\) for all \(y \in K\). By the definition of \(G(y)\) we have
\[\inf_{y \in T(x)} \left( (j(\bar{x}) - \alpha(u - b), 2(y - \bar{x})) + \|\bar{x}\|^2 \right) \leq \|y\|^2 \quad \text{for all } y \in K.\]  

\((a_4)\) By applying minimax Theorem 2.3, we obtain
\[\sup_{y \in K} \inf_{y \in T(x)} \left( (j(\bar{x}) - \alpha(u - b), 2(y - \bar{x})) + \|\bar{x}\|^2 - \|y\|^2 \right) \leq 0.\]  

Put \(p(u, y) = (j(\bar{x}) - \alpha(u - b), 2(y - \bar{x})) + \|\bar{x}\|^2 - \|y\|^2\). Then the functional \(p(\cdot, y)\) is lower semicontinuous and convex. Also the functional \(p(u, \cdot)\) is upper semicontinuous and concave. Therefore, (3.2) becomes
\[\inf_{y \in T(x)} \sup_{y \in K} \inf_{y \in T(x)} p(u, y) = \inf_{y \in T(x)} \inf_{y \in K} \sup_{y \in T(x)} p(u, y) = \inf_{y \in T(x)} p(u, y) \leq 0.\]

Since the functional \(u \mapsto \sup_{y \in K} p(u, y)\) is lower semicontinuous and \(T(\bar{x})\) is weak * compact, there exists \(\bar{u} \in T(\bar{x})\) such that
\[\sup_{y \in K} p(\bar{u}, y) = \inf_{y \in T(x)} p(u, y) \leq 0.\]

Hence we have
\[\inf_{y \in T(x)} p(u, y) \leq 0.\]  

By expanding (3.4), we obtain
\[V(j(\bar{x}) - \alpha(u - b), x) \leq V(j(\bar{x}) - \alpha(u - b), y), \quad \text{for all } y \in K.\]

This implies that \(\bar{x} = \pi_K[j(\bar{x}) - \alpha(u - b)]\). By Proposition 3.1, \(\bar{x}\) is a solution of GVIP\((T - b, K)\). The proof is complete. \(\square\)

It is noticed that in Theorem 3.1, we did not assume the continuity of \(T\). For the case of upper semicontinuous mapping, we have the following result.

**Theorem 3.2.** Let \(E\) be a reflexive and smooth Banach space, \(K\) be a closed convex set in \(E\) and \(b \in E^*\). Let \(\alpha > 0\) and \(T : K \to 2^{E^*}\) be an upper semicontinuous multivalued map such that the set \(T(x)\) is weak * compact and convex for each \(x \in K\). Assume that there exists \(y_0 \in K\) such that the set
\[G(y_0) = \{x \in K : \inf_{y \in T(x)} ((j(x) - \alpha(u - b), 2(y - x)) + \|x\|^2) \leq \|y_0\|^2\}

is compact. Then GVIP\((T - b, K)\) has a solution.

**Proof.** In view of the proof of Theorem 3.1, it is sufficient show that for each \(y \in K\), the set
\[G(y) = \{x \in K : \inf_{y \in T(x)} ((j(x) - \alpha(u - b), 2(y - x)) + \|x\|^2) \leq \|y\|^2\}

is closed.

Let \(\{x_n\}\) be a sequence in \(G(y)\) such that \(x_n \to x_0\) in the norm topology. We define a set \(S = \{x_1, x_2, \ldots, x_n, \ldots\} \cup \{x_0\}\) of points of the sequence \(\{x_n\}\) and its limit point \(x_0\). Then there exists \(u_n \in T(x_n)\) such that
\[\inf_{y \in T(x_n)} ((j(x_n) - \alpha(u - b), 2(y - x_n)) + \|x_n\|^2) \leq \|y\|^2.\]
Since $T(S)$ is compact, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightarrow u_0$, where $u_0 \in T(S)$. By the upper semicontinuity of $T$, we have $u_0 \in T(x_0)$. Thus, without loss of generality, we may assume that $u_n \rightarrow u_0$ and observe that
\[
(J(x_n) - \alpha(u_n - b), 2(y - x_n)) + \|x_n\|^2 \text{ converges to } (J(x_0) - \alpha(u_0 - b), 2(y - x_0)) + \|x_0\|^2.
\]
Therefore,
\[
\inf_{u \in T(x_0)} ((J(x_0) - \alpha(u - b), 2(y - x_0)) + \|x_0\|^2) \leq (J(x_0) - \alpha(u_0 - b), 2(y - x_0)) + \|x_0\|^2 \leq \|y\|^2.
\]
Thus, $x_0 \in G(y)$ and so $G(y)$ is closed. This completes the proof. □

If $T$ is a single-valued map, then from Theorem 3.2, we derive the following result.

**Corollary 3.1** ([5, Theorem 2.1]). Let $E$ be a reflexive and smooth Banach space, $K$ be a closed convex set in $E$ and $b \in E^*$. Let $\alpha > 0$ and $T : K \rightarrow E^*$ be a single-valued and continuous mapping. Assume that there exists $y_0 \in K$ such that the set
\[
\{x \in K : (J(x) - \alpha(T(x) - b), 2(y_0 - x)) + \|x\|^2 \leq \|y_0\|^2\}
\]
is compact. Then $GVI(T - b, K)$ has a solution.

**References**