Robust synchronization of Lur’è networks with incremental nonlinearities

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Abstract—This paper deals with robust synchronization problems for networks of diffusively interconnected identical Lur’è systems subject to incrementally passive nonlinearities and incrementally sector bounded nonlinearities, respectively. Whereas in stabilization of one single Lur’è system the conditions of passivity and sector boundedness for the nonlinear feedback loop are commonly assumed, in our context of networked Lur’è systems we adopt the stronger assumptions of incremental passivity and incremental sector boundedness. Throughout this paper the interconnection topologies among these Lur’è systems are assumed to be undirected and connected. The conventional a priori assumption on the existence of a synchronization manifold is not employed in this paper. Both for the case of incrementally passive as well as incrementally sector bounded nonlinearities we obtain necessary and sufficient conditions for the existence of the sufficient distributed robustly synchronizing protocols. The static feedback gains are computed by solving LMI’s in terms of the matrices defining the individual agent dynamics.

I. INTRODUCTION

Synchronization of complex dynamical networks has attracted a lot of attention in the systems and control literature over the last decade, e.g. [3], [5], [6], [13], [21] to name a few. This is due to the fact that collective behaviors of multiple interconnected dynamical systems are widespread in nature, human society and technology. The essence of synchronization is the collective objective of the agents in a network to reach agreement about certain variables of interest. A special case of this is the consensus problem in which the agreement value is required to be constant.

Consensus and synchronization problems for linear multi-agent networks have been extensively studied, see [9], [16], [17] and the references therein. Recent research interests have also been aimed at nonlinear multi-agent networks, possibly with model uncertainties, time delays and data dropouts etc. [1], [5], [8], [10], [11], [20]. The research in these directions is motivated by practical applications such as assembling networks of mobile robots (e.g. smart sensors, unmanned aerial vehicles and satellites) to work together.

In this paper, we consider nonlinear multi-agent networks in which the individual agent dynamics is described by the Lur’è system, i.e. a nonlinear system consisting of a nominal linear system with an uncertain nonlinear negative feedback loop around it [7]. Such dynamics model is used in many control system applications, e.g. Chua’s circuits, chaotic systems and aircrafts [19]. The nonlinear loop can represent many kinds of nonlinearities such as saturation and dead zone. Often, we can reconfigure a linear system with nonlinearities at the input and the output as a Lur’è system [14]. In the present paper we assume the uncertainty nonlinearities to be incrementally passive or incrementally sector bounded. In contrast with [5], the uncertainties we consider here are not additive perturbations and the networks are nonlinear.

Conditions for global asymptotic stability of a single Lur’è system are of course well known, see e.g. [7], [12], [19]. In [22] pinning synchronization of a Lur’è network is converted into global asymptotic stability of a set of Lur’è systems, and frequency-domain and time-domain synchronization criteria are derived in virtue of the main result in [12]. We note that the time-domain criterion is a corollary of the frequency-domain one and can not be obtained directly in [22]. In contrast to [22], our setting does not start with any assumption on the existence of a global synchronization manifold (or isolated agent, virtual leader, exosystem), a condition which is frequently employed in nonlinear multi-agent coordination. The main contributions of this paper are necessary and sufficient conditions for the existence of the sufficient robustly synchronizing static distributed protocols, both for incrementally passive and for incrementally sector bounded nonlinearities.

The rest of this paper is organized as follows. Section II introduces some preliminaries and formulates the robust synchronization problems we are interested in. The main results for the incrementally passive nonlinearity case and for the incrementally sector bounded nonlinearity case are presented in Sections III and IV, respectively. The paper closes with some concluding remarks in Section V.

II. PRELIMINARIES AND PROBLEM STATEMENT

Let \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of real and complex numbers, respectively. \( \mathbb{R}^+:=[0, \infty) \), \( \mathbb{R}^{m_1 \times m_2} \ (\mathbb{C}^{m_1 \times m_2}) \) denotes the space of \( m_1 \) by \( m_2 \) real (complex) matrices. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscript \((\cdot)^T\) denotes the transpose of a real matrix, and the superscript \((\cdot)^*\) denotes the conjugate transpose of a complex matrix. \( \text{diag}(M_1, M_2, \cdots, M_d) \) represents a block-diagonal matrix with matrices \( M_b, b=1, 2, \cdots, d \), on its diagonal. \( M_1 \otimes M_2 \) denotes the Kronecker product of the matrices \( M_1 \) and \( M_2 \). An important property of the Kronecker product is that \( (M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1M_3)\otimes(M_2M_4) \). \( \mathbf{0} \) denotes the zero matrix of compatible
dimension. \( 1_N \) denotes the column vector of dimension \( N \) with all its components equal to one. \( I \) denotes the identity matrix of compatible dimension.

The interconnection topology of a network of coupled dynamical systems is represented by an undirected graph \( \mathcal{G} \) that consists of a nonempty and finite node set \( V = \{1, 2, \ldots, N\} \) and an edge set \( \mathcal{E} \subset V \times V \) with the property that \((i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}\) for all \( i, j = 1, 2, \ldots, N \) and \( j \neq i \). Assume that the graph \( \mathcal{G} \) is simple, i.e. it does not contain any self-loop \((i, i)\) and there is at most one undirected edge between any two different nodes. An undirected path connecting nodes \( i_0 \) and \( i_l \) is a sequence of undirected edges of the form \((i_{p−1}, i_p)\), \( p = 1, \ldots, l \). The graph \( \mathcal{G} \) is connected if there is an undirected path between any pair of distinct nodes. The adjacency matrix \( A \) associated with the graph \( \mathcal{G} \) is defined as \( [A]_{ij} = a_{ij} \) if \((j, i) \in \mathcal{E}\) and \([A]_{ij} = 0\) otherwise, where \( a_{ij} > 0 \) is the weight of \((j, i)\). The degree of node \( i \) is given by \( d_i = \sum_{j=1}^{N} a_{ji} \). \( D := \text{diag}(d_1, d_2, \ldots, d_N) \) is the degree matrix of the graph \( \mathcal{G} \). The Laplacian matrix \( L \) of the graph \( \mathcal{G} \) is defined by \( L := D - A \). It is well known that \( L1_N = 0 \).

Let \( \mathcal{G} \) be an undirected graph with \( N \) nodes, where \( N \geq 2 \). The graph \( \mathcal{G} \) is connected if and only if the Laplacian eigenvalue \( 0 \) has multiplicity one [16]. In this case, these Laplacian eigenvalues can be ordered as

\[
\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N.
\]

Furthermore, there exists an orthogonal matrix

\[
U = \begin{bmatrix} 1_N & U_2 \end{bmatrix},
\]

where \( U_2 \in \mathbb{R}^{N \times (N−1)} \), such that

\[
U^T L U = \begin{bmatrix} 0 & \Lambda \end{bmatrix},
\]

where \( \Lambda = \text{diag}(\lambda_2, \ldots, \lambda_N) \).

All the notation in this section will be used in the following sections without redefinition or indication due to space limitations.

The following lemma plays a crucial role in our main results:

**Lemma 1:** For any two vectors \( a = [a_1, a_2, \ldots, a_N]^T \in \mathbb{R}^N \) and \( b = [b_1, b_2, \ldots, b_N]^T \in \mathbb{R}^N \) with \( N \geq 2 \), we have

\[
a^T U_2 L U_2^T b = \frac{1}{N} \sum_{1 \leq i < j \leq N} (a_i - a_j)(b_i - b_j).
\]

**Proof:** The result follows straightforwardly and can be omitted here.

Below we give the definition of minimal left annihilator of a given matrix:

**Definition 1:** [15] For a matrix \( B \in \mathbb{C}^{n \times m} \) with rank \( r \), let \( B^\perp \in \mathbb{C}^{(n−r) \times m} \) be any matrix of full row rank such that \( B^\perp B = 0 \). Note that such matrix \( B^\perp \) exists if and only if \( B \) has linearly dependent rows. The set of all such matrices is captured by \( B^\perp = TU_2^T \), where \( T \) is an arbitrary nonsingular matrix and \( U_2 \) is obtained from the singular value decomposition

\[
B = [U_1 \ U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & \tilde{V}_2 \end{bmatrix}.
\]

Also note that, for a given \( B, B^\perp \) is not unique. Throughout this paper, \( B^\perp \) denotes any choice from this set of matrices.

In this paper, we consider a network of \( N(\geq 2) \) identical dynamical agents described by the following Lur’e systems (see Fig. 1)

\[
\begin{aligned}
\dot{x}_i &= A x_i + B u_i + E z_i \\
y_i &= C x_i, \quad i = 1, 2, \ldots, N, \\
\phi_i(t) &= -\phi(y_i(t), t)
\end{aligned}
\]

where \( x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R} \) are the state to be synchronized, diffusive coupling input and output of the \( i \)th agent, respectively. \( \phi_i(t) \) is the nonlinear negative feedback loop. \( \phi_i(t) : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R} \) is an uncertain memoryless function. \( A, B, C \) and \( E \) are system matrices of compatible dimensions. Assume that the pair \((A, B)\) is stabilizable and \( B^\perp \) exists, without loss of generality. The interconnection topology among these agents is represented by the connected undirected graph \( \mathcal{G} \).

The agents in the network are interconnected by means of the following static distributed protocol

\[
u_i = F \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i = 1, 2, \ldots, N,
\]

where \( F \) is a common feedback gain matrix to be determined later.

**Definition 2:** The network of agents (1) with the protocol (2) is robustly synchronized if \( x_i(t) - x_j(t) \rightarrow 0 \) as \( t \rightarrow \infty \), \( \forall i, j = 1, 2, \ldots, N \), for all initial conditions and all uncertainties \( \phi_i(t) \).

By interconnecting (1) and (2) we get the Lur’e dynamical network

\[
\begin{aligned}
\dot{x} &= (I_N \otimes A + L \otimes B F)x - (I_N \otimes E)\Phi(y, t) \\
y &= (I_N \otimes C)x
\end{aligned}
\]

where \( x = [x_1^T, x_2^T, \ldots, x_N^T]^T, y = [y_1, y_2, \ldots, y_N]^T \) and \( \Phi(y, t) = [\phi(y_1, t)^T, \phi(y_2, t)^T, \ldots, \phi(y_N, t)^T]^T \).

In the next two sections, we will discuss robust synchronization of the Lur’e network (3) where \( \phi_i(t) \) is assumed to
be incrementally passive and incrementally sector bounded, respectively.

III. INCREMENTALLY PASSIVE NONLINEARITIES

In this section, we assume that the uncertain functions \( \phi(\cdot, t) \) are incrementally passive. Incremental passivity for static systems of the form

\[
    z = \phi(y, t)
\]

with input \( y \in \mathbb{R} \) and output \( z \in \mathbb{R} \) is defined as follows.

Definition 3: [2] The system (4) is called incrementally passive if the function \( \phi(\cdot, t) \) satisfies

\[
    (y_1 - y_2)(\phi(y_1, t) - \phi(y_2, t)) \geq 0
\]

for all \( y_1, y_2 \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \).

Incremental passivity is stronger than the property of passivity which is defined as \( y\phi(y, t) \geq 0 \) for all \( y \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \).

Theorem 1: If there exist matrices \( P > 0 \) and \( F \) such that

\[
    (A + \lambda_i BF)^T P + P(A + \lambda_i BF) < 0
\]

for all \( i = 2, \ldots, N \), and

\[
    PE = CT,
\]

then the network of Lur’e systems (1) with the protocol (2) is robustly synchronized, i.e. the Lur’e network (3) is synchronized for all incrementally passive \( \phi(\cdot, t) \).

Proof: Let \( \mathcal{U} \) be an orthogonal matrix such that \( \mathcal{U}^T \mathcal{L} \mathcal{U} = \Lambda \) as mentioned in Section II. Let

\[
    \tilde{x} = (\mathcal{U}_1^T \otimes I_n) x,
\]

\[
    \tilde{x} = (\mathcal{U}_2^T \otimes I_n) x,
\]

and partition \( \tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \ldots, \tilde{x}_N^T]^T, \tilde{x} = [\tilde{x}_1^T, \ldots, \tilde{x}_N^T]^T \). Then we get

\[
    \dot{\tilde{x}} = (I_{N-1} \otimes A + \Lambda \otimes BF)\tilde{x} - (U_2^T \otimes E) \Phi(y, t).
\]

By Lemma 1, we have

\[
    \frac{1}{N} \sum_{1 \leq i < j \leq N} (y_i - y_j)(\phi(y_i, t) - \phi(y_j, t)).
\]

We also know that \( x_i(t) - x_j(t) \to 0 \) as \( t \to +\infty \), \( \forall i, j = 1, 2, \ldots, N \), if and only if \( \tilde{x}(t) \to 0 \) as \( t \to +\infty \), see Lemma 3.2 in [5].

Choose a quadratic Lyapunov function candidate

\[
    V(\tilde{x}) = \frac{1}{2} \tilde{x}^T (I_{N-1} \otimes P) \tilde{x},
\]

where \( P > 0 \). Then \( V(\tilde{x}) \) is positive definite and radially unbounded. The time derivative of \( V(\tilde{x}) \) along the trajectories of the system (8) is given by

\[
    \dot{V}(\tilde{x}) = \tilde{x}^T (I_{N-1} \otimes P) \dot{\tilde{x}} = \tilde{x}^T (I_{N-1} \otimes P) \left[ (A + \Lambda \otimes BF)\tilde{x} - (U_2^T \otimes E) \Phi(y, t) \right] - \frac{1}{N} \sum_{1 \leq i < j \leq N} (y_i - y_j)(\phi(y_i, t) - \phi(y_j, t))
\]

which is negative definite. Thus the system (8) is globally asymptotically stable, i.e. the Lur’e network (3) is robustly synchronized. This completes the proof.

Remark 1: Here the robust synchronization is also global and the Lur’e network (3) is self-synchronized without any pinning control input from a virtual leader as in [22]. In general, if synchronization is achieved, the possible synchronization manifold must be the trajectory of any of the agents [18]. However, the true synchronization manifold could differ from the trajectory of a virtual leader. In practice, for a given dynamical network, we can not introduce an extra leader agent and thus we are interested in the self-synchronizability of the network.

In view of the Lefschetz-Kalman-Yakubovich lemma [4], the existence of \( P > 0 \) and \( F \) for (6) and (7) is equivalent to the existence of a common state feedback control law \( u_i = Fx_i \) for the following \( N - 1 \) systems

\[
    \left\{ \begin{array}{l}
    \dot{x}_i = Ax_i + \lambda_i Bu_i + Ez_i \\
    y_i = Cx_i
    \end{array} \right., \quad i = 2, \ldots, N
\]

that renders all the resulting closed-loop systems strictly positive real from \( z_i \) to \( y_i \), with the common storage function \( \frac{1}{2} \tilde{x}_i^T P \tilde{x}_i \).

We will now first study the conditions under which a single system \( \dot{x} = Ax + Bu + Ez, \quad y = Cx \) can be rendered strictly positive real from \( z \) to \( y \) by \( u \) = \( Fx \).

Theorem 2: There exist matrices \( P > 0 \) and \( F \) such that

\[
    (A + BF)^T P + P(A + BF) < 0,
\]

if and only if there exists a matrix \( Q > 0 \) such that

\[
    B^T (QA^T + AQ) (B^T)^T < 0,
\]

\[
    E = QC^T.
\]
In this case, a suitable $P$ is given by $P = Q^{-1}$, and a suitable $F$ is given by
\[
F = -\mu B^T Q^{-1},
\]
where $\mu$ is any positive real number satisfying
\[
Q A^T + A Q - 2 \mu B B^T < 0.
\]

Proof: For the ‘only if’ part, let $Q = P^{-1}$, Then $E = QC^T$ and
\[
Q A^T + A Q + Q F B^T + B F Q < 0,
\]
which yields
\[
B^T (Q A^T + A Q) (B^T)^T < 0.
\]

For the ‘if’ part, by Finsler’s lemma [15], there exists $\mu > 0$ such that
\[
Q A^T + A Q - 2 \mu B B^T < 0.
\]

Let $P = Q^{-1}$ and $F := -\mu B^T Q^{-1}$. Then we get $PE = C^T$ and
\[
(A + \lambda_i B F)^T P + P(A + \lambda_i B F) =
(A - k \lambda_i B B^T Q^{-1})^T Q^{-1} + Q^{-1} (A - k \lambda_i B B^T Q^{-1}) =
A^T Q^{-1} + Q^{-1} A - 2 k \lambda_i B B^T Q^{-1} < 0.
\]

This completes the proof.

Remark 2: If the conditions of Theorem 2 above, then the state feedback control law $u = F x$ renders the system $\dot{x} = A x + B y + E z$, $y = C x$ strictly positive real from $z$ to $y$, equivalently the closed-loop system is robustly stabilized against passive uncertain feedback nonlinearities $z = -\phi(y, t)$ [7].

Next we will focus on the conditions for the existence of the protocol (2) that robustly synchronizes the Lur’e network (3).

Theorem 3: There exist matrices $P > 0$ and $F$ such that (6) and (7) hold for all $i = 2, \cdots, N$ if and only if there exists a matrix $Q > 0$ such that
\[
B^T (Q A^T + A Q) (B^T)^T < 0,
\]
\[
E = QC^T.
\]

In this case, a suitable $P$ is given by $P = Q^{-1}$, and a suitable $F$ is given by
\[
F = -k B^T Q^{-1},
\]
where the positive real number $k$ satisfies
\[
Q A^T + A Q - 2 k \lambda_2 B B^T < 0.
\]
The proof follows the idea in Theorem 2. Proof: The ‘only if’ part is obvious.

For the ‘if’ part, by Finsler’s lemma, there exists $k > 0$ such that
\[
Q A^T + A Q - 2 k \lambda_2 B B^T < 0.
\]

Let $P = Q^{-1}$ and $F := -k B^T Q^{-1}$. Then we get $PE = C^T$ and
\[
(A + \lambda_i B F)^T P + P(A + \lambda_i B F) =
(A - k \lambda_i B B^T Q^{-1})^T Q^{-1} + Q^{-1} (A - k \lambda_i B B^T Q^{-1}) =
A^T Q^{-1} + Q^{-1} A - 2 k \lambda_i B B^T Q^{-1} < 0.
\]

This completes the proof.

Remark 3: To the best of our knowledge, the necessary and sufficient conditions obtained above for the existence of the sufficient static distributed protocols that achieve robust synchronization are new. Note that our conditions are equivalent to the existence of $P > 0$ and $F$ for a single agent. From a computational point of view, it is advantageous. It reduces the computation of a synchronization protocol for a possible large network ($N$ big) to the computation of a state feedback controller that renders a single agent strictly positive real.

IV. INCREMENTALLY SECTOR BOUNDED NONLINEARITIES

In this section, we consider feedback nonlinearities $\phi(\cdot, t)$ given by incrementally sector bounded functions in the sector $[\alpha, \beta]$ with $0 \leq \alpha < \beta < +\infty$. This condition is expressed as
\[
[z_1 - z_2 - \alpha(y_1 - y_2)] [z_1 - z_2 - \beta(y_1 - y_2)] \leq 0
\]
for all $y_1, y_2 \in \mathbb{R}$ and $t \in \mathbb{R}^+$, where $z_1 = \phi(y_1, t)$ and $z_2 = \phi(y_2, t)$. Any function $\phi(\cdot, t)$ satisfying the incremental sector boundedness condition (9) also satisfies the ordinary sector boundedness condition, i.e., $\phi(y, t) - \alpha y \leq 0$ for all $y \in \mathbb{R}$ and $t \in \mathbb{R}^+$. We note that the slope-restricted nonlinearity condition, $\alpha \leq (z_1 - z_2)/(y_1 - y_2) \leq \beta$, $y_1 \neq y_2$, is equivalent to the incremental sector boundedness condition. In contrast to [22], the incremental sector boundedness condition (9) allows us to explore robust network synchronization via linear matrix inequalities immediately.

Theorem 4: If there exist matrices $P > 0$, $F$ and a positive real number $\tau$ such that
\[
\begin{bmatrix}
(A + \lambda_i B F)^T P + P(A + \lambda_i B F) & -PE \\
-\tau \alpha \beta C^T C & -\tau
\end{bmatrix} < 0
\]
for all $i = 2, \cdots, N$, then the network of Lur’e systems (1) with the protocol (2) is robustly synchronized for all $\phi(\cdot, t)$ satisfying the incremental sector boundedness condition (9).

Proof: As in the proof of Theorem 1, choose the same quadratic Lyapunov function candidate
\[
V(\bar{x}) = \frac{1}{2} \bar{x}^T (I_{N-1} \otimes P) \bar{x},
\]
where $P > 0$. We have the following
\[
\dot{V}(\bar{x}) = \bar{x}^T \left[ (I_{N-1} \otimes PA + \Lambda \otimes PBF)\bar{x} - (I_{N-1} \otimes PE)U_2^T \Phi(y,t) \right] = \\
\frac{1}{2} \left[ \bar{x}^T \left[ \begin{array}{cc}
I_{N-1} \otimes (A^T P + PA) & + \Lambda \otimes (F^T B^T P + PBF) \\
- (I_{N-1} \otimes PE) & 0
\end{array} \right] \right] \bar{x} - \frac{1}{2} \Phi^2(y,t)\leq 0,
\]
i.e.
\[
\begin{bmatrix}
A - \frac{\alpha + \beta}{2} EC + \lambda_i BF \\
P \left( A - \frac{\alpha + \beta}{2} EC + \lambda_i BF \right) + \frac{\tau(\alpha - \beta)^2}{4} C^T C + \frac{1}{\tau} PEE^T P
\end{bmatrix} < 0,
\]
for all $i = 2, \cdots, N$. Therefore, in view of the bounded real lemma, the existence of $P > 0, F$ and $\tau > 0$ for (10) is equivalent to the existence of a common state feedback $u_i = F x_i$ for the following $N - 1$ systems
\[
\begin{cases}
\dot{x}_i = \left( A - \frac{\alpha + \beta}{2} EC \right) x_i + \lambda_i Bu_i + E z_i \\
y_i = \sqrt{\tau(\beta - \alpha)} C x_i
\end{cases}
\]
for all $i = 2, \cdots, N$ such that the $\mathcal{H}_\infty$ gains from $z_i$ to $y_i$ are less than or equal to $\sqrt{\tau}$.

Using the same idea as in Theorem 2, we obtain the following result.

**Theorem 5:** There exists $P > 0, F$ and $\tau > 0$ such that (10) holds for all $i = 2, \cdots, N$ if and only if there exist a matrix $Q > 0$ and a positive real number $\rho$ such that the following LMI holds:
\[
\begin{bmatrix}
B^T \\
0
\end{bmatrix} \begin{bmatrix}
Q \left( A - \frac{\alpha + \beta}{2} EC \right)^T + (A - \frac{\alpha + \beta}{2} EC) Q + \rho E E^T \\
C Q
\end{bmatrix} \begin{bmatrix}
B^T \\
0
\end{bmatrix}^T < 0. \tag{11}
\]

In this case, a suitable $P$ is given by $P = Q^{-1}$, a suitable $\tau$ is given by $\tau = 1/\rho$, and a suitable $F$ is given by
\[
F = -k B^T Q^{-1},
\]
where $k > 0$ satisfies
\[
\begin{bmatrix}
Q \left( A - \frac{\alpha + \beta}{2} EC \right)^T + (A - \frac{\alpha + \beta}{2} EC) Q + \rho E E^T - 2k \lambda_2 BB^T \\
C Q
\end{bmatrix} \begin{bmatrix}
B^T \\
0
\end{bmatrix}^T < 0. \tag{12}
\]

**Proof:** For the "only if" part, by Schur complement lemma, (10) is also equivalent to
\[
\begin{bmatrix}
A - \frac{\alpha + \beta}{2} EC + \lambda_i BF \\
P \left( A - \frac{\alpha + \beta}{2} EC + \lambda_i BF \right) + \frac{\tau(\alpha - \beta)^2}{4} C^T C + \frac{1}{\tau} PEE^T P
\end{bmatrix} < 0.
\]
Let $Q = P^{-1}$ and $\rho = 1/\tau$, we get
\[
\begin{bmatrix}
Q \left( A - \frac{\alpha + \beta}{2} EC \right)^T + \left( A - \frac{\alpha + \beta}{2} EC \right) Q + \lambda_i \left( QFTBT + BFQ \right) + \rho EE^T \\
CQ \\
- \frac{4\rho}{(\alpha - \beta)^2}
\end{bmatrix} < 0.
\]
Without loss of generality, we have
\[
\begin{bmatrix}
B \parallel \alpha \parallel \tau \times 1 \\
0_{1 \times n}
\end{bmatrix} = \begin{bmatrix}
B^\perp \parallel \alpha \parallel \tau \times 1 \\
0_{1 \times n}
\end{bmatrix}.
\]
By premultiplying and postmultiplying with (13), (11) is obtained.

For the ‘if’ part, again by Schur complement lemma, (11) implies
\[
B^\perp \left[ Q \left( A - \frac{\alpha + \beta}{2} EC \right)^T + \left( A - \frac{\alpha + \beta}{2} EC \right) Q + \rho EE^T + \frac{(\alpha - \beta)^2}{4\rho} QC^T CQ \right] (B^\perp)^T < 0.
\]
By Finsler’s lemma, it follows that there exists a $k > 0$ such that
\[
Q \left( A - \frac{\alpha + \beta}{2} EC \right)^T + \left( A - \frac{\alpha + \beta}{2} EC \right) Q + \rho EE^T + \frac{(\alpha - \beta)^2}{4\rho} QC^T CQ - 2k\lambda_2 BB^T < 0,
\]
i.e. (12). Let $P = Q^{-1}$, $\tau = \frac{1}{\rho}$ and $F := -kBT^P$, we get
\[
\Theta_i \leq \left( A - \frac{\alpha + \beta}{2} EC \right)^T P + P \left( A - \frac{\alpha + \beta}{2} EC \right) - 2k\lambda_2 PBB^T P + \frac{\tau(\alpha - \beta)^2}{4} C^T C + \frac{1}{\tau} PEE^T P < 0,
\]
for all $i = 2, \cdots, N$. This completes the proof.

V. CONCLUSIONS

In this paper we have discussed the roles of incremental passivity and incremental sector boundedness conditions in robust synchronization of homogeneous Lur’e networks. The necessary and sufficient conditions for the existence of the sufficient static distributed protocols to robustly synchronize Lur’e networks have been given. The required feedback gain matrices are computed by solving LMI’s, which can be easily done using the LMI Control Toolbox in MATLAB. As a possible topic for future research we intend to investigate robust synchronization problems for networks of Lur’e systems subject to heterogeneous nonlinear uncertainties within identical sector bounds.

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