Uniform Global Practical Asymptotic Stability for Time-Varying Cascaded Systems

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Abstract

This paper aims to give sufficient conditions for a cascade composed of nonlinear time-varying systems that are uniformly globally practically asymptotically stable (UGPAS) to be UGPAS. These conditions are expressed as relations between the Lyapunov function of the driven subsystem and the interconnection term. Our results generalise previous theorems that establish the uniform global asymptotic stability of cascades and its application is illustrated by characterising the effect of smoothing control laws in disturbance rejection.

Key words: Lyapunov stability, robustness analysis, nonlinear time-varying systems, practical stability, disturbance rejection.

1 Introduction

Cascaded dynamical systems appear in many control applications whether naturally or intentionally due to control design. Cascades-based control consists in designing the control law so that the closed loop system has a cascaded structure (see e.g. [20, 12, 15, 14]). From a theoretical viewpoint the problem of global stability analysis of cascaded systems has attracted the interest of the community since the seminal paper [28], see also [25] and references therein. A fundamental result states that the cascade of uniformly globally asymptotically stable systems (UGAS) remains UGAS if and only if its solutions are uniformly globally bounded (UGB). This has been proved in [23, 26] for the case of autonomous systems and in [19] for time-varying systems.

In many concrete applications, asymptotic stability is conservative: unmodeled dynamics, measurement noises and other disturbances often prevent the error signals from tending to zero. The only property that is often established for such systems is ultimate boundedness, meaning that the errors remain in some neighbourhood of the origin after a sufficiently long time. However, many

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of these systems present the property that the vicinity of the origin to which solutions converge can be made arbitrarily small by tuning some parameters of the system (typically some control gains). If, in addition, the solutions remain arbitrarily close to the small chosen neighbourhood of the origin at all time provided that its initial state was sufficiently near from zero, then this property is referred to as \textit{practical} stability.

In this paper, we establish sufficient conditions for uniform global practical asymptotic stability (UGPAS). Interestingly, these conditions are not much more restrictive than those commonly invoked to establish the \textit{weaker} property of global ultimate boundedness. Our main result extends previous results as [19] and [4] by establishing the UGPAS of cascaded systems. We emphasise that it is not a direct generalisation of the mentioned references. For instance, the typical assumptions to guarantee the UGB of the solutions of the overall system (for instance ISS cf. \textit{e.g.} [26], integrability of solutions, etc.) do not hold for UGPAS systems as their solutions may not converge to the origin. On the other hand, in [4] we established sufficient conditions for the uniform \textit{semiglobal} practical asymptotic stability (USPAS), which is a more general property than UGPAS. However, establishing UGPAS is not a direct corollary of the main result in [4] and the methods of proof are fundamentally different. They rely on auxiliary results on “reshaping” of Lyapunov functions, which have interest on their own right. This approach yields easy-to-check sufficient conditions.

Indeed, while the main requirement in many previous similar results is the (uniform) boundedness of the solutions (see \textit{e.g.} [23, 26, 4]), our statement does not impose such a property which remains sometimes hard to check in practice. Instead, similarly to [19, 1, 24, 3], it restricts the admissible interconnection term that links the two UGPAS subsystems. This restriction is expressed as an order comparison between the gradient, bounds and dissipation rate of a Lyapunov function whose bounds may depend on the tuning parameter. Our main results are presented in Section 3 and their proofs are given in Section 5. We present a simple example of application in Section 4 in which we rigorously characterise the effect of smoothing control laws in disturbance rejection. We conclude with some remarks in Section 6.

\section*{2 Definitions and preliminary results}

\textbf{Notation.} A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ ($\sigma \in \mathcal{L}$) if it is decreasing and tends to zero as its argument tends to infinity. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class $\mathcal{KL}$ function if, $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot) \in \mathcal{L}$ for any $s \in \mathbb{R}_{\geq 0}$. We denote by $x(\cdot, t_0, x_0)$ the solutions of the differential equation $\dot{x} = f(t, x)$ with initial conditions $(t_0, x_0)$. We use $|\cdot|$ for the Euclidean norm of vectors and the induced $L_2$ norm of matrices. We denote by $B_{\delta}$ the \textit{closed} ball in $\mathbb{R}^n$ of radius $\delta$ centred at the origin, \textit{i.e.} $B_{\delta} := \{x \in \mathbb{R}^n : |x| \leq \delta\}$. We define $|x|_\delta := \inf_{z \in B_{\delta}} |x - z|$. We designate by $\mathbb{N}_{\leq N}$ the set of all nonnegative integers less than or equal to $N$. Given $a \in [-\infty, +\infty]$ and two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, we write $f_1(s) = \mathcal{O}(f_2(s))$ as $s \to a$ if there exists a nonnegative constant $b$ such that $|f_1(s)| \leq b|f_2(s)|$ in a neighbourhood of $a$. When the context is sufficiently explicit, we may omit to write the arguments of a function.
2.1 Global asymptotic stability of balls

We start by recalling some definitions for nonlinear time-varying systems of the form
\[ \dot{x} = f(t, x), \]  
where \( x \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0} \) and \( f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \).

**Definition 1 (UGAS of a ball)** Let \( \delta \) be a positive number. The ball \( B_\delta \) is said to be Uniformly Globally Asymptotically Stable for (1) if there exists a class \( \mathcal{KL} \) function \( \beta \) such that the solution of (1) from any initial state \( x_0 \in \mathbb{R}^n \) and initial time \( t_0 \in \mathbb{R}_{\geq 0} \) satisfies
\[ |x(t, t_0, x_0)| \leq \delta + \beta(|x_0|, t - t_0), \quad \forall t \geq t_0. \]

The property of UGAS with respect to a ball defined above uses two measures\(^1\) (namely, \(|·|_\delta\) and \(|·|_\delta\)) and is therefore less restrictive than some other versions of stability of sets encountered in the literature (see e.g. [13]), where the argument of the \( \mathcal{KL} \) estimate is \(|x_0|\) instead of its Euclidean norm (i.e. stability with respect to the single measure \(|·|_\delta\)). For example, we do not impose here that the ball \( B_\delta \) be positively invariant. The type of stability introduced by Definition 1 is often satisfied by perturbed systems when the nominal system is UGAS, which motivated our choice. Also, as more clearly shown in the next section, it often allows, for a perturbed system, to use the same Lyapunov function as for the nominal one, which constitutes an interesting feature in practice.

**Remark 1** While the definition given above implies the property of ultimate boundedness (cf. e.g. [8]) with any \( \delta' > \delta \) as ultimate bound, it actually constitutes a stronger property. Notably, the transient is guaranteed to remain arbitrarily near \( B_\delta \) for any sufficiently small initial state, which is a stability property not covered by the notion of ultimate boundedness.

2.2 Global practical asymptotic stability

Our main result addresses the problem of uniform global practical asymptotic stability (UGPAS) for parameterised nonlinear time-varying systems of the form
\[ \dot{x} = f(t, x, \theta), \]  
where \( x \in \mathbb{R}^n, t \in \mathbb{R}_{\geq 0}, \theta \in \mathbb{R}^m \) is a constant free parameter and \( f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is locally Lipschitz in \( x \) and piecewise continuous in \( t \) for all \( \theta \) under consideration. Systems of this form result, for instance, from control systems in closed loop. The parameter \( \theta \) typically contains control gains (cf. e.g. [18, 4]), but may also represent other design parameters (cf. e.g. [30, 29, 27]).

**Definition 2 (UGPAS)** Let \( \Theta \subset \mathbb{R}^m \) be a set of parameters. The system (3) is said to be Uniformly Globally Practically Asymptotically Stable on \( \Theta \) if, given any positive \( \delta \), there exists \( \theta \in \Theta \) such that \( B_\delta \) is UGAS for the system \( \dot{x} = f_\theta(t, x) := f(t, x, \theta) \).

In other words, we say that (3) is UGPAS if the size of the ball which is UGAS can be arbitrarily diminished by a convenient choice of \( \theta \). Such a situation is fairly common in control practice, notably in the case of UGAS (with respect to the origin) controlled systems perturbed by bounded external disturbances. Note also that the origin does not need to be an equilibrium.

\(^1\)To see this more clearly, notice that (2) is equivalent to \(|x(t, t_0, x_0)|_\delta \leq \beta(|x_0|, t - t_0)\).
We underline that many definitions of practical stability already exist in the literature. Many of them are similar to ultimate boundedness (see e.g. [8]), as they only require that solutions eventually enter a ball without leaving it anymore (cf. Remark 1 above). Others, as the input-to-state practical stability property introduced in [7], require the transients to be “proportional” to the initial state (and the magnitude of the input signal). However, neither of these definitions impose that the ball to which solutions converge be reducible at will by tuning a parameter. Hence, they constitute a much weaker property than UGPAS. Other definitions are more conservative than Definition 2, as they require that the $\mathcal{KL}$ estimate, or at least its dependency on the initial state, be the same for all parameters $\theta \in \Theta$. While the latter property is satisfied in many contexts (see e.g. [30, 16, 29]), it may fail when dealing with perturbed systems: see [4] for an example in robot control. It should therefore be clear that, in Definition 2, “uniform” refers only to the initial conditions, and not to the tuning parameter. Finally, we stress that other definitions, such as in [30, 17], require that the tuning parameter be a positive scalar that needs to be diminished in order to get a better precision. No such tuning procedure is imposed by Definition 2 and the parameter does not need to be scalar, making possible to take into account multiple gains tuning (see e.g. [5, 10, 9] for applications involving mechanical systems). Note that, most of the time, a tuning procedure follows directly from the Lyapunov analysis (cf. Proposition 1 below). In a word, the good compromise between generality and strength offered by the above definition motivated its use.

The following result gives a sufficient condition, in terms of a Lyapunov function defined out of a ball centred at the origin, for the dynamical parameterised system (3) to be uniformly globally practically asymptotically stable on a given set of parameters. See Section 5.1 for the proof.

**Proposition 1 (Lyapunov sufficient condition for UGPAS)** Suppose that, given any $\delta > 0$, there exist a parameter $\theta(\delta) \in \Theta$, a continuously differentiable Lyapunov function $V_\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_\delta$, $\overline{\alpha}_\delta$, $\alpha_\delta$ such that, for all $x \in \mathbb{R}^n \setminus B_\delta$ and all $t \in \mathbb{R}_{\geq 0},$

$$\underline{\alpha}_\delta(|x|) \leq V_\delta(t,x) \leq \overline{\alpha}_\delta(|x|)$$  \hspace{1cm} (4)

$$\frac{\partial V_\delta}{\partial t}(t,x) + \frac{\partial V_\delta}{\partial x}(t,x)f(t,x,\theta) \leq -\alpha_\delta(|x|)$$  \hspace{1cm} (5)

$$\lim_{\delta \rightarrow 0} \underline{\alpha}_\delta^{-1} \circ \overline{\alpha}_\delta(\delta) = 0.$$  \hspace{1cm} (6)

Then the system (3) is UGPAS on the parameter set $\Theta$.

It is worth mentioning that, for perturbed systems, conditions (4) and (5) may often be verified with the Lyapunov function that serves in establishing the UGAS of the nominal system. This would not be the case if we used a stability definition based on the single measure $|\cdot|_\delta$, as $V_\delta$ would then have to vanish on the whole $B_\delta$, cf. [2] for further details.

Compared to classical results for Lyapunov stability, we see that an additional assumption (6) is required that links the bounds on the Lyapunov function. Indeed, the Lyapunov function may here depend on the tuning parameter $\theta$, and consequently on the radius $\delta$. Hence, as opposed to previously cited definitions of practical stability, this dependency may also affect the bounds on the Lyapunov function. As it appears more clearly in the proof, this dependency may prevent the size of the vicinity of the origin towards which solutions converge to be diminished at will. To the best of our knowledge, Proposition 1 is the first such result that allows to cope with a parameter-dependency of $\underline{\alpha}$ and $\overline{\alpha}$, which is often the case when dealing with energy-based Lyapunov functions, cf. e.g. [5, 10, 9]. Note that condition (6) is little conservative and satisfied in many concrete applications, as for instance in the latter references. Notably, it trivially holds when the bounds on the Lyapunov function can be chosen disregarding $\theta$. 

3 Main results

We consider cascaded systems of the form

\[ \begin{align*}
\dot{x}_1 &= f_1(t, x_1, \theta_1) + g(t, x, \theta) \\
\dot{x}_2 &= f_2(t, x_2, \theta_2)
\end{align*} \tag{7a} \tag{7b} \]

where \( x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \), \( t \in \mathbb{R}_{\geq 0} \), \( \theta = (\theta_1^\top, \theta_2^\top)^\top \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \), \( f_1, f_2 \) and \( g \) are locally Lipschitz in state and piecewise continuous in time. In order to simplify the statement of our main result, we first recall the following notation from [4].

**Definition 3** (θ-set) For any \( \delta > 0 \), the \( \mathcal{D}^\infty \)-set of (3) is defined as:

\[ \mathcal{D}^\infty_f(\delta) := \{ \theta \in \mathbb{R}^m : B_\delta \text{ is UGAS for } (3) \}. \]

Roughly speaking, \( \mathcal{D}^\infty_f(\delta) \) contains all the values of the tuning parameter that allows to asymptotically reach a given precision \( \delta \). Note that we have the property: \( \delta' \leq \delta \Rightarrow \mathcal{D}^\infty_f(\delta') \subset \mathcal{D}^\infty_f(\delta) \). Please refer to [4, 2] for further details.

We now state our main result.

**Theorem 1** Under Assumptions 1–3 below, the cascaded system (7) is UGPAS on \( \Theta_1 \times \Theta_2 \).

**Assumption 1** (Bound on the interconnection term) There exists a continuous function \( g_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) and, for any \( \theta = (\theta_1^\top, \theta_2^\top)^\top \in \Theta \), there exists a class \( K \) function \( G_{\theta_2} \) independent of \( \theta_2 \) and such that, for all \( x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and all \( t \in \mathbb{R}_{\geq 0} \),

\[ |g(t, x, \theta)| \leq g_0(|x_1|)G_{\theta_2}(|x_2|). \]

**Assumption 2** (Lyapunov UGPAS of the \( x_1 \)-subsystem) Given any \( \delta_1 > 0 \), there exist a parameter \( \theta_1(\delta_1) \in \Theta_1 \), a continuously differentiable Lyapunov function \( V_{\delta_1} \), class \( K_\infty \) functions \( \omega_{\delta_1}, \overline{\omega}_{\delta_1}, \alpha_{\delta_1} \), and a continuous positive nondecreasing function \( c_{\delta_1} \) such that, for all \( x_1 \in \mathbb{R}^{n_1} \setminus B_{\delta_1} \) and all \( t \in \mathbb{R}_{\geq 0} \), bounds (4) and (5) hold with these functions and

\[ \left| \frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1) \right| \leq c_{\delta_1}(|x_1|). \tag{8} \]

\[ \lim_{\delta_1 \to 0} \omega_{\delta_1}^{-1} \circ \overline{\omega}_{\delta_1}(\delta_1) = 0. \tag{9} \]

In addition, for all \( \delta_1 > 0 \) and as \( s \) tends to \( +\infty \),

\[ \begin{align*}
c_{\delta_1}(s)g_0(s) &= \mathcal{O}(\alpha_{\delta_1} \circ \overline{\omega}_{\delta_1}^{-1} \circ \omega_{\delta_1}(s)) \tag{10a} \\
\alpha_{\delta_1}(s) &= \mathcal{O}(\overline{\omega}_{\delta_1}(s)) \tag{10b}
\end{align*} \]

**Assumption 3** The system (7b) is UGPAS on \( \Theta_2 \).

In view of Proposition 1, it is clear that Assumption 2 implies that the zero-input \( x_1 \)-subsystem is UGPAS on \( \Theta_1 \). We state the main result under the assumption that we know a Lyapunov function \( V_{\delta_1} \) satisfying (8), (9) and (10), as these requirements implicitly impose uniform global boundedness of the solutions of cascade.

Regarding Assumption 1, it is worth emphasising that, with the exception of few articles, as [24], it is typically required that the dependency of the interconnection term in \( x_1 \) be at most linear (i.e. \( g_0 \) affine); see e.g. [25]. In fact, such a behaviour of \( g \) is implicitly imposed by (10), but only when \( |x_1| \) tends to infinity. In this respect, we underline the similarity existing between the requirements (10) and [19, Assumption 4] (which borrows from [22]): they are even equivalent in the case when, for all \( \delta_1 > 0 \), \( \lim_{s \to +\infty} \overline{\omega}_{\delta_1}(s)/\omega_{\delta_1}(s) < \infty \).

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Remark 2 For clarity, Theorem 1 is stated under the assumption that the bound on the interconnection term is independent of $\theta_2^2$, hence of $\delta_2$. For the case that this does not hold, as it appears more clearly along the proof, it is sufficient to additionally impose that, for all $\theta_1 \in \Theta_1$,

$$\lim_{\delta_2 \to 0} \left( \sup \{ G_{\theta_1, \theta_2}(\delta_2) : \theta_2 \in D_f^\infty(\delta_2) \cap \Theta_2 \} \right) = 0.$$ 

The proof of Theorem 1, presented in Section 5.2, consists in constructing a ball $B_\delta$ and a $KL$ estimate for the solutions of the cascaded system, based on the respective balls for the $x_1$ (i.e. (7a) with $x_2 \equiv 0$) and the $x_2$ subsystems, and to show that this $\delta$ can be arbitrarily reduced by a convenient choice of parameters. It relies on the following two results, that may also have interest on their own. The first one provides sufficient conditions to transform a “classical” Lyapunov function into another that presents useful properties of its gradient and its total derivative, while preserving interesting properties for its bounds. Its proof is given in Section 5.3.

**Lemma 1** Let $\delta > 0$ be some given constant and let $X$ be a subset of $\mathbb{R}^n \setminus B_\delta$. Suppose that there exist a continuously differentiable function $V : \mathbb{R}_\geq \times \mathbb{R}^n \to \mathbb{R}_\geq$ and class $K\infty$ functions $\alpha, \overline{\alpha}, \alpha$ such that, for all $x \in X$ and all $t \in \mathbb{R}_\geq$, (4) and (5) hold with these functions. Then, for any positive $k$, there exists a continuously differentiable function $\tilde{V} : \mathbb{R}_\geq \times \mathbb{R}^n \to \mathbb{R}_\geq$, and class $K\infty$ functions $\tilde{\alpha}$ and $\overline{\alpha}$ such that, for all $x \in X$ and all $t \in \mathbb{R}_\geq$,

$$\tilde{\alpha}(|x|) \leq V(t, x) \leq \tilde{\alpha}(|x|)$$

and, for any $s \in [0; 1]$, it holds that

$$\tilde{\alpha}^{-1} \circ \overline{\alpha}(s) = \alpha^{-1} \circ \alpha(s).$$

If, in addition, there exist continuous functions $c, \mu : \mathbb{R}_\geq \to \mathbb{R}_\geq$, with $c$ nondecreasing, such that (8) holds for $V$ and $c$ and

$$c(s)\mu(s) = O(\alpha \circ \overline{\alpha}^{-1} \circ \alpha(s))$$

$$\alpha(s) = O(\overline{\alpha}(s))$$

as $s$ tends to $+\infty$, then there exists a nonnegative constant $\eta$ such that, for all $x \in X$ and all $t \in \mathbb{R}_\geq$, the gradient of $V$ satisfies

$$\left| \frac{\partial V}{\partial x}(t, x) \right| \mu(|x|) \leq \eta V(t, x).$$

**Remark 3** If, in the statement of Theorem 1, the Lyapunov function $V_{\delta_1}$ (directly) satisfies (11), (12) and (15) with $\mu = g_0$ and some positive constant $\eta$, then (10) is not required anymore. This can be seen by noticing that the proof of Theorem 1 starts by transforming the original Lyapunov function into an other one satisfying (11), (12) and (15) thanks to the previous result.

The second auxiliary result may be viewed as a comparison theorem for differential inequalities that hold only out of a ball centred at zero. It provides a $KL$ estimate of all solutions starting in $\mathbb{R}^n$. See Section 5.4 for the proof.
Lemma 2 Let $\delta$ be a nonnegative constant. Assume that there exists a continuously differentiable function $V : \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}_{>0}$, class $\mathcal{K}_\infty$ functions $\alpha$ and $\overline{\alpha}$, and a constant $k \in \mathbb{R}$ such that bounds (11) and (12) hold for all $x \in \mathbb{R}^n \setminus B_1$ and all $t \in \mathbb{R}_{\geq 0}$. Then, for all $x_0 \in \mathbb{R}^n$, all $t_0 \in \mathbb{R}_{\geq 0}$ and all $t \geq t_0$, the solution of $\dot{x} = f(t, x)$ satisfies

\[ |x(t, t_0, x_0)| \leq \alpha^{-1} \circ \overline{\alpha}(\delta) + \alpha^{-1}\left(\overline{\alpha}(|x_0|) e^{-k(t-t_0)}\right). \]

\[ \square \]

4 Example: Approximated Discontinuous Feedback

Consider a control system affected by a non-vanishing perturbation, i.e.,

\[ \dot{x}_1 = f_1(t, x_1) + h_1(t, x_1)[u + d(t, x_1)] \quad (16) \]

where $d$ is a bounded function measurable in $t$ and locally Lipschitz in $x_1$. In general, we have $d(t, 0) \neq 0$, which justifies the denomination “non-vanishing perturbation” – cf. [8]. Consider the control problem of finding a control $u(t, x_1)$, possibly discontinuous in $x_1$, such that the closed-loop system is uniformly globally asymptotically stable within the following setting.

Let $u^*(t, x_1)$ be such that the closed-loop system that makes (16) UGAS provided that $d \equiv 0$. Let $V_1$ be a strict Lyapunov function for this nominal closed-loop system, that is, assume that there exist $\overline{\alpha}_1, \alpha_1 \in \mathcal{K}_\infty$ such that, for all $t \in \mathbb{R}_{\geq 0}$ and $x_1 \in \mathbb{R}^n_1$,

\[ \alpha_1(|x_1|) \leq V_1(t, x_1) \leq \overline{\alpha}_1(|x_1|) \quad (17) \]

\[ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}(f_1(t, x_1) + h_1(t, x_1)u^*(t, x_1)) \leq -\alpha_1(|x_1|). \quad (18) \]

Then, it is well-accepted that the controller $u^*(t, x_1)$ plus discontinuous terms of the state (roughly of the same size as the perturbation) still achieves UGAS and, in certain cases, finite-time stabilisation – cf. [31, 6]. More precisely, the system (16) may be rendered UGAS via the discontinuous feedback

\[ u(t, x_1) = u^*(t, x_1) - d_M \text{sign} \left( \frac{\partial V_1}{\partial x_1}(t, x_1)h_1(t, x_1) \right). \quad (19) \]

Indeed, a straightforward calculation yields

\[ \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}(f_1(t, x_1) + h_1(t, x_1)(u(t, x_1) + d(t, x))) \leq -\alpha_1(|x_1|). \]

Under appropriate regularity properties on the control, the disturbance and the Lyapunov function, and embedding the differential equation (16) in a Filippov differential inclusion, we can conclude from (17) and (18), that the closed-loop system is UGAS.

We stress that, even though when placed in the right theoretical setting, one may show that the UGAS property of the nominal system is conserved in the case of non-vanishing disturbances, this is at the price of an infinite-gain controller that induces undesirable phenomena such as chattering. A common remedy adopted in control practice is to replace $\text{sign}(\cdot)$ by a saturation function $\text{sat}(\cdot)$ with “high slope”, e.g. the function $\text{sat}(sx) := sx$ for all $|x| \leq 1/\sigma$, $\text{sat}(sx) := \text{sign}(x)$ for all $|x| > 1/\sigma$, with a choice $\sigma \gg 1$. In more general terms, we can define a saturation function as follows: $\text{sat} : \mathbb{R} \to [-1; 1]$ is taken to be locally Lipschitz, nondecreasing and satisfying

\[ \lim_{|s| \to \infty} |\text{sat}(s)| = 1, \quad \text{and} \quad \text{sat}(s)s > 0, \quad \forall s \neq 0. \]
Typical examples of saturation functions are \( \tanh(s) \), \( 2 \arctan(s)/\pi \), \( s^2 \text{sign}(s)/(1 + s^2) \) and the function \( \text{sat}(\sigma s) \) defined above.

For a number of specific applications, for instance mechanical systems with friction, it may be observed in simulations that the use of \( \text{sat}(\sigma \cdot) \) in place of \( \text{sign}(\cdot) \) in (19) as an approximation of the ideal discontinuous term impedes the asymptotic convergence of the trajectories to the origin. Instead, a steady-state error is commonly observed.

The purpose of this section is to underline the utility of Theorem 1 under this scenario. Consider the case when the system (16) is interconnected in cascade with a second subsystem; in particular, due to a cascaded-based design (cf. [25, 15]). That is, consider the system

\[
\begin{align*}
\dot{x}_1 &= f_1(t, x_1) + h(t, x_1)(u + d(t, x)) + g(t, x) \\
\dot{x}_2 &= f_2(t, x_2).
\end{align*}
\]

(20a)

(20b)

where \( u \in \mathbb{R}^m \) is the control and \( d : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a non measured perturbation satisfying \( |d(t, x)| \leq d_M \) for all \( x \in \mathbb{R}^n \) and all \( t \in \mathbb{R}_{\geq 0} \). Let \( f_1, f_2, h, d \) and \( g \) be locally Lipschitz continuous in \( x \) and measurable in \( t \). For instance, consider the case of an electro-mechanical system: the \( x_1 \) dynamics may be thought of as that of a mechanical system; the perturbation \( d \) may represent external disturbances, actuator deficiency, etc.; the subsystem (20b) may then represent the closed-loop dynamics of the actuators which may in turn include disturbances.

The control problem of interest is the following: assume that the \( x_2 \)-subsystem (20b) is UGAS (or even UGPAS). Suppose that we know a control \( u^* \) which makes the nominal \( x_1 \)-subsystem (i.e. (20a) with \( g \equiv 0 \) and \( d \equiv 0 \)) UGAS. More precisely, let \( u^* \) be such that (18) holds. Design a locally Lipschitz control \( u \) (possibly smooth) such that the system (20) is uniformly globally practically asymptotically stable.

We propose a solution to this problem with a continuous (or even smooth) approximation of the control law (19), obtained by replacing the sign function by a sufficiently stiff saturation. The stability analysis follows a cascades-based reasoning: showing that each subsystem in the cascade is UGAS (i.e. when \( g \equiv 0 \)) and, then, showing that the cascaded interconnection does not destroy stability. In this regard we stress that, as we remarked earlier, UGAS may be achievable for each subsystem of the cascade (20), however, the theorems for cascades of UGAS systems that we are aware of do not apply for two main reasons: 1) they rely on the assumption that solutions are unique and the right-hand side term is sufficiently smooth; 2) by smoothening the discontinuous terms in the control, UGAS is lost. Hence, we rely on Theorem 1.

**Proposition 2 (Disturbance rejection by smooth control)** Let \( V_1 \) be any smooth Lyapunov function for the UGAS nominal system \( \dot{x}_1 = f_1(t, x_1) + h_1(t, x_1)u^*(t, x_1) \), i.e., for all \( x_1 \in \mathbb{R}^{n_1} \) and all \( t \in \mathbb{R}_{\geq 0} \),

\[
\left| \frac{\partial V_1}{\partial x_1}(t, x_1) \right| \leq c_1(|x_1|),
\]

and (17), (18) hold where \( c_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is a continuous nondecreasing function. Assume further that there exists \( G \in \mathcal{K} \) and a continuous function \( g_0 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) such that

\[
|g(t, x)| \leq g_0(|x_1|)G(|x_2|), \quad \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad \forall t \in \mathbb{R}_{\geq 0}, \tag{21}
\]

and that, as \( s \) tends to \(+\infty\),

\[
\begin{align*}
g_0(s)c_1(s) &= \mathcal{O}(\alpha_1 \circ \sigma_1^{-1} \circ \alpha_1(s)) \tag{22} \\
\alpha_1(s) &= \mathcal{O}(\sigma_1(s)). \tag{23}
\end{align*}
\]
Assume finally that (20b) is UGAS. Then, for any saturation function sat and any positive constant ε, the overall system (20) in closed loop with

\[
u(t, x_1) := u^*(t, x_1) - (1 + ε)d_M \text{ sat} \left( \theta \frac{\partial V_1(t, x_1)h_1(t, x_1)}{\partial x_1} \right)
\]
is UGPAS on Θ := ℝ > 0, with θ as tuning parameter.

In particular, if sat is chosen as a smooth function, then the control u inherits the same regularity properties as u*. Notice also that, for the case of an autonomous system and if u* is a state feedback, then u is independent of time as well. Furthermore, the magnitude of the additional control law is only required to be strictly greater than d_M; in particular, if u* can be designed as a bounded control, then u is bounded too. Note finally that a similar result holds if (20b) is UGPAS.

**Proof of Proposition 2.** For all \(x_1 \in \mathbb{R}^{n_1}\) and all \(t \in \mathbb{R}_{>0}\), let

\[
L_{h_1}V_1(t, x_1) := \frac{\partial V_1(t, x_1)h_1(t, x_1)}{\partial x_1}.
\]

When considering \(g(t, x) \equiv 0\), the system (20a) in closed loop with (24) is

\[
\dot{x}_1 = f_1(t, x_1) + h_1(t, x_1) \left[ u^*(t, x_1) - (1 + ε)d_M \text{ sat} (\theta L_{h_1}V_1(t, x_1)) + d(t, x) \right].
\]

Using (18), the assumed properties of sat and the boundedness of the perturbation, the derivative of \(V_1\) along the trajectories of (20a) when disconnected yields

\[
\dot{V}_1(t, x_1) \leq -\alpha_1(|x_1|) - (1 + ε)d_M L_{h_1}V_1(t, x_1) \text{ sat} (\theta L_{h_1}V_1(t, x_1)) + L_{h_1}V_1(t, x_1)d(t, x)
\]

\[
\leq -\alpha_1(|x_1|) - d_M |L_{h_1}V_1(t, x_1)| \left( (1 + ε)|\text{ sat} (\theta L_{h_1}V_1(t, x_1))| - 1 \right).
\]

(25)

Consider any arbitrary \(δ_1 > 0\), and choose \(θ(δ_1)\) large enough so that

\[
\text{sat} \left( \frac{θα_1(δ_1)}{2d_M} \right) \geq \frac{1}{1 + ε},
\]

which is always possible since \(α_1\) is independent of \(θ\) and since \(\text{sat}\) is continuous and tends to 1 as its argument tends to +\(∞\). We claim that, with this choice of parameter,

\[
|x_1| \geq δ_1 \quad \Rightarrow \quad \dot{V}_1(t, x_1) \leq -\frac{1}{2}\alpha_1(|x_1|).
\]

(27)

To see this, assume that \(|x_1| \geq δ_1\) and distinguish the following two cases:

- Case 1: \(|L_{h_1}V_1(t, x_1)| \leq α_1(δ_1)/2d_M\): we then get from (25) that

\[
\dot{V}_1(t, x_1) \leq -\alpha_1(|x_1|) + d_M |L_{h_1}V_1(t, x_1)| \leq -\alpha_1(|x_1|) + \frac{α_1(δ_1)}{2},
\]

and (27) follows.

- Case 2: \(|L_{h_1}V_1(t, x_1)| > α_1(δ_1)/2d_M\): it then follows from (26) that

\[
|\text{sat} (\theta L_{h_1}V_1(t, x_1))| \geq \frac{1}{1 + ε},
\]

and (27) directly follows from (25).

In view of (17) and (27), and noticing that the functions \(α\) and \(σ\) are independent of \(δ_1\) (which makes (9) trivial), we conclude with (21), (22) and (23) that Assumption 2 holds, and the conclusion follows applying Theorem 1.
5 Proofs

5.1 Proof of Proposition 1

For any $\delta > 0$, let $k = 1$ and $V_{\delta}$ generate, via Lemma 1, a continuously differentiable function $V_{\delta}$ such that, for all $x \in \mathbb{R}^n \setminus B_{\delta}$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\tilde{\alpha}_{\delta}(|x|) \leq V_{\delta}(t, x) \leq \tilde{\sigma}_{\delta}(|x|),$$

$$\frac{\partial V_{\delta}}{\partial t}(t, x) + \frac{\partial V_{\delta}}{\partial x}(t, x)f(t, x, \theta) \leq -V_{\delta}(t, x)$$

(28)

hold with class $\mathcal{K}_\infty$ functions $\tilde{\alpha}_{\delta}$, $\tilde{\sigma}_{\delta}$, and $\tilde{\alpha}_{\delta}$, satisfying

$$\tilde{\alpha}_{\delta}^{-1} \circ \tilde{\sigma}_{\delta}(s) = \tilde{\alpha}_{\delta}^{-1} \circ \tilde{\sigma}_{\delta}(s), \quad \forall s \in [0, 1].$$

From the latter and (6), we have

$$\lim_{\delta \to 0} \tilde{\alpha}_{\delta}^{-1} \circ \tilde{\sigma}_{\delta}(\delta) = 0.$$ (29)

Furthermore, writing $x(\cdot, t_0, x_0, \theta)$ as simply $x(\cdot)$, we get from (28) and Lemma 2 that, for all $t \geq t_0$,

$$|x(t)| \leq \tilde{\alpha}_{\delta}^{-1} \circ \tilde{\sigma}_{\delta}(\delta) + \tilde{\alpha}_{\delta}^{-1} \left(\tilde{\sigma}_{\delta}(|x_0|)e^{-(t-t_0)}\right).$$

Define $\tilde{\delta} := \tilde{\alpha}_{\delta}^{-1} \circ \tilde{\sigma}_{\delta}(\delta)$ and, for all $s, t \in \mathbb{R}_{\geq 0}$, $\beta_{\delta}(s, t) := \tilde{\alpha}_{\delta}^{-1}(\tilde{\sigma}_{\delta}(s)e^{-t})$. Then we have that, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|x(t)| \leq \tilde{\delta} + \beta_{\delta}(|x_0|, t-t_0), \quad \forall t \geq t_0,$$

and it is easy to see that $\beta_{\delta}$ is a $\mathcal{K}_\mathcal{L}$ function for all positive $\delta$. Furthermore, it follows from (29) that $\tilde{\delta}$ can be made arbitrarily small by picking a parameter $\theta(\delta) \in \Theta$ corresponding to a sufficiently small $\delta$. UGPAS of (3) follows. Note that the dependency of $\beta$ in $\delta$ is not in contradiction with Definition 2. The fact of allowing the $\mathcal{K}_\mathcal{L}$ estimate to depend on the size of the ball to which solutions converge constitutes indeed an important feature of UGPAS.

5.2 Proof of Theorem 1

For any positive number $\delta_1$, let $V_{\delta_1}$ and $\theta_1(\delta_1) \in \Theta_1$ be generated by Assumption 2. Then, apply Lemma 1 to $V_{\delta_1}$ on the set $\mathbb{R}^{n_1} \setminus B_{\delta_1}$ with $\mu = g_0$ and $k = 2$. It follows that there exist a function $V_{\delta_1}$, class $\mathcal{K}_\infty$ functions $\tilde{\alpha}_{\delta_1}$, $\tilde{\sigma}_{\delta_1}$, and a nonnegative constant $\eta_{\delta_1}$ such that, for all $x_1 \in \mathbb{R}^{n_1} \setminus B_{\delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\tilde{\alpha}_{\delta_1}(|x_1|) \leq V_{\delta_1}(t, x_1) \leq \tilde{\sigma}_{\delta_1}(|x_1|)

\frac{\partial V_{\delta_1}}{\partial t}(t, x_1) + \frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1)f_1(t, x_1, \theta_1) \leq -2V_{\delta_1}(t, x_1)

\left|\frac{\partial V_{\delta_1}}{\partial x_1}(t, x_1)\right| g_0(|x_1|) \leq \eta_{\delta_1} V_{\delta_1}(t, x_1),$$

(30)

with the property that (see the reasoning done for (29)):

$$\lim_{\delta_1 \to 0} \tilde{\alpha}_{\delta_1}^{-1} \circ \tilde{\sigma}_{\delta_1}(\delta_1) = 0.$$ (31)

Next, let $G_{\theta_1}$ be given by Assumption 1 and choose $\delta_2$ small enough so that

$$G_{\theta_1}(\delta_2) \leq \frac{1}{\eta_{\delta_1}}.$$ (32)
which is always possible since $G_{\theta_1}$ is a class $\mathcal{K}$ function and neither $G_{\theta_1}$ nor $\eta_{\delta_1}$ depend on $\delta_2$. Finally, let $\theta_2$ be any parameter in $\mathcal{P}_{\delta_2}(\beta_2) \cap \Theta_2$.

We proceed in three steps. We first show that, for this choice of $\theta = (\theta_1^T, \theta_2^T)^T$, the cascade (7) is forward complete. We then use this property to prove that a certain ball $B_\delta$ is U GAS, with $\delta$ defined based on $\delta_1$ and $\delta_2$. Finally, we show that the size of this ball $B_\delta$ can be arbitrarily diminished by a convenient choice of $\theta$.

5.2.1 Proof of forward completeness

The total time derivative of $\mathcal{V}_{\delta_1}$ along (7) yields

$$\dot{\mathcal{V}}_{\delta_1} = \frac{\partial \mathcal{V}_{\delta_1}}{\partial t} + \frac{\partial \mathcal{V}_{\delta_1}}{\partial x_1}(f_1(t, x_1, \theta_1) + g(t, x, \theta)).$$

Therefore, in view of Assumption 1 and (30), it holds that, for all $x_1 \in \mathbb{R}^{n_1} \setminus \mathcal{B}_{\delta_1}$ and all $t \in \mathbb{R}_{\geq 0}$,

$$\dot{\mathcal{V}}_{\delta_1} \leq -2\mathcal{V}_{\delta_1} + \left| \frac{\partial \mathcal{V}_{\delta_1}}{\partial x_1} \right| |g(t, x, \theta)|$$

$$\leq -2\mathcal{V}_{\delta_1} + \left| \frac{\partial \mathcal{V}_{\delta_1}}{\partial x_1} \right| g_0(|x_1|)G_{\theta_1}(|x_2|)$$

$$\leq - \left( 2 - \eta_{\delta_1}G_{\theta_1}(|x_2|) \right) \mathcal{V}_{\delta_1}. \tag{33}$$

Let Assumption 3 generate a class $\mathcal{KL}$ function $\beta_{\delta_2}$ such that for any $x_{20} \in \mathbb{R}^{n_2}$ and any $t \geq t_0$,

$$|x_2(t_0, x_{20}, \theta_2)| \leq \beta_{\delta_2}(|x_{20}|, t - t_0) + \delta_2. \tag{34}$$

Let $x_0 = (x_{10}^T, x_{20}^T)^T \in \mathbb{R}^n$ be any given initial state and $t_0 \in \mathbb{R}_{\geq 0}$ be any given initial time. In order to simplify the notations, we refer to $x(\cdot, t_0, x_{20}, \theta)$ as simply $x(\cdot)$ and we define $v_1(\cdot) := \mathcal{V}_{\delta_1}(\cdot, x_1(\cdot))$. It follows from (33) that

$$|x_1(t)| > \delta_1 \Rightarrow \dot{v}_1(t) \leq \eta_{\delta_1}G_{\theta_1}(\beta_{\delta_2}(|x_{20}|, 0))v_1(t).$$

Hence, from Lemma 2, we conclude that, for all $t \geq t_0$,

$$|x_1(t)| \leq \tilde{a}_{\delta_1}^{-1} \circ \tilde{\alpha}_{\delta_1}(\delta_1) + \tilde{\alpha}_{\delta_1}^{-1} \left( \tilde{\alpha}_{\delta_1}(|x_0|) \exp (\eta_{\delta_1}G_{\theta_1}(\beta_{\delta_2}(|x_{20}|, 0))(t - t_0)) \right).$$

Thus, defining

$$\delta_3 := \tilde{a}_{\delta_1}^{-1} \circ \tilde{\alpha}_{\delta_1}(\delta_1), \tag{35}$$

and$^3$ for all $s, t \in \mathbb{R}_{\geq 0}$,

$$\rho(s, t) := \tilde{a}_{\delta_1}^{-1} \left( \tilde{\alpha}_{\delta_1}(s) \exp (\eta_{\delta_1}G_{\theta_1}(\beta_{\delta_2}(s, 0))t) \right),$$

we obtain that, for all $x_0 \in \mathbb{R}^n$ and all $t_0 \in \mathbb{R}_{\geq 0}$,

$$|x_1(t)| \leq \delta_3 + \rho(|x_0|, t - t_0), \quad \forall t \geq t_0. \tag{36}$$

Notice that $\rho(\cdot, t) \in \mathcal{K}_\infty$ for all $t \in \mathbb{R}_{\geq 0}$ and that $\rho(s, \cdot)$ is a continuous nondecreasing function for all $s \in \mathbb{R}_{\geq 0}$. This ensures the forward completeness of (7a), and consequently of the whole cascade (7).

$^2$Note that, for the case that $G$ depends on $\theta_2$ (and therefore on $\delta_2$), (32) remains achievable for $\delta_2$ small enough under the additional condition of Remark 2.

$^3$It should be clear that $\rho$ and $T$ depend on the choice of $\delta_1$ and $\delta_2$. We do not exhibit this dependency in our notation for clarity.
5.2.2 Proof of UGAS of a ball

For all \( x_0 \in \mathbb{R}^n \), consider the time\(^3\) \( T(|x_{20}|) \) such that\(^4\)

\[
\eta_{\delta_1} G_{\theta_1} \left( \beta_{\delta_2}(|x_{20}|, T(|x_{20}|)) + \delta_2 \right) = 1,
\]

Note that, in view of (32), \( T(|x_{20}|) \) is finite and nonnegative for all \( x_{20} \in \mathbb{R}^{n_2} \). Also, \( T(\cdot) \) can be picked as a nondecreasing function. In view of (33) and (34), we have that

\[
|x(t)| > \delta_1 \implies |\dot{x}(t)| \leq -\gamma(x(t), T(|x_{20}|)) + \eta T(|x_{20}|), \quad \forall t \geq t_0 + T(|x_{20}|).
\]

Invoking again Lemma 2, we get that, for all \( t \geq t_0 + T(|x_{20}|) \),

\[
|x(t)| \leq \tilde{\alpha}_{\delta_1}^{-1} \circ \tilde{\alpha}_1 (\delta_1) + \tilde{\alpha}_1^{-1} \left( \tilde{\alpha}_1 (|x_0|) e^{-(t-T(|x_{20}|)-t_0)} \right) .
\]

Therefore, since \( T(\cdot) \) is nondecreasing and \( |x_{20}| \leq |x_0| \), we obtain in view of (35) that, for all \( t \geq t_0 + T(|x_{20}|) \),

\[
|x(t)| \leq \delta_3 + \tilde{\alpha}_1^{-1} \left( \tilde{\alpha}_1 (|x_0|) e^{T(|x_{20}|)} e^{-(t-t_0)} \right) .
\]

In addition, it follows from (36) and the fact that \( \rho(s, \cdot) \) is nondecreasing for all \( s \in \mathbb{R}_{\geq 0} \), that, for all \( t \in [t_0, t_0 + T(|x_0|)] \),

\[
|x(t)| \leq \delta_3 + \rho(|x_0|, T(|x_{20}|)) e^{T(|x_{20}|)} e^{-(t-t_0)} .
\]

From these two latter bounds, we conclude that, for all \( x_0 \in \mathbb{R}^n \) and all \( t_0 \in \mathbb{R}_{\geq 0} \),

\[
|x(t)|_{\delta_3} \leq \tilde{\beta}(|x_0|, t - t_0), \quad \forall t \geq t_0,
\]

where, for all \( s, t \in \mathbb{R}_{\geq 0} \),

\[
\tilde{\beta}_{\delta_1, \delta_2}(s, t) := \tilde{\alpha}_1^{-1} \left( \tilde{\alpha}_1 (s) e^{T(s)} e^{-t} + \rho(s, t) e^{T(s)} e^{-t} \right) .
\]

Observe that \( \tilde{\beta}_{\delta_1, \delta_2} \) is a class \( \mathcal{KL} \) function. Finally, let

\[
\delta := \max\{\delta_2; \delta_3\},
\]

and, for all \( s, t \in \mathbb{R}_{\geq 0} \), \( \beta_{\delta_1, \delta_2}(s, t) := \sqrt{\beta_{\delta_1, \delta_2}(s, t)^2 + \beta_{\delta_2}(s, t)^2} \). We then conclude from (34) and (37) that, for all \( x_0 \in \mathbb{R}^n \) and all \( t_0 \in \mathbb{R}_{\geq 0} \), the solutions of (7) satisfy

\[
|x(t)| \leq \delta + \beta_{\delta_1, \delta_2}(|x_0|, t - t_0), \quad \forall t \geq t_0 .
\]

UGAS of \( \mathcal{B}_\delta \) follows by observing that \( \beta_{\delta_1, \delta_2} \) is a \( \mathcal{KL} \) function for all \( \delta_1, \delta_2 > 0 \).

5.2.3 Proof of UGPAS

It is only left to show that \( \delta \) can be arbitrarily reduced. In view of (31) and (35), \( \delta_3 \) can be picked arbitrarily small by choosing \( \delta_1 \) small enough. It follows from (38) that \( \delta \) can be made arbitrarily small by taking both \( \delta_1 \) and \( \delta_2 \) small enough. Thus, it suffices to pick the parameters \( \theta_1 \) and \( \theta_2 \) generated by these chosen \( \delta_1 \) and \( \delta_2 \), to conclude that, for any \( \delta > 0 \), there exist some parameters \( \theta_1 \in \Theta_1 \) and \( \theta_2 \in \Theta_2 \) such that (7) is UGAS with respect \( \mathcal{B}_\delta \), which establishes the result.

\(^3\) \( T(|x_{20}|) \) is taken as zero if \( \eta_{\delta_1} G_{\theta_1} (\beta_{\delta_2}(|x_{20}|, 0) + \delta_2) \leq 1.\)
5.3 Proof of Lemma 1

The proof is inspired by [21, Proposition 13], originally presented in [11]. We first consider
the case when (8), (13) and (14) do not hold. In this situation, we see with (11) that the bound
\( V(t, x) \leq \hat{\alpha}(|x|) \) holds for all \( x \in \mathbb{R}^n \) and all \( t \in \mathbb{R}_{\geq 0} \) if \( \hat{\alpha} \) designates any class \( \mathcal{K}_\infty \) function satisfying
\[
\hat{\alpha}(s) = \begin{cases} 
\overline{\alpha}(s) & \text{if } s \in [0, 1] \\
\max\{\overline{\alpha}(s), \alpha(s)\} & \text{if } s \geq 2.
\end{cases}
\]

Next, let \( \alpha \) be a class \( \mathcal{K} \) function with the following properties:
\[
\begin{align*}
\alpha(s) &= \frac{1}{2}\alpha \circ \hat{\alpha}^{-1}(s) \text{ for all } s \geq \alpha(\delta), \\
\alpha(s) &\leq s \text{ for all } s \leq \alpha(\delta)/2, \\
\alpha'(0) &= 0,
\end{align*}
\]
and define \( \rho \) as the following function
\[
\rho(s) = \exp \left( \int_1^s \frac{d\tau}{a(\tau)} \right), \quad \forall s \in \mathbb{R}_{\geq 0}.
\]

First observe that, since \( \alpha(s) = \mathcal{O}(\hat{\alpha}(s)) \) as \( s \) tends to infinity, the integral in the exponential diverges. In the same way, since \( a(s) \leq s \) in a neighbourhood of zero, the very integral tends to \(-\infty \) when \( s \) tends to zero. It can also be seen that \( \rho \) is continuous and increasing, which makes it a class \( \mathcal{K}_\infty \) function. Also, based on [21, Lemma 12], \( \rho \) is continuously differentiable too. Hence, by operating the transformation \( V := \rho \circ V \), we see that \( V \) is continuously differentiable as well, and that (11) can be established with the following class \( \mathcal{K}_\infty \) functions: \( \tilde{\alpha} := \rho \circ \alpha \) and \( \tilde{\pi} := \rho \circ \hat{\alpha} \). In turn, we have that, for all \( s \in [0, 1] \),
\[
\tilde{\alpha}^{-1} \circ \tilde{\pi}(s) = (\alpha^{-1} \circ \rho^{-1}) \circ (\rho \circ \pi)(s) = \alpha^{-1} \circ \pi(s).
\]

In addition, for all \( x \in X, V(t, x) \geq \alpha(\delta) \), hence
\[
\begin{align*}
\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x) &\leq -\rho'(V(t, x))\alpha(|x|) \\
&\leq -\frac{V(t, x)}{a(V(t, x))} \alpha \circ \hat{\alpha}^{-1}(V(t, x)) \\
&\leq -kV(t, x),
\end{align*}
\]
which establishes the first part of the result.

Now we consider that (8), (13) and (14) hold. Then we can see that the whole previous reasoning remains valid by picking \( \hat{\alpha} \) as simply \( \pi \) (notably, \( \rho \) is still a class \( \mathcal{K}_\infty \) function in view of (14)). In addition, for all \( x \in X, \) we have that
\[
\left| \frac{\partial V}{\partial x} \right| \mu(|x|) = \rho'(V)c(|x|)\mu(|x|) \leq \frac{V}{a(V)} c(|x|)\mu(|x|) .
\]

The bound (15) then follows by recalling that \( X \subset \mathbb{R}^n \setminus \mathcal{B}_\delta \) and by noticing that (13) together with
the continuity of \( \mu \) ensures the existence of a nonnegative constant \( \eta \) such that
\[
\sup_{\|x\| \geq \delta} \frac{c(|x|)\mu(|x|)}{a(V(t, x))} \leq \sup_{\|x\| \geq \delta} \frac{kc(|x|)\mu(|x|)}{\alpha \circ \pi^{-1} \circ \alpha(|x|)} \leq \eta .
\]
5.4 Proof of Lemma 2

For simplicity, we write \( x(\cdot; t_0, x_0) \) as \( x(\cdot) \) and we define \( v(\cdot) := V(\cdot, x(\cdot)) \). We distinguish two cases: whether the trajectories start from outside or inside \( B_3 \).

**Case 1:** \( |x_0| > \delta \).
In this case, there exists \( T_0 \in (0; \infty] \) such that \( |x(t)| > \delta \) for all \([t_0; t_0 + T_0)\) and \( |x(t + T_0)| = \delta \). Hence, using the comparison lemma, we get that
\[
v(t) \leq v(t_0)e^{-k(t-t_0)}, \quad \forall t \in [t_0; t_0 + T_0).
\]
Using the bounds on \( V \), it follows that
\[
|x(t)| \leq \alpha^{-1}\left(\overline{\pi}(|x_0|)e^{-k(t-t_0)}\right), \quad \forall t \in [t_0; t_0 + T_0).
\]
In addition, for each \( t \geq t_0 + T_0 \), either \( |x(t)| \leq \delta \) in which case \( x(\cdot) \leq \alpha^{-1} \circ \overline{\pi}(\delta) \), or \( |x(t)| > \delta \).

In this second case, we can again invoke the continuity of the solution to see that there exists a nonempty time-interval \([\tau; \tau + T]\), with \( T \in (0; \infty] \), containing \( t \) and such that \( |x(s)| > \delta \) for all \( s \in [\tau; \tau + T] \), with \( |x(\tau)| = \delta \). Hence, integrating from \( \tau \) to \( t \in [\tau; \tau + T] \), we obtain in the same way as before that, whenever \( |x(t)| > \delta \), it holds that
\[
|x(t)| \leq \alpha^{-1}\left(\overline{\pi}(\delta)e^{-k(t-\tau)}\right) \leq \alpha^{-1}\left(\overline{\pi}(|x_0|)e^{-k(t-t_0)}\right). \tag{39}
\]
To sum up, for all \( t \geq t_0 \), we have the following:
\[
|x_0| > \delta \quad \Rightarrow \quad |x(t)| \leq \alpha^{-1}\left(\overline{\pi}(|x_0|)e^{-k(t-t_0)}\right). \tag{40}
\]

**Case 2:** \( |x_0| \leq \delta \).
In this case, as long as \( |x(\cdot)| \leq \delta \), we have trivially that \( |x(t)| \leq \alpha^{-1} \circ \overline{\pi}(\delta) \). If \( |x(t)| > \delta \) at some instant \( t > t_0 \), then, again, there exists a nonempty time-interval \([\tau; \tau + T]\), with \( T \in (0; \infty] \) and \( \tau > t_0 \), containing \( t \) and such that \( |x(s)| > \delta \) for all \( s \in [\tau; \tau + T] \), with \( |x(\tau)| = \delta \). Thus, from (39), we obtain that
\[
|x(t)| \leq \alpha^{-1}\left(\overline{\pi}(\delta)e^{-k(t-\tau)}\right) \leq \alpha^{-1}\left(\overline{\pi}(\delta)\right). 
\]
Hence, for all \( t \geq t_0 \),
\[
|x_0| \leq \delta \quad \Rightarrow \quad |x(t)|_\delta \leq \alpha^{-1}\left(\overline{\pi}(\delta)\right). \tag{41}
\]
The conclusion follows from (40) and (41).

6 CONCLUSION

We have presented results for uniform global practical asymptotic stability of nonlinear time-varying systems. Our first result is a Lyapunov sufficient condition for UGPAS. The novelty here mainly consists in authorizing the Lyapunov function to depend on the tuning parameter, which is often the case when dealing with perturbed systems. We then establish that the cascade of two UGPAS systems remains UGPAS provided a specific condition linking the Lyapunov function of the driven subsystem to the term of interconnection between the two subsystems. The application of this second result is illustrated through an example involving saturated controls.

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5 If \( |x(t)| > \delta \) forever after, we consider that \( T_0 = \infty \).
6 This is direct by noticing that \( \alpha(s) \leq \overline{\pi}(s) \) for all \( s \in \mathbb{R}_{\geq 0} \).
REFERENCES


