Domination subdivision numbers of trees

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Abstract

A set $S$ of vertices of a graph $G = (V, E)$ is a dominating set if every vertex of $V(G) \setminus S$ is adjacent to some vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. The domination subdivision number $sd_{\gamma}(G)$ is the minimum number of edges that must be subdivided in order to increase the domination number. Velammal showed that for any tree $T$ of order at least 3, $1 \leq sd_{\gamma}(T) \leq 3$. In this paper, we give two characterizations of trees whose domination subdivision number is 3 and a linear algorithm for recognizing them.

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1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For every vertex $v \in V(G)$, the open neighborhood $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$. Let $S$ be a set of vertices, and let $u \in S$. A vertex $v$ is a private neighbor of $u$ (with respect to $S$) if $N[v] \cap S = \{u\}$. The private neighbor set of $u$, with respect to $S$, is $pn[u, S] = \{v \mid N[v] \cap S = \{u\}\}$. A subset $S$ of vertices of $G$ is a dominating set if $N[S] = V$ and a total dominating set if $N[S] = V$. The (total) domination number $\gamma(G)$ ($\gamma_t(G)$) is the minimum cardinality of a (total) dominating set of $G$. A dominating set of minimum cardinality of $G$ is called a $\gamma$-set of $G$ or $\gamma(G)$-set.

The domination subdivision number $sd_{\gamma}(G)$ of a graph $G$ is the minimum number of edges that must be subdivided, where each edge in $G$ can be subdivided at most once, in order to increase the domination number. (An edge $uv \in E(G)$ is subdivided if the edge $uv$ is deleted, but a new vertex $x$ is added, along with two new edges $ux$ and $vx$. The vertex $x$ is called a subdivision vertex.) Since the domination number of the graph $K_2$ does not change when its only edge is subdivided, we assume that the graph is of order $n \geq 3$. The domination subdivision number, defined in Velammal’s thesis [9], has been studied in [1–3]. A similar concept related to total domination was defined in [4].

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In general, for notation and graph theory terminology we follow [5]. A leaf of a graph $G$ is a vertex of degree 1, while a support vertex of $G$ is a vertex adjacent to a leaf. A support vertex is strong if it is adjacent to at least two leaves. Note that every graph has a $γ$-set containing all of its support vertices. A path on $n$ vertices is denoted by $P_n$. For $t ≥ 1$, a subdivided star $SK_{1,t}$ is obtained by subdividing the $t$ edges of a star $K_{1,t}$. Its domination number is equal to $t$. For $t = 1$, $SK_{1,1} = P_3$.

Here are some well-known results on $γ(G)$ and $sd_G(G)$.

**Theorem A.** For $n ≥ 3$, $γ(P_n) = ⌊\frac{n}{3}\rfloor$.

An immediate consequence of Theorem A now follows.

**Proposition 1.** For a path on $n ≥ 3$ vertices,

$$sd_G = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 2 \pmod{3} \\ 3 & \text{if } n \equiv 1 \pmod{3}. \end{cases}$$

It is shown in [9] that the domination subdivision number of a tree is either 1, 2, or 3, and so trees can be classified as class 1, class 2, or class 3 depending on whether their domination subdivision number is 1, 2 or 3, respectively. Similarly, the authors of [4] showed that the total domination subdivision number of a tree is 1, 2 or 3. A constructive characterization of trees such that $sd_T(T) = 3$ is given in [6].

Our purpose in this paper is to characterize the trees such that $sd_T(T) = 3$. We first give a constructive characterization similar to that one in [6] for $sd_T(T) = 3$, and then a structural one. We begin with two lemmas, the proof of the first one is straightforward.

**Lemma 2.** If the graph $G$ has a strong support vertex then $sd_T(G) = 1$, and if $G$ is a subdivided star $SK_{1,t}$ with $t ≥ 2$ then $sd_T(G) = 2$.

**Lemma 3.** (1) If $G$ is a graph obtained from a graph $G'$ of order at least 2 by adding a subdivided star $SK_{1,t}$ with $t ≥ 1$ and adding an edge joining the center $c$ of the star to a vertex $y$ of $G'$, then $γ(G) = γ(G') + t$. Moreover, if $G'$ has order at least 3, then $sd_T(G) ≤ sd_T(G')$.

(2) If $G$ is a graph obtained from a graph $G'$ containing a pendant edge $ya$ or a pendant path $ybc$ by adding a path $xz$ and an edge joining $x$ to the vertex $y$, then $γ(G) = γ(G') + 1$. Moreover, if $y$ is a support vertex of $G'$ and $G'$ has order at least 3, then $sd_T(G) ≤ sd_T(G')$.

**Proof.** (1) Let $V(SK_{1,t}) = \{c, x_1, w_1, \ldots, x_t, w_t\}$ with $d(x_i) = 2$ and $d(w_i) = 1$ for $1 ≤ i ≤ t$. Clearly every $γ(G)$-set $S$ exactly contains $t$ vertices in $\{x_1, \ldots, x_t, w_1, \ldots, w_t\}$. Moreover, $S \setminus V(SK_{1,t})$ if $c ∉ S$ and $(S \setminus V(SK_{1,t})) \cup \{y\}$ if $c ∈ S$ is a $γ(G)$-set of order $γ(G) - t$.

If $sd_T(G') = k$, consider $k$ edges $e_j ∈ E(G')$ such that their subdivision yields a graph $G''$ satisfying $γ(G'') > γ(G')$. Let $G''$ be obtained from $G'$ by subdividing the $k$ edges $e_j$ and let $S'$ be a $γ(G'')$-set. Then $S' \cup \{x_1, \ldots, x_t\}$ is a $γ(G''-set$ and so $γ(G'') = γ(G'') + t > γ(G') + t = γ(G')$. Therefore $sd_T(G) ≤ k$.

(2) If $S'$ is a $γ(G')$-set, then $S' \cup \{x\}$ is a dominating set of $G$ and so $γ(G) ≤ γ(G') + 1$. Conversely, if $S$ is a $γ(G)$-set containing all the support vertices of $G$ and in particular $y$ or one of its neighbors in $G'$, then $S \setminus \{x\}$ is a dominating set of $G$ and so $γ(G) ≤ γ(G') - 1$. The same argument as in (1) proves that $sd_T(G) ≤ sd_T(G')$.

2. A constructive characterization of trees in class 3

In this section we provide a constructive characterization of all trees in class 3. For this purpose we describe a procedure for building a family $F$ of labeled trees that are of class 3 as follows. The label of a vertex is also called its status, denoted as $sta(v)$.

**Definition.** Let $F$ be the family of labeled trees that:

(1) contains $P_4$ where the two leaves have status $A$, and the two support vertices have status $B$, and
shows that if a tree $T$ has been obtained from $T_1$ and $T_2$ by successive operations, all the sequences have the same length $m = \gamma(T) - 2$.

Lemma 6. Let $T \in \mathcal{F}$ and $z \in A(T)$. There is a $\gamma$-set of $T$, say $S$, such that $z \in S$ and $pn[z, S] = \{z\}$.

Proof. Note that $pn[z, S] = \{z\}$ means that $z$ is isolated in $S$ and has no other $S$ private neighbor. Let $T_0 = P_4$ and $T$ be obtained from $P_4$ by successive operations $\mathcal{S}_1, \ldots, \mathcal{S}_m$. The proof is by induction on $m$. If $m = 0$ then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from $P_4$ with at most $m - 1$ operations. Let $T_{m-1}$ be obtained from $P_4$ by successive operations $\mathcal{S}_1, \ldots, \mathcal{S}_{m-1}$. By Lemma 5, $\gamma(T) = m + 2 = \gamma(T_{m-1}) + 1$. We consider two cases.

Case 1. $\mathcal{S}_m = \mathcal{S}_1$. Then $T$ has been obtained from $T_{m-1}$ by adding a path $xwv$ and an edge $xy$ with $y \in A(T_{m-1})$. Moreover, $\text{sta}(x) = \text{sta}(w) = B$, and $\text{sta}(v) = A$. Let $z \in A(T)$. If $z \in A(T_{m-1})$ then by the inductive hypothesis, there is a $\gamma(T_{m-1})$-set, say $S$, such that $z \in S$ and $pn[z, S] = \{z\}$. Then $S' = S \cup \{w\}$ is a $\gamma$-set of $T$ and $pn[z, S'] = \{z\}$. Now let $z = v$. By the inductive hypothesis there is a $\gamma(T_{m-1})$-set, say $S$, such that $y \in S$ and $pn[y, S] = \{y\}$. Then $S' = (S \setminus \{y\}) \cup \{x, v\}$ is a $\gamma$-set of $T$ and $pn[v, S'] = \{v\}$.

Case 2. $\mathcal{S}_m = \mathcal{S}_2$. Then $T$ has been obtained from $T_{m-1}$ by adding a path $xw$ and an edge $xy$ with $y \in B(T_{m-1})$. Moreover, $\text{sta}(x) = B$ and $\text{sta}(w) = A$. Let $z \in A(T)$. If $z \in A(T_{m-1})$ then an argument similar to that described
for case 1 shows that the statement holds. So let $z = w$. By Observation 4(4), $y$ has exactly one neighbor in $A(T_{m-1})$, say $v$. By the inductive hypothesis there is a $\gamma(T_{m-1})$-set, say $S$, such that $v \in S$ and $pn[v, S] = \{v\}$. Then $S' = (S \setminus \{v\}) \cup \{y, w\}$ is a $\gamma$-set of $T$ and $pn[w, S'] = \{w\}$. This completes the proof. □

**Lemma 7.** Let us have $T \in \mathcal{F}$, $T^*$ obtained from $T$ by subdividing one edge of $T$, and $z \in A(T)$. Then $\gamma(T^*) = \gamma(T)$ and there is a $\gamma$-set of $T^*$ containing $z$.

**Proof.** Let $T \in \mathcal{F}$. Note first that $\gamma(T^*) \geq \gamma(T)$ and that any dominating set of $T^*$ of order $\gamma(T)$ is a $\gamma(T^*)$-set. Let us have $e \in E(T)$ and let $T^*$ be obtained from $T$ by adding a new vertex $e$ subdividing the edge $e$. The proof is by induction on the number $m$ of operations used to construct $T$ from $P_k$. If $m = 0$ then the statement is true since $\gamma(P_2) = \gamma(P_4) = 2$ and $P_5$ admits a $\gamma$-set containing an endvertex. Assume that $m \geq 1$ and that the statement holds for all trees which are obtained from $P_k$ with at most $m - 1$ operations. Let $T$ be obtained from $P_k$ by the $m$ operations $\gamma^1, \ldots, \gamma^{m-1}, \gamma^m$ and let $T_{m-1}$ be the tree obtained after the $m - 1$ first operations $\gamma^1, \ldots, \gamma^{m-1}$. When $e \in E(T_{m-1})$, let $T_{m-1}^*$ be obtained from $T_{m-1}$ by subdividing the edge $e$. We consider two cases.

**Case 1.** $\gamma^m = \gamma_1$. Then $T$ has been obtained from $T_{m-1}$ by adding a path $xwv$ and the edge $xy$ such that $y \in A(T_{m-1})$ and $st(a) = st(w) = B$, $st(v) = A$.

Suppose first that $e \in E(T_{m-1})$. By the inductive hypothesis, $\gamma(T^*_{m-1}) = \gamma(T_{m-1}) = \gamma(T) - 1$ and for any vertex $t \in A(T_{m-1})$ there is a $\gamma(T_{m-1})$-set $S_t$ containing $t$. Let $S_1 = S_2 \cup \{w\}$ if $z \in A(T_{m-1})$, $S_1 = S_2 \cup \{v\}$ if $z = v$. In both cases, $S_1$ is a dominating set of $T^*$ containing $z$ and of order $\gamma(T)$.

Suppose now that we subdivide an edge $e$ of the path $xwv$, say without loss of generality, $e = xw$. By Lemma 6, there exists a $\gamma(T)$-set $S$ such that $v \in S$ and $pn[v, S] = \{v\}$. Necessarily, $x \in S$. If $z \in A(T_{m-1})$, let $S_1 = A(T_{m-1}) \cup \{w\}$. If $z = v$, let $S_1 = S$. In both cases, $S_1$ is a dominating set of $T^*$ containing $z$ and of order $\gamma(T)$.

**Case 2.** $\gamma^m = \gamma_2$. Then $T$ has been obtained from $T_{m-1}$ by adding a path $xw$ and the edge $xy$ such that $y \in B(T_{m-1})$ and $st(a) = st(w) = A$.

Suppose first that $e \in E(T_{m-1})$. By the inductive hypothesis, $\gamma(T^*_{m-1}) = \gamma(T_{m-1}) = \gamma(T) - 1$ and if $z \in A(T_{m-1})$, there is a $\gamma(T_{m-1})$-set $S$ containing $z$. When $z = w$, take for $S$ any $\gamma$-set of $T_{m-1}$. Let $S_1 = S \cup \{w\}$.

Suppose now that we subdivide an edge $e$ of the path $xwv$, say without loss of generality, $e = xy$. By Lemma 6, there exists a $\gamma(T)$-set $S$ such that $w \in S$ and $pn[w, S] = \{w\}$. Necessarily, $y \in S$. If $z \in A(T_{m-1})$, let $S_1 = A(T_{m-1}) \cup \{x\}$. If $z = w$, let $S_1 = S$.

In all cases, $S_1$ is a dominating set of $T^*$ containing $z$ and of order $\gamma(T)$. □

**Theorem 8.** Each tree in family $\mathcal{F}$ is in class 3.

**Proof.** The proof is by induction on the length $m$ of the sequence of operations needed to construct the tree $T$. When $m = 0$, then $T = P_3$ and by Theorem A, $T$ is in class 3. Assume $m \geq 1$ and the result holds for all trees in $\mathcal{F}$ that can be constructed from $P_k$ by a sequence of less than $m$ operations. Let $T \in \mathcal{F}$ be obtained by $\mathcal{F}$ from $\mathcal{F}$ by the inductive hypothesis, $T_{m-1}$ is in class 3.

Let $T^*$ be obtained from $T$ by subdividing any two edges, say $e$ and $f$ of $T$. Clearly $\gamma(T^*) \geq \gamma(T)$. To show that $T$ is in class 3, it is sufficient to show that $\gamma(T^*) \leq \gamma(T)$. We consider two cases.

**Case 1.** $T$ is obtained from $T_{m-1}$ by operation $\mathcal{F}_1$, that is by adding a path $xwv$ and an edge $xy$ with $y \in A(T_{m-1})$.

Consider three subcases.

**Subcase 1.1.** $e, f \in E(T_{m-1})$. Let $T^*_{m-1}$ be obtained from $T_{m-1}$ by subdividing the edges $e, f$. Then $T^*$ is obtained from $T^*_{m-1}$ by adding the path $yvuv$ to the vertex $y \in V(T^*_{m-1})$. By the inductive hypothesis and by Lemma 5, $\gamma(T^*_{m-1}) = \gamma(T_{m-1}) = \gamma(T) - 1$. Let $S$ be a $\gamma(T_{m-1})$-set. Then $S_1 = S \cup \{w\}$ is a dominating set of $T^*$ and so $\gamma(T^*) \leq \gamma(T)$.

**Subcase 1.2.** $\{(e, f) \cap E(T_{m-1})\} = 1$. We may assume that $e \in E(T_{m-1})$ and, without loss of generality, $f = xw$. Let $T^*_{m-1}$ be obtained from $T_{m-1}$ by subdividing $e$. By the inductive hypothesis and by Lemma 5, $\gamma(T^*_{m-1}) = \gamma(T_{m-1}) = \gamma(T) - 1$. By Lemma 7, there exists a $\gamma(T^*_{m-1})$-set $S$ containing $y$. Then $S_1 = S \cup \{w\}$ is a dominating set of $T^*$ and so $\gamma(T^*) \leq \gamma(T)$.

**Subcase 1.3.** $e, f \in E(T) \setminus E(T_{m-1})$. We may assume without loss of generality that $e = xw$ and $f = yw$. Let $w'$ be the new vertex subdividing $uv$. Since $y \in A(T_{m-1})$ and by Lemma 6, there exists a $\gamma(T_{m-1})$-set $S$ such that $y \in S$ and $pn[y, S] = \{y\}$. Then $S_1 = (S \setminus \{y\}) \cup \{x, w'\}$ is a dominating set of $T^*$ and so $\gamma(T^*) \leq \gamma(T_{m-1}) + 1 = \gamma(T)$. H. Aram et al. / Discrete Mathematics 309 (2009) 622–628
Lemma 6

Observation 4

Lemma 2

Observation 4

Lemma 2

Proof. Theorem 10.

Case 2. \( T \) is obtained from \( T_{m-1} \) by operation \( \Sigma_2 \), that is by adding a path \( xw \) and an edge \( xy \) with \( y \in B(T_{m-1}) \). We proceed as in case 1, with the small modifications indicated below.

Subcase 2.2. \(|\{e, f\} \cap E(T_{m-1})| = 1 \). Without loss of generality, \( f = xy \). Take for \( S \) any \( \gamma(T_{m-1}) \)-set and \( S_1 = S \cup \{x\} \).

Subcase 2.3. \( e, f \in E(T - E(T_{m-1})) \). Necessarily, \( e = xy \) and \( f = xw \). Let \( w' \) be the new vertex subdividing \( xw \) and \( y' \) the unique neighbor of status \( A \) of \( y \) in \( T_{m-1} \). By Lemma 6, there exists a \( \gamma(T_{m-1}) \)-set \( S \) such that \( y' \in S \) and \( pn[y', S] = \{y'\} \). Take \( S_1 = (S \setminus \{y'\}) \cup \{y, w'\} \).

In all cases, \( S_1 \) is a dominating set of \( T^* \) of order \( \gamma(T) \), which completes the proof. \[ \Box \]

We now present our first characterization.

**Theorem 9**. A tree \( T \) of order \( n \geq 3 \) is in class 3 if and only if \( T \in \mathcal{F} \).

**Proof.** By Theorem 8, it is sufficient to prove that the condition is necessary. The proof is by induction on the order \( n \) of \( T \). By Lemma 2, the only tree \( T \) of order 3 or 4 and \( sd_T(T) = 3 \) is \( P_4 \), which belongs to \( \mathcal{F} \). Let \( n \geq 5 \) and suppose that the statement holds for every tree in class 3 and order less than \( n \). Let \( T \) be a tree of order \( n \) and \( sd_T(T) = 3 \). By Lemma 2, the support vertices of \( T \) are not strong. Let \( P : v_1 v_2 \ldots v_\ell \) be a longest path in \( T \). Obviously \( \deg(v_1) = \deg(v_\ell) = 1 \) and \( \deg(v_2) = \deg(v_{\ell-1}) = 2 \). Hence \( \ell \geq 5 \). We consider two cases.

Case 1. \( v_3 \) is a support vertex.

Let \( T' = T \setminus \{v_1, v_2\} \). By Lemma 3(2), \( T' \) is in class 3 and so belongs to \( \mathcal{F} \) by the inductive hypothesis. By Observation 4(2), \( v_3 \in B(T') \). Hence \( T \) is obtained from \( T' \) with one operation \( \Sigma_2 \) and belongs to \( \mathcal{F} \).

Case 2. \( v_3 \) is not a support vertex.

Let \( T' \) and \( T'' \) be the components of \( T - v_1 v_4 \) respectively containing \( v_4 \) and \( v_3 \). Since \( P \) is a longest path of \( T \), all the neighbors of \( v_3 \) different from \( v_4 \) are support vertices. Hence \( T'' \) is a subdivided star \( SK_{1,t+1} \) with \( t \geq 1 \).

Moreover \( |V(T')| \geq 3 \) for otherwise \( T \) is a subdivided star \( SK_{1,t+1} \) which contradicts \( sd_T(T) = 3 \) by Lemma 2.

By Lemma 3(1), \( T' \) is in class 3 and so belongs to \( \mathcal{F} \) by the inductive hypothesis. If \( v_4 \) is a support vertex or has a neighbor which is a support vertex, let \( T^* \) (\( T''^* \) respectively) be obtained from \( T \) (\( T'' \) respectively) by subdividing the two edges \( v_1 v_2 \) and \( v_2 v_3 \). Let \( S \) be a \( \gamma(T) \)-set and \( S^* \) be a \( \gamma(T^*) \)-set. The set \( S \) contains \( v_4 \) or one of its neighbors in \( T' \), and does not contain \( v_3 \). Hence, \( |S^* \cap V(T''^*)| = t + 1 = |S \cap V(T'')| + 1 \) and \( |S^* \cap V(T')| = |S \cap V(T')| \).

Therefore \( \gamma(T^*) > \gamma(T) \), in contradiction to \( sd_T(T) = 3 \). Hence either \( \deg(v_4) = 2 \) and \( v_4 \) is a leaf of \( T' \), or \( \deg(v_4) \geq 3 \) and all the neighbors of \( v_4 \) in \( T' - v_5 \) are at distance exactly 2 (since \( P \) is a longest path) from a leaf of \( T' \). In the first case \( v_4 \in A(T') \) by Observation 4(1). In the second one, all the neighbors of \( v_4 \) in \( T' - v_5 \) are in \( B(T') \) by Observation 4(1) and (5), and have no neighbors in \( A(T') \) except possibly \( v_4 \). By Observation 4(4), \( v_4 \in A(T') \).

Then \( T \) can be obtained from \( T' \) with one operation \( \Sigma_1 \) and \( t - 1 \) operations \( \Sigma_2 \), which completes the proof. \[ \Box \]

3. A structural characterization of trees in class 3

We first recall two classical definitions. An independent set (respectively 2-packing or, for short, packing) of a graph \( G \) is a subset of vertices mutually at distance more than 1 (respectively 2). Clearly every packing is independent. The minimum cardinality of a maximal independent set, or equivalently of a dominating independent set, of \( G \) is denoted \( i(G) \) and the maximum cardinality of a packing of \( G \) is denoted \( \rho(G) \). It is well known that \( \rho(G) \leq \gamma(G) \leq i(G) \) for every graph and that \( \rho(T) = \gamma(T) \) for every tree \( T \) [8].

We will say that a tree \( T \) has Property \( \mathcal{P} \) if it admits a packing which is dominating and contains all its leaves.

**Theorem 10**. A tree \( T \) contains at most one dominating packing containing all its leaves.

**Proof.** Let \( S \) and \( S' \) be two different dominating packings containing the set \( L \) of leaves of \( T \) and let \( R = V(T) \setminus S, R' = V(T) \setminus S' \). Since \( \rho(T) = \gamma(T), |S| = |S'| = \gamma(T) \).

Let \( S' \setminus S = \{v_1, v_2, \ldots, v_p\} \subseteq R' \) and \( S \setminus S = \{w_1, w_2, \ldots, w_p\} \subseteq R \). Each vertex \( v_i \) belongs to \( R' \cap S \). Hence, since \( S' \) is a dominating packing and \( S \) is independent, each \( v_i \) is adjacent to exactly one vertex of \( S \setminus S \). Without loss of generality we may suppose that the edges between \( \{v_1, v_2, \ldots, v_p\} \) and \( \{w_1, w_2, \ldots, w_p\} \) form a matching \( \{v_1w_1, \ldots, v_pw_p\} \). For \( 1 \leq i \leq p \), the vertex \( v_i \) is not a leaf since \( L \subseteq S \), and has no neighbor in \( N[S \cap S'] \cup S' \) since \( S' \) is a packing. Hence each \( w_i \) has at least one neighbor in \( N[v_1] \cup \cdots \cup N[v_p] \setminus \{w_1, \ldots, w_p\} \). Therefore the subgraph induced in \( T \) by \( N[v_1] \cup \cdots \cup N[v_p] \) contains a cycle, a contradiction which completes the proof. \[ \Box \]
Let $T$ be a tree of $\mathcal{F}$ obtained from an initial $P_4$ by a sequence of operations $\Xi_1$ or $\Xi_2$, and let $A(T)$ and $B(T)$ be the vertices of respective status $A$ and $B$ in the corresponding construction of $T$. By Observation 4(1, 4, 5), the set $A(T)$ is a dominating packing containing all the leaves of $T$. In particular, the minimum dominating set $A(T)$ is independent which proves that $\mathcal{F}$ is a subclass of the class of $(\gamma - i)$-trees, which are trees for which $\gamma(T) = i(T)$. Moreover every tree in $\mathcal{F}$ has Property $\mathcal{P}$ and the following result is a consequence of Theorem 10.

**Theorem 11.** The set of vertices with status $A$ of a tree $T$ in $\mathcal{F}$ does not depend on the construction of $T$ and is its unique dominating packing containing all the leaves.

We can now give a second characterization of the trees in class 3.

**Theorem 12.** A tree $T$ of order $n \geq 3$ is in class 3 if and only if it has Property $\mathcal{P}$.

**Proof.** By Theorem 9, we have to show that $T$ is in $\mathcal{F}$ if and only if it has Property $\mathcal{P}$. Since every tree in $\mathcal{F}$ has Property $\mathcal{P}$, we prove by induction on $n$ that every tree with Property $\mathcal{P}$ is in $\mathcal{F}$. A tree with Property $\mathcal{P}$ has order at least 4 and if $n = 4$, then the only tree with $\mathcal{P}$ is $P_4$ which belongs to $\mathcal{F}$. For $n \geq 5$, suppose that every tree having $\mathcal{P}$ and of order less than $n$ is in $\mathcal{F}$ and let $T$ be a tree of order $n$ with Property $\mathcal{P}$. Let $S$ be the unique dominating packing of $T$ containing all its leaves and let $P = v_1v_2 \cdots v_p$ be a longest path of $T$. The leaves $v_1$ and $v_p$ belong to $S$. The vertices $v_2$ and $v_{p-1}$ have degree 2 since all the leaves are in the packing $S$. Hence $p \geq 5$. By the definition of a packing, $v_2$ and $v_3$ are not in $S$ and so if $d(v_3) = 2$, then $v_4 \notin S$. Let $T' = T - \{v_1, v_2, v_3\}$ if $d(v_3) = 2$, and let $S' = S \cap V(T')$. In both cases the set $S'$ is a dominating packing of the tree $T'$ containing all its leaves. By the inductive hypothesis and Theorem 11, $T' \in \mathcal{F}$ and $A(T') = S'$. If $d(v_3) = 2$, then $v_4 \in S$ and hence $v_4 \in A(T') = S' = S \cap V(T')$. If $d(v_3) \geq 3$ then, since $v_3 \notin S$, $v_3 \notin S' = A(T')$ and thus $v_3 \in B(T')$. Therefore $T$ can be obtained from $T'$ by an operation $\Xi_1$ when $d(v_3) = 2$, $\Xi_2$ when $d(v_3) \geq 3$. Hence $T \in \mathcal{F}$, which completes the proof. □

We finish the paper with the informal description of a linear algorithm for deciding whether a given tree $T$ is in class 3 and if it is, getting the unique partition $V(T) = A \cup B$ such that

- every leaf is in $A$,
- every neighbor of a vertex in $A$ is in $B$,
- exactly one neighbor of a vertex in $B$ is in $A$.

We proceed by a DFS from a leaf $x$. Each vertex is examined once, either because it is a leaf different from $x$ or after all its children have been examined. Every vertex $v$ receives a mark $A$, $B$ or $C$ and a label $(a, b, c)$ where $a$, $b$, $c$ are non-negative integers. A mark $A$ or $B$ is definitive. A mark $C$ at $v$ is temporary and either it will be transformed into $B$, if the father of $v$ later receives the mark $A$, or we will stop the algorithm with FALSE, i.e., $T$ is not in class 3, otherwise. The label $(a, b, c)$ of a vertex indicates the number of its children respectively marked $A$, $B$ or $C$. When we mark a vertex with $A$, $B$ or $C$, we increase by 1 the corresponding term $a$, $b$ or $c$ of its father’s label. For the initialization, all the labels are $(0, 0, 0)$ and no vertex is marked. The rule for marking a vertex $v$ labelled $(a, b, c)$ or stopping the algorithm with FALSE is as follows (we leave the reader to check that these rules correspond to the three properties on $A$ and $B$ recalled above):

- if $a \geq 1$ then FALSE,
- if $(a = 1$ and $c \geq 1$) then FALSE,
- if $(a = 1$ and $c = 0$) then mark($v$) = $B$,
- if $(a = 0$ and $b \geq 1$ and $c \geq 1$) then FALSE,
- if $(a = 0$ and $b \geq 1$ and $c = 0$) then mark($v$) = $C$,
- if $(a = 0$ and $b = 0$) then mark($v$) = $A$

(hence the first marked vertex, a leaf different from $x$, receives the mark $A$).

The algorithm either stops with FALSE somewhere, or runs until every vertex is marked.

- If mark($x$) $\neq A$, then FALSE; otherwise $T$ is in class 3 and changing all the marks $C$ into $B$ gives the unique partition $V = A \cup B$.

Now that trees in class 3 can be easily recognized, the question is how to decide when a tree not in $\mathcal{F}$ is in class 1 or 2. This problem is more difficult. In [7], the authors give a characterization of the trees in class 1 by the means of three tree properties which are not easy to check.
References