Non-Fermi liquid theory of a compactified Anderson single-impurity model

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Abstract

We consider a version of the symmetric Anderson impurity model (compactified) which has a non-Fermi liquid weak coupling regime. We find that in the Majorana fermion representation the perturbation theory can be conveniently developed in terms of Pfaffian determinants and we use this formalism to calculate the impurity free energy, self energies, and vertex functions. We also derive expressions for the impurity and the local conduction electron charge and spin dynamical susceptibilities in terms of the impurity self-energies and vertex functions. In the second-order perturbation theory, a linear temperature dependence of the electrical resistivity is obtained, and the leading corrections to the impurity specific heat are found to behave as $T \ln T$. The impurity static susceptibilities have terms in $\ln T$ to zero, first, and second order, and corrections of $\ln^2 T$ to second order as well. The conduction electron static susceptibilities, and the singlet superconducting paired static susceptibility at the impurity site, are found to have second-order corrections $\ln T$, which we interpret as an indication that a singlet conduction electron pairing resonance forms at the Fermi level (the chemical potential). When the perturbation theory is extended to third order logarithmic divergences are found in the only vertex function $\Gamma_{0,1,2,3}(0,0,0,0)$, which is non-vanishing in the zero frequency limit. We use the multiplicative renormalization-group (RG) method to sum all the leading order logarithmic contributions. These give rise

to a new weak-coupling low-temperature energy scale $T_c = \Delta \exp\left[-\frac{1}{9}\left(\frac{\pi\Delta}{U}\right)^2\right]$, which is the combination of the two independent coupling parameters. The RG scaling equation is also derived and shows that the dimensionless coupling constant $\bar{U} = \frac{U}{\pi\Delta}$ is increased as the high-energy scale Δ is reduced, so our perturbational results can be justified in the regime $T > T_c$. Below T_c the perturbation theory breaks down.

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I. INTRODUCTION

Strongly correlated electron systems, and especially the high T_c cuprate superconductors, display features which have been difficult to reconcile with conventional theories of Fermi liquids and superconductivity. The various ideas that have been put forward to explain the behavior of these systems remain controversial. The unusual normal state of the high T_c materials has been interpreted as some form of non-Fermi liquid (non-FL), but there have been no fully convincing microscopic derivations of such a state in two dimensional systems with short range interactions. Understanding the normal state is rather important as it may be the key to understanding the nature and origin of the superconductivity. Anomalous behavior that has been observed experimentally in certain heavy fermion U-based superconductors has also been interpreted as some form of non-FL behavior [1]. One of the most striking characteristics of the non-FL in both these types of systems is the linear temperature dependence of the electrical resistivity. A phenomenological marginal-FL spectrum for the spin and charge fluctuations was put forward by Varma *et al.* [2] as a unified way of interpreting the diverse anomalies observed in the cuprate superconductors. The essential point in this theory is that the frequency dependence of the polarization is singular, while the momentum dependence is taken to be smooth. This picture is very similar to the two dimensional Luttinger liquid hypothesis used by Anderson [3]. There is no generally accepted microscopic theory to explain the linear temperature dependence of the electrical resistivity.

The lack of progress in developing a microscopic theory is due to the difficulty in solving the lattice models in strong coupling, and it may be quite a time before suitable methods can be developed to overcome these difficulties. There are impurity models, however, which display non-FL behavior; these are more accessible, and may provide valuable insights. For many of these models we have exact solutions [4], and in this context, the non-FL thermodynamic behavior of the two-channel Kondo model has been extensively investigated [5–10]. However the exact solution for this model obtained from conformal field theory [7]

gives a resistivity that behaves as $\rho(T) = \rho(0)(1 + a\sqrt{T})$ in the low temperature limit, so it is not clear that the experimental observations can be interpreted using this theoretical model. The question then arises as to whether the linear temperature dependence of the resistivity, and the marginal-FL behavior, can be found in any single-impurity model.

In this paper we examine a model, originally put forward by Coleman and Schofield [10], which we will show displays just this type of behavior, at least in weak coupling perturbation theory. We can obtain the model starting from usual symmetric Anderson single-impurity model in the Majorana representation. The usual model has O(4) symmetry arising from a product of the SU(2) spin and the SU(2) particle-hole symmetry. The modified model is obtained by reducing the O(4) symmetry of the hybridization term to O(3) symmetry. We then construct a systematic non-FL perturbation theory around the weak coupling limit. In contrast to the two-channel Kondo model, the linear temperature dependence of the electrical resistivity is obtained from second order perturbation theory. The result would appear to be valid at temperatures above a characteristic temperature T_c. At T_c, the sum of the leading order singular perturbational contributions to the electrical resistivity diverge indicating a breakdown of the perturbation theory at this energy scale. A brief report of this work was given in our previous paper [11].

The paper is arranged as follows. In section II, the modified Anderson single-impurity model is introduced and the general features of this model are discussed, in particular, there is an important relation between the conduction electron and impurity Green functions. In section III, we fully investigate the unperturbed Hamiltonian, and show that it displays a non-FL behavior in the spin and charge density-density spectrum. In section IV, we present the a new perturbation formalism in which the impurity free energy, single-particle Green functions, and two-particle vertex functions can be expressed in terms of Pfaffian determinants, a specific determinant defined from the ordinary antisymmetric determinant. In section V, using this formalism, expressions are derived for both impurity and local conduction electron spin and charge dynamical susceptibilities, including the conduction

electron singlet superconducting paired susceptibility, in terms of the impurity self energy and vertex functions. In section VI, the lower order results are given and discussed in detail. In section VII, multiplicative renormalization-group method is applied to this model to sum all the leading logarithmic contributions and obtain the scaling equation. Finally, some discussions and conclusions are given in section VIII.

II. THE MODEL HAMILTONIAN

The ordinary symmetric Anderson single-impurity model can be expressed in the form:

$$H = it\sum_{n,\sigma}\left[C_\sigma^\dagger(n+1)C_\sigma(n) - H.c.\right] + iV\sum_\sigma\left[C_\sigma^\dagger(0)d_\sigma - H.c.\right] + U\left(d_\uparrow^\dagger d_\uparrow - \frac{1}{2}\right)\left(d_\downarrow^\dagger d_\downarrow - \frac{1}{2}\right), \tag{1}$$

where the symmetric condition $\epsilon_d = -U/2$ has been used, and the chemical potential is set to zero, the Fermi level. The Hamiltonian has $O(4)$ symmetry due to the $SU(2)$ symmetry from the spin rotational invariance and an additional $SU(2)$ from particle-hole symmetry, giving $O(4) \sim SU(2) \otimes SU(2)$.

It is important to observe that the $O(4)$ symmetry can be explicitly displayed when the fermions of each type of spin are expressed in terms of four Majorana (real) fermions [10]:

$$C_\uparrow(n) = \frac{1}{\sqrt{2}}\left[\Psi_1(n) - i\Psi_2(n)\right], \qquad C_\downarrow(n) = \frac{1}{\sqrt{2}}\left[-\Psi_3(n) - i\Psi_0(n)\right];$$

$$d_\uparrow = \frac{1}{\sqrt{2}}(d_1 - id_2), \qquad d_\downarrow = \frac{1}{\sqrt{2}}(-d_3 - id_0),$$

these new operators satisfy

$$\Psi_\alpha(n) = (\Psi_\alpha(n))^\dagger, \qquad d_\alpha = (d_\alpha)^\dagger;$$

$$\{\Psi_\alpha(n), \Psi_\beta(n')\} = \delta_{\alpha,\beta}\delta_{n,n'}, \qquad \{d_\alpha, d_\beta\} = \delta_{\alpha,\beta}. \tag{2}$$

Then the Hamiltonian becomes

$$H = it\sum_n\sum_{\alpha=0}^{3}\Psi_\alpha(n+1)\Psi_\alpha(n) + iV\sum_{\alpha=0}^{3}\Psi_\alpha(0)d_\alpha + Ud_1d_2d_3d_0. \tag{3}$$

When the $O(4)$ symmetry breaks down to $O(3)$ symmetry in the hybridization term, the model Hamiltonian becomes

$$H = it\sum_{n}\sum_{\alpha=0}^{3}\Psi_\alpha(n+1)\Psi_\alpha(n) + iV_0\Psi_0(0)d_0 \\ +iV\sum_{\alpha=1}^{3}\Psi_\alpha(0)d_\alpha + Ud_1d_2d_3d_0, \tag{4}$$

where $V_0 \neq V$. In the large U limit ($U > 0$), a Schrieffer-Wolff canonical transformation can be applied to generate a s-d type of model, the so-called 'compactified two-channel Kondo model',

$$H = it\sum_{n}\sum_{\alpha=0}^{3}\Psi_\alpha(n+1)\Psi_\alpha(n) + [J_1\vec{\sigma}(0) + J_2\vec{\tau}(0)]\cdot\vec{S}_d, \tag{5}$$

with $J_1 = 2V(V+V_0)/U$ and $J_2 = 2V(V-V_0)/U$, where the local impurity spin couples to both the conduction electron spin $\vec{\sigma}(0)$ and the conduction electron 'isospin' density operator $\vec{\tau}(0)$ at the impurity site. When $V_0 = 0$ the two exchange couplings are identical, *i.e.*, $J_1 = J_2$, and it had been conjectured that this form of the model has the same low-energy excitations as the two-channel Kondo model [9].

To distinguish the Anderson model of Eq.(4) from other Anderson impurity models we will refer to it as the 'compactified' Anderson impurity model. Here we concentrate on the $V_0 = 0$ case:

$$H = H_0 + H_I, \\ H_0 = it\sum_{n}\sum_{\alpha=0}^{3}\Psi_\alpha(n+1)\Psi_\alpha(n) + iV\sum_{\alpha=1}^{3}\Psi_\alpha(0)d_\alpha, \\ H_I = Ud_1d_2d_3d_0. \tag{6}$$

This will be used as our model Hamiltonian. The Hamiltonian for $V_0 \neq 0$ is considered in a separate paper [12].

Fourier transforms for the conduction electron operators can be introduced as usual

$$\Psi_\alpha(n) = \frac{1}{\sqrt{N}}\sum_{k}\Psi_\alpha(k)e^{ikna}, \qquad \alpha = 0,1,2,3, \tag{7}$$

here N is the total number of the sites and the lattice spacing is a. The anticommutation relation for the conduction electrons is converted into

$$\{\Psi_\alpha(k), \Psi_\beta(-k')\} = \delta_{\alpha,\beta}\delta_{k,k'}. \tag{8}$$

Substituting these expressions in our model Hamiltonian, up to a constant we get

$$H = \sum_{k>0}\sum_{\alpha=0}^{3} \epsilon_k \Psi_\alpha(-k)\Psi_\alpha(k) + \frac{iV}{\sqrt{N}}\sum_{k}\sum_{\alpha=1}^{3} \Psi_\alpha(k)d_\alpha + U d_1 d_2 d_3 d_0, \tag{9}$$

where $\epsilon_k = 2t\sin(ka)$ is the dispersion relation of the conduction electrons.

In the present model, the new O(3) symmetry in the hybridization is the key feature. Since the scalar field ($\alpha = 0$) decouples from the local impurity, its single-particle Green functions, defined in terms of retarded double-time correlation function, is easily found to be a free propagator:

$$G^0_{k,k'}(\omega_n) \equiv -\int_0^\beta d\tau \langle T_\tau \Psi_0(k,\tau)\Psi_0(-k',0)\rangle e^{i\omega_n\tau} = \frac{\delta_{k,k'}}{i\omega_n - \epsilon_k}, \tag{10}$$

where $\omega_n = (2n+1)\pi/\beta$, β is the inverse of the temperature. Meanwhile, the vector field $\Psi_\alpha(k)$ ($\alpha = 1,2,3$) hybridizes with the corresponding impurity vector field d_α, and the scattering of the conduction electrons from the local impurity is given by the following relation:

$$G^\alpha_{k,k'}(\omega_n) = \frac{\delta_{k,k'}}{i\omega_n - \epsilon_k} + \frac{V^2}{N}\frac{G_{\rm vec}(\omega_n)}{(i\omega_n - \epsilon_k)(i\omega_n - \epsilon_{-k'})}, \tag{11}$$

where $G_{\rm vec}(\omega_n)$ is the Fourier transform of the impurity vector propagator defined by $-\langle T_\tau d_\alpha(\tau)d_\alpha(\tau')\rangle_H$. Then the conduction electron t-matrix is thus expressed as

$$t_{k,-k'}(\omega_n) = \frac{V^2}{N}G_{\rm vec}(\omega_n). \tag{12}$$

It is easily seen that this t-matrix *only* depends on the Green function for the impurity vector field $G_{\rm vec}(\tau)$. The implication of this relation is that the conduction electron resistivity will be determined by the $G_{\rm vec}(\tau)$ only. Moreover, there is also a general relation between the impurity vector propagator and the cross correlation function,

$$G_{k,d}(\omega_n) \equiv \langle\langle \Psi_\alpha(k)|d_\alpha\rangle\rangle = \frac{iV}{\sqrt{N}} \frac{G_{\rm vec}(\omega_n)}{(i\omega_n - \epsilon_k)} \qquad \alpha = 1,2,3. \tag{13}$$

This relation becomes useful when we calculate various two-particle correlation functions for the conduction electrons, because we can relate the conduction electron correlation functions at the impurity site to the impurity correlation functions.

In addition, we note that under the simple transformation $S = \sqrt{2}d_0$, the model Hamiltonian transforms according to

$$SH(U)S^{-1} = H(-U). \tag{14}$$

Since the sign change of the on-site Hubbard interaction corresponds to the exchange of the impurity charge and spin degrees of freedom, or the exchange of the impurity spin and 'isospin' density operators, the particle-hole symmetry is kept in the present model, and the the chemical potential is pinned at the Fermi level.

III. THE UNPERTURBED HAMILTONIAN

Before considering the effects of interactions, it is very constructive to examine the unperturbed model Hamiltonian.

$$H_0 = it \sum_n \sum_{\alpha=0}^{3} \Psi_\alpha(n+1)\Psi_\alpha(n) + iV \sum_{\alpha=1}^{3} \Psi_\alpha(0) d_\alpha. \tag{15}$$

This is an exactly soluble Hamiltonian in which only the vector field of the conduction electrons hybridizes with the impurity vector field, while the scalar field is free. The impurity Green functions are defined as follows:

$$G_0(\tau) = -\langle T_\tau d_0(\tau) d_0(0)\rangle, \qquad G_\alpha(\tau) = -\langle T_\tau d_\alpha(\tau) d_\alpha(0)\rangle. \tag{16}$$

It is straight forward to calculate their Fourier counterparts,

$$G_0(\omega_n) = \frac{1}{i\omega_n}, \qquad G_\alpha(\omega_n) = \frac{1}{i\omega_n + i\Delta {\rm sgn}\omega_n}, \tag{17}$$

where $\Delta = \pi\rho V^2$ is the hybridization width, $\rho = (hv_f)^{-1}$ is the conduction electron density of states, and $\omega_n = (2n+1)\pi/\beta$ is the fermionic Matsubara frequency. Here we find that

the impurity scalar propagator is a fermionic zero mode with $G_0(\tau) = -\mathrm{sgn}\tau/2$. Moreover, both impurity scalar and vector field propagators are odd in their arguments.

The local impurity spectral function can be evaluated as

$$A_d(\omega) = \frac{3}{2\pi}\frac{\Delta}{\omega^2+\Delta^2} + \frac{1}{2}\delta(\omega), \tag{18}$$

which reveals the basic physics of the unperturbed Hamiltonian that half of the impurity degree of freedom is free, while the rest of the degrees of freedom couple to the conduction electrons. A similar impurity spectral function was also found in the single-impurity two-channel Kondo model [8]. The change of free energy due to the hybridization with the impurity can be calculated as

$$F^{(0)}_{\mathrm{imp}} = \frac{3}{2\pi}\int_{-\infty}^{\infty} f(\omega)\tan^{-1}\left(\frac{\Delta}{\omega}\right)d\omega - \frac{T}{2}\ln 2, \tag{19}$$

where $f(\omega)$ is Fermi distribution function, and the impurity residual entropy is found to be $\ln\sqrt{2}$, revealing that the ground state of H_0 is a two-fold degenerate state, which is a consequence of the spin-isospin symmetry with $[H_0, S] = 0$ for $S = \sqrt{2}d_0$.

In addition, we can calculate the spin and charge density-density correlation functions to find out what kind of low-energy behavior is described by H_0. In the Majorana representation, the impurity spin and charge density operators are expressed as

$$S^z_d = \frac{1}{2}(d^\dagger_\uparrow d_\uparrow - d^\dagger_\downarrow d_\downarrow) = -\frac{i}{2}(d_1 d_2 - d_0 d_3),$$
$$n_d = \frac{1}{2}(d^\dagger_\uparrow d_\uparrow + d^\dagger_\downarrow d_\downarrow - 1) = -\frac{i}{2}(d_1 d_2 + d_0 d_3).$$

The spin and charge density-density correlation functions are equal to

$$\langle T_\tau S^z_d(\tau) S^z_d(0)\rangle = \langle T_\tau n_d(\tau) n_d(0)\rangle = \frac{1}{4}\left[G^2_\alpha(\tau) + G_\alpha(\tau)G_0(\tau)\right]. \tag{20}$$

Their corresponding Fourier transform is

$$\chi^{\mathrm{imp}}_{\rho,\sigma}(\omega_n) = \frac{1}{4\beta}\sum_{\omega_{n'}}\left[G_\alpha(\omega_{n'})G_\alpha(\omega_n - \omega_{n'}) + G_\alpha(\omega_{n'})G_0(\omega_n - \omega_{n'})\right], \tag{21}$$

where $\omega_n = 2n\pi/\beta$ is the boson-type Matsubara frequency. The first term in the brackets corresponds to the normal FL-like density-density correlation forms, but the second term is

singular. As far as the singularity is concerned, the imaginary part of the spectral functions is obtained by completing the summation over frequency:

$$\mathrm{Im}\chi^{\mathrm{imp}}_{\rho,\sigma}(\omega,T) = -\frac{1}{8}\frac{\Delta}{\omega^2+\Delta^2}\tanh\left(\frac{\omega}{2T}\right) \sim \begin{cases} \frac{\omega}{T}, & \omega \ll T \\ \text{const.}, & \Delta \gg \omega \gg T \end{cases}$$

This kind of behavior was assumed in the marginal FL phenomenology [2] to describe the normal state of the high temperature superconductors, where the standard FL theory does not appear to explain the experimental observations. The conduction electron charge and spin density-density correlation functions at the impurity site can also be calculated, but their spectral functions do not display any kind of singular behavior.

It is clear that the Hamiltonian with $U = 0$ has singular behavior due to the degeneracy of the ground state and does not describe a non-interacting Fermi liquid. In the perturbation theory we develop in the next section, we make an expansion about this unusual weak-coupling limit.

IV. PERTURBATION FORMALISM OF THE MODEL HAMILTONIAN

A. Partition function and free energy

Now we consider the full Hamiltonian for the compactified Anderson single-impurity model,

$$\begin{aligned} H &= H_0 + H_I, \\ H_0 &= it\sum_n \sum_{\alpha=0}^{3} \Psi_\alpha(n+1)\Psi_\alpha(n) + iV\sum_{\alpha=1}^{3}\Psi_\alpha(0)d_\alpha, \\ H_I &= -Ud_0d_1d_2d_3. \end{aligned} \tag{22}$$

The partition function for this Hamiltonian can be expanded as follows:

$$\begin{aligned} Z &= Z_0\sum_{n=0}^{\infty}(-1)^n\int_0^\beta d\tau_n\int_0^{\tau_n}d\tau_{n-1}....\int_0^{\tau_2}d\tau_1\langle H_I(\tau_n)H_I(\tau_{n-1})...H_I(\tau_1)\rangle \\ &= Z_0\sum_{n=0}^{\infty}U^n\int_0^\beta d\tau_n\int_0^{\tau_n}d\tau_{n-1}....\int_0^{\tau_2}d\tau_1 F_n(\tau_n,\tau_{n-1},...,\tau_1), \end{aligned} \tag{23}$$

where Z_0 denotes the partition function for the unperturbed Hamiltonian, and $\langle ... \rangle$ means the thermodynamic average is carried out over the unperturbed part H_0. Noting that in H_0 the four Majorana fermion components of the local impurity decouple completely, $F_n(\tau_n, \tau_{n-1}, ..., \tau_1)$ can be factorized as

$$\langle d_0(\tau_n) d_0(\tau_{n-1}) ... d_0(\tau_1) \rangle \prod_{\alpha=1}^{3} \langle d_\alpha(\tau_n) d_\alpha(\tau_{n-1}) ... d_\alpha(\tau_1) \rangle. \tag{24}$$

Here both the impurity scalar and vector field single-particle Green function $G_0(\tau)$ and $G_\alpha(\tau)$ ($\alpha = 1, 2, 3$) are free propagators. When the Wick theorem is implemented, it can be verified order by order, that each Majorana fermion expectation average can be represented by a Pfaffian determinant [13]. For the expectation values of the impurity vector operators, the Pfaffian determinant is defined by the square root of an antisymmetric determinant composed of the impurity vector propagator $G_\alpha(\tau)$,

$$\begin{vmatrix} 0, & G_\alpha(\tau_1 - \tau_2), & G_\alpha(\tau_1 - \tau_3), & ....., & G_\alpha(\tau_1 - \tau_n) \\ G_\alpha(\tau_2 - \tau_1), & 0, & G_\alpha(\tau_2 - \tau_3), & ....., & G_\alpha(\tau_2 - \tau_n) \\ \vdots & \vdots & \vdots & & \vdots \\ G_\alpha(\tau_n - \tau_1), & G_\alpha(\tau_n - \tau_2), & G_\alpha(\tau_n - \tau_3), & ....., & 0 \end{vmatrix} = \backslash D_n(\tau_1, \tau_2, ..., \tau_n)|^2.$$

As shown in Section II, the impurity vector field propagators are odd in their arguments, $G_\alpha(\tau_r - \tau_s) + G_\alpha(\tau_s - \tau_r) = 0$ and $G_\alpha(\tau_r = \tau_s) = 0$. The Pfaffian determinant $\backslash D_n(\tau_1, \tau_2, .., \tau_n)|$ is given by

$$\begin{vmatrix} | G_\alpha(\tau_1 - \tau_2), & G_\alpha(\tau_1 - \tau_3), & ...., & G_\alpha(\tau_1 - \tau_n) \\ & G_\alpha(\tau_2 - \tau_3), & ...., & G_\alpha(\tau_2 - \tau_n) \\ & & & \vdots \\ & & & G_\alpha(\tau_{n-1} - \tau_n) \end{vmatrix} = \sum \pm G_\alpha(\tau_1 - \tau_a) G_\alpha(\tau_b - \tau_c) ..... G_\alpha(\tau_l - \tau_m),$$

where the subscripts $1, a, b, c, ..., l, m$ of each term under the summation are a permutation of the first n integers, each Green function $G_\alpha(\tau_r - \tau_s)$ has $s > r$, all different terms of this type are included, and the total number of terms is $(n-1)(n-3) \cdots 5 \cdot 3 \cdot 1$. The sign attached to each term is positive or negative depending on whether the permutation is

even or odd. In addition, the basic property of Pfaffian determinant is that all odd-order determinants identically vanish, and all the expansion terms we should consider are even-order determinants. For instance, the fourth Pfaffian determinant $\backslash D_4(\tau_1,\tau_2,\tau_3,\tau_4)|$ is given by

$$\begin{vmatrix} G_\alpha(\tau_1,\tau_2), & G_\alpha(\tau_1,\tau_3), & G_\alpha(\tau_1,\tau_4) \\ & G_\alpha(\tau_2,\tau_3), & G_\alpha(\tau_2,\tau_4) \\ & & G_\alpha(\tau_3,\tau_4) \end{vmatrix}$$
$$= G_\alpha(\tau_1,\tau_2)G_\alpha(\tau_3,\tau_4) - G_\alpha(\tau_1,\tau_3)G_\alpha(\tau_2,\tau_4) + G_\alpha(\tau_1,\tau_4)G_\alpha(\tau_2,\tau_3).$$

On the other hand, the impurity scalar field propagator has a special form, $G_0(\tau) = -\mathrm{sgn}\tau/2$, its corresponding expectation is trivial to calculate because the imaginary time sequence has been assumed $\beta > \tau_{2n} > \tau_{2n-1} > \cdots > \tau_2 > \tau_1 > 0$.

$$\langle d_0(\tau_{2n}) d_0(\tau_{2n-1}) ... d_0(\tau_1)\rangle = \left(\frac{1}{2}\right)^n . \tag{25}$$

Therefore, the partition function for the compactified Anderson single-impurity model H can be expressed as the cube of the Pfaffian determinant with elements corresponding to the impurity vector field single-particle Green function at different times:

$$Z/Z_0 = \sum_{n=0}^{\infty} \left(\frac{U}{\sqrt{2}}\right)^{2n} \int_0^\beta d\tau_{2n} \int_0^{\tau_{2n}} d\tau_{2n-1} .... \int_0^{\tau_2} d\tau_1 \left\{\backslash D_{2n}(\tau_1,\tau_2,...,\tau_{2n})|\right\}^3 . \tag{26}$$

For the ordinary symmetric Anderson model with $V_0 = V$, the power of the Pfaffian determinant in the partition function is four rather than three, and in fact there is no need to introduce Pfaffian determinant in that case [14,15]. Thus one power of the Pfaffian determinant in the partition function corresponds to each of the Majorana fermions involved in the hybridization. Using the linked cluster theorem, we find that the free energy change due to the local impurity can be expressed in power series in the parameter U

$$F_{\mathrm{imp}} = F_{\mathrm{imp}}^{(0)} - \sum_{n=1}^{\infty} \left(\frac{U}{\sqrt{2}}\right)^{2n} \frac{1}{\beta} \int_0^\beta d\tau_{2n} \int_0^{\tau_{2n}} d\tau_{2n-1} .... \int_0^{\tau_2} d\tau_1 \left\{\backslash D_{2n}(\tau_1,\tau_2,...,\tau_{2n})|\right\}_l^3 , \tag{27}$$

where the subscript l on the bracket in the above equation indicates that only linked diagrams are to be considered. The extra impurity specific heat can be calculated systematically from this free energy expression.

B. Single-particle Green functions and self energies

Now we consider the perturbed impurity single-particle correlation functions. For the impurity scalar field d_0, the perturbed Green function is defined as

$$G_{\rm sc}(\tau,\tau') = -\frac{\langle T_\tau d_0(\tau) d_0(\tau') exp\left[-\int_0^\beta H_I(\tau) d\tau\right]\rangle}{\langle T_\tau exp\left[-\int_0^\beta H_I(\tau) d\tau\right]\rangle}, \tag{28}$$

and its Fourier transform is given by

$$G_{\rm sc}(\omega_n) = \frac{1}{\beta}\int_0^\beta d\tau \int_0^\beta d\tau' G_{\rm sc}(\tau,\tau') e^{i\omega_n(\tau-\tau')}. \tag{29}$$

In a similar way to our early derivation of the partition function, the perturbed scalar field propagator can be expanded in powers of U:

$$G_{\rm sc}(\omega_n) = G_0(\omega_n) + \sum_{n=1}^{\infty}\frac{U^{2n}}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\int_0^\beta d\tau_{2n}\int_0^{\tau_{2n}} d\tau_{2n-1}...\int_0^{\tau_2} d\tau_1 e^{i\omega_n(\tau-\tau')}$$
$$\left\{\langle T_\tau d_0(\tau) d_0(\tau') d_0(\tau_{2n})....d_0(\tau_1)\rangle \backslash D_{2n}(\tau_1,\tau_2,...,\tau_{2n})|^3\right\}_l. \tag{30}$$

For the special form of $G_0(\tau) = -{\rm sgn}\tau/2$, the expectation value of the impurity scalar field propagator can be calculated as

$$-\langle T_\tau d_0(\tau) d_0(\tau') d_0(\tau_{2n})....d_0(\tau_1)\rangle$$
$$= \left(\frac{1}{2}\right)^{n-1}\sum_{i<j}(-1)^{i+j}[G_0(\tau-\tau_i)G_0(\tau'-\tau_j) - G_0(\tau-\tau_j)G_0(\tau'-\tau_i)] + \left(\frac{1}{2}\right)^n G_0(\tau-\tau'). \tag{31}$$

Then completing the integrals over τ and τ', we obtain the scalar field Green function in terms of the following equation,

$$G_{\rm sc}(\omega_n) = G_0(\omega_n) + G_0(\omega_n)\Sigma'_{\rm sc}(\omega_n)G_0(\omega_n), \tag{32}$$

where an improper self-energy is represented as

$$\Sigma'_{\rm sc}(\omega_n) = \sum_{n=1}^{\infty}\frac{4i}{\beta}\left(\frac{U}{\sqrt{2}}\right)^{2n}\int_0^\beta d\tau_{2n}\int_0^{\tau_{2n}} d\tau_{2n-1}....\int_0^{\tau_2} d\tau_1$$
$$\left\{\sum_{i<j}(-1)^{i+j}[\sin\omega_n(\tau_i-\tau_j)]\left\{\backslash D_{2n}(\tau_1,\tau_2,...,\tau_{2n})|\right\}^3\right\}_l. \tag{33}$$

For the impurity vector field d_α with $(\alpha = 1, 2, 3)$, the definition of its Green function $G_{\rm vec}(\tau, \tau')$ is similar to the scalar field, and its Fourier counterpart $G_{\rm vec}(\omega_n)$ can also be expanded in a power series in U:

$$G_{\rm vec}(\omega_n) = G_\alpha(\omega_n) + \sum_{n=1}^{\infty} \left(\frac{U}{\sqrt{2}}\right)^{2n} \frac{1}{\beta} \int_0^\beta d\tau \int_0^\beta d\tau' \int_0^\beta d\tau_{2n} .... \int_0^{\tau_2} d\tau_1 e^{i\omega_n(\tau-\tau')} \left\{ \langle T_\tau d_\alpha(\tau) d_\alpha(\tau') d_\alpha(\tau_{2n}) .... d_\alpha(\tau_1) \rangle \backslash D_{2n}(\tau_1, \tau_2, ..., \tau_{2n}) |^2 \right\}_l . \quad (34)$$

The remaining expectation average over d_α can be calculated as

$$-\langle T_\tau d_\alpha(\tau) d_\alpha(\tau') d_\alpha(\tau_{2n}) .... d_\alpha(\tau_1) \rangle$$
$$= \sum_{i<j} (-1)^{i+j} [G_\alpha(\tau - \tau_i) G_\alpha(\tau' - \tau_j) - G_\alpha(\tau - \tau_j) G_\alpha(\tau' - \tau_i)] \backslash D_{2n}^{ij}| + G_\alpha(\tau - \tau') \backslash D_{2n}|, \quad (35)$$

where $\backslash D_{2n}^{ij}|$ is the so-called *cofactor* of Pfaffian determinant $\backslash D_{2n}|$, which is a Pfaffian determinant of the same type but with τ_i and τ_j removed from the time sequence. Then we can perform the integrals over τ and τ' and get the impurity vector field Green function,

$$G_{\rm vec}(\omega_n) = G_\alpha(\omega_n) + G_\alpha(\omega_n) \Sigma'_{\rm vec}(\omega_n) G_\alpha(\omega_n), \quad (36)$$

from which the corresponding improper self-energy is extracted as

$$\Sigma'_{\rm vec}(\omega_n) = \sum_{n=1}^{\infty} \frac{2i}{\beta} \left(\frac{U}{\sqrt{2}}\right)^{2n} \int_0^\beta d\tau_{2n} \int_0^{\tau_{2n}} d\tau_{2n-1} .... \int_0^{\tau_2} d\tau_1 \left\{ \sum_{i<j} (-1)^{i+j} [\sin \omega_n(\tau_i - \tau_j)] \backslash D_{2n}^{ij}(\tau_1, \tau_2, ..., \tau_{2n})| \left\{ \backslash D_{2n}(\tau_1, \tau_2, ..., \tau_{2n})| \right\}^2 \right\}_l . \quad (37)$$

The self-energies usually used are the proper self energies, which are related to the improper ones by

$$\Sigma_{\rm sc}(\omega_n) = \Sigma'_{\rm sc}(\omega_n) [1 + G_0(\omega_n) \Sigma'_{\rm sc}(\omega_n)]^{-1};$$
$$\Sigma_{\rm vec}(\omega_n) = \Sigma'_{\rm vec}(\omega_n) [1 + G_\alpha(\omega_n) \Sigma'_{\rm vec}(\omega_n)]^{-1}. \quad (38)$$

As far as the leading-order perturbative contributions are concerned, it is not necessary to distinguish the improper self energies from the proper ones.

C. Two-particle Green functions and vertex functions

The method of evaluating the single-particle correlation functions can be applied further to the impurity two-particle Green functions. Then the two-particle vertex functions can be calculated to give a more complete picture of the physical behavior of this model. Generally, three different two-particle correlation functions can be defined. The first one is given by

$$G^{II}_{0,1,2,3}(\omega,\omega',\omega'',\omega''') = \int_0^\beta ... \int_0^\beta d\tau d\tau' d\tau'' d\tau''' \langle T_\tau d_0(\tau) d_1(\tau') d_2(\tau'') d_3(\tau''')\rangle e^{i(\omega\tau+\omega'\tau'+\omega''\tau''+\omega'''\tau''')}. \quad (39)$$

Using the same strategies we used earlier in calculating the impurity single-particle Green functions, we find that

$$G^{II}_{0,1,2,3}(\omega,\omega',\omega'',\omega''') = -\sqrt{2}G_0(\omega)G_\alpha(\omega')G_\alpha(\omega'')G_\alpha(\omega''')\sum_{n=0}^{\infty}\left(\frac{U}{\sqrt{2}}\right)^{2n+1}\int_0^\beta d\tau_{2n+1}\int_0^{\tau_{2n+1}} d\tau_{2n}....\int_0^{\tau_2} d\tau_1$$
$$\left\{\sum_{i,j}\sum_{i',j'}(-1)^{i+j+i'+j'}e^{i(\omega\tau_i+\omega'\tau_j+\omega''\tau_{i'}+\omega'''\tau_{j'})}\backslash D^j_{2n+1}|\backslash D^{i'}_{2n+1}|\backslash D^{j'}_{2n+1}|\right\}_l . \quad (40)$$

The corresponding improper vertex function can be defined as

$$\Gamma'_{0,1,2,3}(\omega,\omega',\omega'',\omega''') = -\sum_{n=0}^{\infty}\frac{\sqrt{2}}{\beta}\left(\frac{U}{\sqrt{2}}\right)^{2n+1}\int_0^\beta d\tau_{2n+1}\int_0^{\tau_{2n+1}} d\tau_{2n}....\int_0^{\tau_2} d\tau_1$$
$$\left\{\sum_{i,j}\sum_{i',j'}(-1)^{i+j+i'+j'}e^{i(\omega\tau_i+\omega'\tau_j+\omega''\tau_{i'}+\omega'''\tau_{j'})}\backslash D^j_{2n+1}|\backslash D^{i'}_{2n+1}|\backslash D^{j'}_{2n+1}|\right\}_l . \quad (41)$$

When all the frequencies are set to zero, this expression simplifies to

$$\Gamma'_{0,1,2,3}(0,0,0,0) = \sum_{n=0}^{\infty}\frac{\sqrt{2}}{\beta}\left(\frac{U}{\sqrt{2}}\right)^{2n+1}\int_0^\beta d\tau_{2n+1}\int_0^{\tau_{2n+1}} d\tau_{2n}....\int_0^{\tau_2} d\tau_1\left\{\sum_i(-1)^i\backslash D^i_{2n+1}(\tau_1,...,\tau_{2n+1})|\right\}^3_l . \quad (42)$$

This vertex function will play an important role when we apply the multiplicative renormalization group to our perturbation expansion in section VII.

The above treatment can be applied to the following two two-particle Green functions straight forwardly:

$$\langle T_\tau d_0(\tau) d_0(\tau') d_\alpha(\tau'') d_\alpha(\tau''')\rangle \quad \text{and} \quad \langle T_\tau d_\alpha(\tau) d_\alpha(\tau') d_\beta(\tau'') d_\beta(\tau''')\rangle,$$

here $\alpha, \beta = 1, 2, 3$, and $\alpha \neq \beta$. Their resulting equations of motion turn out to be

$$
\begin{aligned}
&G^{II}_{0,0,\alpha,\alpha}(\omega, \omega', \omega'', \omega''') \\
&= \beta^2 G_0(\omega) G_\alpha(\omega'') \delta_{\omega',-\omega} \delta_{\omega''',-\omega''} + \beta G_0(\omega) G_0(\omega') G_\alpha(\omega'') G_\alpha(\omega''') \Gamma'_{0,0,\alpha,\alpha}(\omega, \omega', \omega'', \omega'''); \\
&G^{II}_{\alpha,\alpha,\beta,\beta}(\omega, \omega', \omega'', \omega''') \\
&= \beta^2 G_\alpha(\omega) G_\alpha(\omega'') \delta_{\omega',-\omega} \delta_{\omega''',-\omega''} + \beta G_\alpha(\omega) G_\alpha(\omega') G_\alpha(\omega'') G_\alpha(\omega''') \Gamma'_{\alpha,\alpha,\beta,\beta}(\omega, \omega', \omega'', \omega'''), \quad (43)
\end{aligned}
$$

where the improper vertex functions are given by

$$
\begin{aligned}
\Gamma'_{0,0,\alpha,\alpha}(\omega, \omega', \omega'', \omega''') = \sum_{n=0}^{\infty} \left(\frac{U}{\sqrt{2}} \right)^{2n} \frac{2}{\beta} \int_0^\beta d\tau_{2n} \int_0^{\tau_{2n}} d\tau_{2n-1} .... \int_0^{\tau_2} d\tau_1 \left\{ \sum_{i<j} \sum_{i'<j'} (-1)^{i+j+i'+j'} \right. \\
\left. \left[e^{i(\omega\tau_i + \omega'\tau_j)} - e^{i(\omega\tau_j + \omega'\tau_i)} \right] \left[e^{i(\omega''\tau_{i'} + \omega'''\tau_{j'})} - e^{i(\omega''\tau_{j'} + \omega'''\tau_{i'})} \right] \setminus D^{i'j'}_{2n} | \setminus D_{2n} |^2 \right\}_l ; \quad (44)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma'_{\alpha,\alpha,\beta,\beta}(\omega, \omega', \omega'', \omega''') = \sum_{n=0}^{\infty} \left(\frac{U}{\sqrt{2}} \right)^{2n} \frac{1}{\beta} \int_0^\beta d\tau_{2n} \int_0^{\tau_{2n}} d\tau_{2n-1} .... \int_0^{\tau_2} d\tau_1 \left\{ \sum_{i<j} \sum_{i'<j'} (-1)^{i+j+i'+j'} \right. \\
\left. \left[e^{i(\omega\tau_i + \omega'\tau_j)} - e^{i(\omega\tau_j + \omega'\tau_i)} \right] \left[e^{i(\omega''\tau_{i'} + \omega'''\tau_{j'})} - e^{i(\omega''\tau_{j'} + \omega'''\tau_{i'})} \right] \setminus D^{ij}_{2n} | \setminus D^{i'j'}_{2n} | \setminus D_{2n} | \right\}_l . \quad (45)
\end{aligned}
$$

These two improper vertex functions are antisymmetric in their arguments and

$$
\Gamma'_{0,0,\alpha,\alpha}(0, 0, \omega'', \omega''') = \Gamma'_{0,0,\alpha,\alpha}(\omega, \omega', 0, 0) = \Gamma'_{\alpha,\alpha,\beta,\beta}(0, 0, \omega'', \omega''') = \Gamma'_{\alpha,\alpha,\beta,\beta}(\omega, \omega', 0, 0) = 0.
$$

Since the last two vertex functions vanish in the zero frequency limit, we expect only $\Gamma_{0,1,2,3}$ to be important in determining the two-particle interactions in the low-energy regime. It is straightforward to relate the improper and proper vertex function, for example,

$$
\Gamma_{0,1,2,3}(\omega, \omega', \omega'', \omega''') = \frac{1}{z_{\rm sc}} \frac{1}{z^3_{\rm vec}} \Gamma'_{0,1,2,3}(\omega, \omega', \omega'', \omega'''), \quad (46)
$$

where $z_{\rm sc}(\omega_n) = G_{\rm sc}(\omega_n)/G_0(\omega_n)$ and $z_{\rm vec}(\omega_n) = G_{\rm vec}(\omega_n)/G_\alpha(\omega_n)$ denote the wave function renormalization factors for the impurity scalar and vector components, respectively.

D. Dyson equations of the impurity self-energies

The impurity scalar and vector self energies $\Sigma_{\rm sc}$ and $\Sigma_{\rm vec}$ have already been derived in terms of the Pfaffian determinants and their co-factors, which are functions of the unperturbed propagators. However, when it is necessary to sum an infinite series, instead of just

calculating the first few diagrams, it is usually more convenient to express the impurity self energies in terms of the vertex function $\Gamma_{0,1,2,3}$. On the other hand, these two impurity self energies are also dependent of the vertex function $\Gamma_{0,1,2,3}$. The procedures are illustrated by considering $\Sigma_{\rm sc}$ as an example.

Since $\Sigma_{\rm sc}$ is even function of U, there are no first-order perturbation corrections. Also there are no higher-order diagrams in which the self energy is connected to the basic propagator $G_{\rm sc}$ by a single first-order vertex function $\Gamma^{(1)}_{0,1,2,3}(\omega,\omega',\omega'',\omega''') = -U\delta_{\omega+\omega'+\omega''+\omega''',0}$. The second-order perturbative corrections to the self energy $\Sigma_{\rm sc}$ will be derived in Section VII, here we just use the result,

$$\Sigma^{(2)}_{\rm sc}(\omega_n) = -\frac{U^2}{\beta^2}\sum_{\omega_1,\omega_2} G_\alpha(\omega_1)G_\alpha(\omega_2)G_\alpha(\omega_n-\omega_1-\omega_2). \tag{47}$$

It is not hard to see that all the higher order diagrams for $\Sigma_{\rm sc}$ can be obtained from this second-order diagram by insertion of self energy parts into internal unperturbed propagators G_α and replacement of the square on the right by the proper vertex function $\Gamma_{0,1,2,3}$. This diagram is displayed in Fig.1a. Then the Dyson equation of the impurity scalar self energy is derived in terms of the impurity vertex function,

$$\begin{aligned}&\Sigma_{\rm sc}(\omega_n)\\ &= \frac{U}{\beta^2}\sum_{\omega_1,\omega_2} G_{\rm vec}(\omega_1)G_{\rm vec}(\omega_2)G_{\rm vec}(\omega_n-\omega_1-\omega_2)\Gamma_{0,1,2,3}(\omega_2,\omega_n-\omega_1-\omega_2;\omega_1,-\omega_n).\end{aligned} \tag{48}$$

In a similar way, we can build up the Dyson equation for the impurity vector self energy $\Sigma_{\rm vec}$ from its second order perturbative corrections, which will also be derived in section VII,

$$\Sigma^{(2)}_{\rm vec}(\omega_n) = -\frac{U^2}{\beta^2}\sum_{\omega_1,\omega_2} G_\alpha(\omega_1)G_\alpha(\omega_2)G_0(\omega_n-\omega_1-\omega_2). \tag{49}$$

The corresponding Dyson equation is given by

$$\begin{aligned}&\Sigma_{\rm vec}(\omega_n)\\ &= \frac{U}{\beta^2}\sum_{\omega_1,\omega_2} G_{\rm vec}(\omega_1)G_{\rm vec}(\omega_2)G_{\rm sc}(\omega_n-\omega_1-\omega_2)\Gamma_{0,1,2,3}(\omega_2,\omega_n-\omega_1-\omega_2;\omega_1,-\omega_n),\end{aligned} \tag{50}$$

the corresponding diagram is described by Fig.1b. Eq.(48) and Eq.(50) are two basic equations of the model Hamiltonian and also exhibit the relations between the impurity self energies and the two-particle vertex function which is finite in the zero frequency limit.

V. DYNAMICAL SUSCEPTIBILITIES

A. Impurity spin and charge dynamical susceptibilities

As noted earlier the impurity charge and spin density operators are defined by

$$S_d^z = -\frac{i}{2}(d_1 d_2 - d_0 d_3), \quad n_d = -\frac{i}{2}(d_1 d_2 + d_0 d_3). \tag{51}$$

Using these expression the impurity spin and charge density-density correlation functions in the Fourier space can be expressed in the form,

$$\begin{aligned}
\chi_\sigma^{\rm imp}(\omega_n) &= \left(\frac{g\mu}{2}\right)^2 \{\tilde{\chi}_{\rm even}(\omega_n) + \tilde{\chi}_{\rm odd}(\omega_n)\}, \\
\chi_\rho^{\rm imp}(\omega_n) &= \frac{1}{4}\{\tilde{\chi}_{\rm even}(\omega_n) - \tilde{\chi}_{\rm odd}(\omega_n)\},
\end{aligned} \tag{52}$$

with

$$\begin{aligned}
\tilde{\chi}_{\rm even}(\omega_n) &= \frac{1}{\beta}\int_0^\beta d\tau \int_0^\beta d\tau' \{\langle T_\tau d_1(\tau) d_1(\tau') d_2(\tau) d_2(\tau')\rangle + \langle T_\tau d_0(\tau) d_0(\tau') d_3(\tau) d_3(\tau')\rangle\}, \\
\tilde{\chi}_{\rm odd}(\omega_n) &= \frac{2}{\beta}\int_0^\beta d\tau \int_0^\beta d\tau' \langle T_\tau d_0(\tau) d_1(\tau') d_2(\tau') d_3(\tau)\rangle.
\end{aligned} \tag{53}$$

Expressions for these correlation functions in terms of the Pfaffian determinants are rather complicated. We can express them however in terms of the impurity improper self-energies and vertex functions, which is a more convenient form for the calculation of the low-order perturbation corrections. For the even part, the form is given by

$$\begin{aligned}
&\tilde{\chi}_{\rm even}(\omega_n) \\
&= \frac{1}{\beta}\sum_{\omega_1} G_\alpha(\omega_1) G_\alpha(\omega_n - \omega_1) + \frac{1}{\beta}\sum_{\omega_1} G_\alpha(\omega_1) G_0(\omega_n - \omega_1) - \frac{2}{\beta}\sum_{\omega_1} G_\alpha^2(\omega_1) G_\alpha(\omega_n - \omega_1)\Sigma'_{\rm vec}(\omega_1) \\
&\quad -\frac{1}{\beta}\sum_{\omega_1} G_\alpha^2(\omega_1) G_0(\omega_n - \omega_1)\Sigma'_{\rm vec}(\omega_1) - \frac{1}{\beta}\sum_{\omega_1} G_0^2(\omega_1) G_\alpha(\omega_n - \omega_1)\Sigma'_{\rm sc}(\omega_1) \\
&\quad -\frac{1}{\beta^2}\sum_{\omega_1,\omega_2} G_\alpha(\omega_1) G_\alpha(\omega_2) G_\alpha(\omega_n - \omega_1) G_\alpha(\omega_n + \omega_2)\Gamma'_{\alpha,\alpha,\beta,\beta}(\omega_1, \omega_2; \omega_n - \omega_1, -\omega_n - \omega_2) \\
&\quad -\frac{1}{\beta^2}\sum_{\omega_1,\omega_2} G_0(\omega_1) G_0(\omega_2) G_\alpha(\omega_n - \omega_1) G_\alpha(\omega_n + \omega_2)\Gamma'_{0,0,\beta,\beta}(\omega_1, \omega_2; \omega_n - \omega_1, -\omega_n - \omega_2),
\end{aligned} \tag{54}$$

while the odd part is given by

$$\tilde{\chi}_{\rm odd}(\omega_n)$$
$$= \frac{2}{\beta^2}\sum_{\omega_1,\omega_2} G_0(\omega_1)G_\alpha(\omega_2)G_\alpha(\omega_n-\omega_1)G_\alpha(\omega_n+\omega_2)\Gamma'_{0,1,2,3}(\omega_1,\omega_2;\omega_n-\omega_1,-\omega_n-\omega_2), \quad (55)$$

where G_0 and G_α correspond to the unperturbed impurity scalar and vector propagators, respectively.

If the bosonic Matsubara frequency ω_n is taken to zero, the static charge and spin susceptibilities are obtained. In fact, there is another way to derive the impurity static susceptibilities. When an additional term δH is added to the Hamiltonian corresponding to a coupling with an external field, such as a coupling to a uniform magnetic field $\delta H_I = -i(g\mu_B h/2)(d_1 d_2 - d_0 d_3)$ or a chemical potential term $\delta H = -i(\mu/2)(d_1 d_2 + d_0 d_3)$, the partition function can be written in the form

$$Z = \mathrm{Tr}\left\{ e^{-\beta H_0} T_\tau exp\left[-\int_0^\beta H_I(\tau)d\tau \right] exp\left[-\int_0^\beta \delta H_I(\tau')d\tau' \right] \right\}. \quad (56)$$

Then one can expand this partition function to second-order in δH in a power series in U, and derive expressions for the static spin and charge susceptibilities:

$$\chi_\sigma^{\rm imp} = \left(\frac{g\mu_B}{2}\right)^2 \left\{ \tilde{\chi}_{\rm even} + \tilde{\chi}_{\rm odd} \right\}, \qquad \chi_\rho^{\rm imp} = \frac{1}{4}\left\{ \tilde{\chi}_{\rm even} - \tilde{\chi}_{\rm odd} \right\},$$

where $\tilde{\chi}_{\rm even}$ and $\tilde{\chi}_{\rm odd}$ are expressed in terms of the Pfaffian determinant and its co-factors, respectively. Since we do not use these forms to calculate the static susceptibilities, we shall not write them here.

It should be pointed out that the above expressions for $\tilde{\chi}_{\rm even}(\omega_n)$ and $\tilde{\chi}_{\rm odd}(\omega_n)$ can also be converted into alternative forms in terms of the perturbed propagators $G_{\rm sc}$ and $G_{\rm vec}$, and the proper vertex functions $\Gamma_{\alpha,\alpha,\beta,\beta}$ and $\Gamma_{0,0,\alpha,\alpha}$, which are general expressions for the model Hamiltonian.

$$\tilde{\chi}_{\rm even}(\omega_n) = \frac{1}{\beta}\sum_{\omega_1} G_\alpha(\omega_1)G_\alpha(\omega_n-\omega_1) + \frac{1}{\beta}\sum_{\omega_1} G_\alpha(\omega_1)G_0(\omega_n-\omega_1)$$
$$-\frac{2}{\beta}\sum_{\omega_1} G_{\rm vec}(\omega_1)G_\alpha(\omega_n-\omega_1) - \frac{1}{\beta}\sum_{\omega_1} G_{\rm vec}(\omega_1)G_0(\omega_n-\omega_1) - \frac{1}{\beta}\sum_{\omega_1} G_{\rm sc}(\omega_1)G_\alpha(\omega_n-\omega_1)$$
$$+\frac{2}{\beta}\sum_{\omega_1} G_{\rm vec}(\omega_1)G_{\rm vec}(\omega_n-\omega_1) + \frac{1}{\beta}\sum_{\omega_1} G_{\rm vec}(\omega_1)G_{\rm sc}(\omega_n-\omega_1) + \frac{1}{\beta}\sum_{\omega_1} G_{\rm sc}(\omega_1)G_{\rm vec}(\omega_n-\omega_1)$$

$$-\frac{1}{\beta^2}\sum_{\omega_1,\omega_2} G_{\rm vec}(\omega_1)G_{\rm vec}(\omega_2)G_{\rm vec}(\omega_n-\omega_1)G_{\rm vec}(\omega_n+\omega_2)\Gamma_{\alpha,\alpha,\beta,\beta}(\omega_1,\omega_2;\omega_n-\omega_1,-\omega_n-\omega_2)$$

$$-\frac{1}{\beta^2}\sum_{\omega_1,\omega_2} G_{\rm sc}(\omega_1)G_{\rm sc}(\omega_2)G_{\rm vec}(\omega_n-\omega_1)G_{\rm vec}(\omega_n+\omega_2)\Gamma_{0,0,\beta,\beta}(\omega_1,\omega_2;\omega_n-\omega_1,-\omega_n-\omega_2);$$

$$\tilde{\chi}_{\rm odd}(\omega_n)$$

$$=\frac{2}{\beta^2}\sum_{\omega_1,\omega_2} G_{\rm sc}(\omega_1)G_{\rm vec}(\omega_2)G_{\rm vec}(\omega_n-\omega_1)G_{\rm vec}(\omega_n+\omega_2)\Gamma_{0,1,2,3}(\omega_1,\omega_2;\omega_n-\omega_1,-\omega_n-\omega_2). \quad (57)$$

These two expressions can be described diagrammatically in Fig.2, where each term corresponds to one Feynman diagram.

By the way, the free energy (27) can also be converted into a function of the impurity perturbed propagators and vertex function $\tilde{\chi}_{\rm odd}(\omega_n)$ when we use the relation

$$\backslash D_{2n}(\tau_1,\tau_2,...,\tau_{2n})| = -\sum_i (-1)^i G_\alpha(\tau_{2n}-\tau_i)\backslash D^i_{2n-1}|,$$

and then integrate out the imaginary time τ_{2n} The result is

$$(F-F_0)_{\rm imp} = \frac{1}{\beta}\sum_{\omega_n}\int_0^U \tilde{\chi}_{\rm odd}(\omega_n)dU$$

$$=\frac{2}{\beta^3}\int_0^U dU \sum_{\omega_1,\omega_2,\omega_n} G_{\rm sc}(\omega_1)G_{\rm vec}(\omega_2)G_{\rm vec}(\omega_n-\omega_1)G_{\rm vec}(\omega_n+\omega_2)$$
$$\Gamma_{0,1,2,3}(\omega_1,\omega_2;\omega_n-\omega_1,-\omega_n-\omega_2). \quad (58)$$

B. Local conduction electron dynamical susceptibilities and singlet superconducting pairing correlation function

The conduction electron charge and spin density operators are defined in terms of the corresponding Majorana fermions,

$$S^z_c(n) = \frac{1}{2}\left[C^\dagger_\uparrow(n)C_\uparrow(n) - C^\dagger_\downarrow(n)C_\downarrow(n)\right] = -\frac{i}{2}\left[\Psi_1(n)\Psi_2(n)-\Psi_0(n)\Psi_3(n)\right];$$
$$n_c(n) = \frac{1}{2}\left[C^\dagger_\uparrow(n)C_\uparrow(n) + C^\dagger_\downarrow(n)C_\downarrow(n) - 1\right] = -\frac{i}{2}\left[\Psi_1(n)\Psi_2(n)+\Psi_0(n)\Psi_3(n)\right], \quad (59)$$

and their density-density correlation functions at the impurity site are given by

$$\langle T_\tau S^z_c(0,\tau)S^z_c(0,\tau')\rangle = \langle T_\tau n_c(0,\tau)n_c(0,\tau')\rangle$$

$$=\frac{1}{4}\left\{\langle T_\tau\Psi_1(0,\tau)\Psi_1(0,\tau')\Psi_2(0,\tau)\Psi_2(0,\tau')\rangle + \langle T_\tau\Psi_0(0,\tau)\Psi_0(0,\tau')\rangle\langle T_\tau\Psi_3(0,\tau)\Psi_3(0,\tau')\rangle\right\}, \quad (60)$$

where the conduction electron scalar operators decouple from the vector operators because they are free Majorana fermions in the model. Due to the hybridization between the impurity vector field and the conduction electron vector field, the spectral function of these two density-density correlations can be derived from the perturbation expansion series of U in the same way we did for the impurity density-density correlation functions. The final result is expressed as

$$\begin{aligned}\chi^{\rm con}_{\rho,\sigma}(\omega_n) = \frac{1}{4N^2}\sum_{k_1,k_2,k_3,k_4}\Bigg\{&\frac{1}{\beta}\sum_{\omega_1}\left[G^{\alpha}_{k_1,k_2}(\omega_1)+G^{0}_{k_1,k_2}(\omega_1)\right]G^{\alpha}_{k_3,k_4}(\omega_n-\omega_1)\\
&-\frac{1}{\beta^2}\sum_{\omega_1,\omega_2}G_{k_1,d}(\omega_1)G_{k_2,d}(\omega_2)G_{k_3,d}(\omega_n-\omega_1)G_{k_4,d}(\omega_n+\omega_2)\Gamma'_{1,1,2,2}(\omega_1,\omega_2;\omega_n-\omega_1,-\omega_n-\omega_2)\\
&-\frac{1}{\beta}\sum_{\omega_1}G_{k_1,d}(\omega_1)G_{k_2,d}(\omega_1)G^{\alpha}_{k_3,k_4}(\omega_n-\omega_1)\Sigma'_{\rm vec}(\omega_1)\\
&-\frac{1}{\beta}\sum_{\omega_1}G_{k_3,d}(\omega_1)G_{k_4,d}(\omega_1)\left[G^{0}_{k_1,k_2}(\omega_n-\omega_1)+G^{\alpha}_{k_1,k_2}(\omega_n-\omega_1)\right]\Sigma'_{\rm vec}(\omega_1)\Bigg\}. \qquad (61)\end{aligned}$$

Meanwhile, we can consider the singlet superconducting pairing correlation function at the impurity site which is defined by

$$\chi^{\rm con}_{\rm sup}(\omega_n) = \frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\langle T_\tau C_\downarrow(0,\tau)C_\uparrow(0,\tau)C^\dagger_\uparrow(0,\tau')C^\dagger_\downarrow(0,\tau')\rangle e^{i\omega_n(\tau-\tau')}, \qquad (62)$$

where $\omega_n = 2n\pi/\beta$. Using the Majorana fermion representation, we obtain the following expression

$$\begin{aligned}&\langle T_\tau C_\downarrow(0,\tau)C_\uparrow(0,\tau)C^\dagger_\uparrow(0,\tau')C^\dagger_\downarrow(0,\tau')\rangle\\
&=\frac{1}{4}\left\{\langle T_\tau\Psi_1(0,\tau)\Psi_1(0,\tau')\Psi_3(0,\tau)\Psi_3(0,\tau')\rangle+\langle T_\tau\Psi_2(0,\tau)\Psi_2(0,\tau')\Psi_3(0,\tau)\Psi_3(0,\tau')\rangle\right.\\
&\left.+\langle T_\tau\Psi_0(0,\tau)\Psi_0(0,\tau')\rangle\langle T_\tau\Psi_1(0,\tau)\Psi_1(0,\tau')\rangle+\langle T_\tau\Psi_0(0,\tau)\Psi_0(0,\tau')\rangle\langle T_\tau\Psi_2(0,\tau)\Psi_2(0,\tau')\rangle\right\}, \qquad (63)\end{aligned}$$

and, a relation with $\chi^{\rm con}_{\rho,\sigma}$ is found

$$\chi^{\rm con}_{\rm sup}(\omega_n) = 2\chi^{\rm con}_{\rho,\sigma}(\omega_n). \qquad (64)$$

For the present Anderson-type model, we have several general relations between the conduction electron propagators and the impurity propagators. These relations for the Majorana Green functions are

$$G^{\alpha}_{k_1,k_2}(\omega_1) = \frac{\delta_{k_2,k_1}}{i\omega_1 - \epsilon_{k_1}} + \frac{V^2}{N}\frac{G_\alpha(\omega_1)}{(i\omega_1 - \epsilon_{k_1})(i\omega_1 - \epsilon_{k_2})};$$
$$G^{0}_{k_1,k_2}(\omega_1) = \frac{\delta_{k_2,k_1}}{i\omega_1 - \epsilon_{k_1}};$$
$$G_{k_1,d}(\omega_1) = \frac{iV}{\sqrt{N}}\frac{G_\alpha(\omega_1)}{i\omega_1 - \epsilon_{k_1}}. \tag{65}$$

Substituting these relations into the expression (61), we finally get the result

$$\begin{aligned}
&\chi^{\rm con}_{\rho,\sigma}(\omega_n) - \chi^{\rm con.(0)}_{\rho,\sigma}(\omega_n) \\
&= i\frac{3V^2}{4\beta}(\pi\rho)^3\sum_{\omega_1} G^2_\alpha(\omega_1)\Sigma'_{\rm vec}(\omega_1){\rm sgn}(\omega_n - \omega_1) + \frac{V^4}{2\beta}(\pi\rho)^4\sum_{\omega_1} G^2_\alpha(\omega_1)G_\alpha(\omega_n - \omega_1)\Sigma'_{\rm vec}(\omega_1) \\
&\quad -\frac{V^4}{4\beta^2}(\pi\rho)^4\sum_{\omega_1,\omega_2} G_\alpha(\omega_1)G_\alpha(\omega_2)G_\alpha(\omega_n - \omega_1)G_\alpha(\omega_n + \omega_2) \\
&\qquad \Gamma'_{1,1,2,2}(\omega'_1\omega_2; \omega_n - \omega_1, -\omega_n - \omega_2){\rm sgn}\omega_1{\rm sgn}\omega_2{\rm sgn}(\omega_{\rm n} - \omega_1){\rm sgn}(\omega_{\rm n} + \omega_2),
\end{aligned} \tag{66}$$

where the density-density spectra in the case of $U = 0$ are

$$\begin{aligned}
\chi^{\rm con.(0)}_{\rho,\sigma}(\omega_n) &= \frac{1}{2\beta}(\pi\rho)^2\sum_{\omega_1}{\rm sgn}\omega_1{\rm sgn}(\omega_n - \omega_1) + i\frac{3V^2}{4\beta}(\pi\rho)^3\sum_{\omega_1} G_\alpha(\omega_1){\rm sgn}(\omega_n - \omega_1) \\
&\quad + \frac{V^4}{4\beta}(\pi\rho)^4\sum_{\omega_1} G_\alpha(\omega_1)G_\alpha(\omega_n - \omega_1),
\end{aligned} \tag{67}$$

which is a regular contribution. When the Matsubara frequency is taken to zero, the corresponding static susceptibilities are also obtained. With the help of the relation (64), the conduction electron singlet superconducting pairing correlation function at the impurity site can also be obtained.

VI. MAIN PERTURBATIONAL RESULTS OF THE MODEL

A. Leading specific heat correction

In the section IV, an expression for the impurity free energy was derived in a power series in U. Since $F^{(0)}_{\rm imp}$ is the corresponding free energy in the case of $U = 0$ and is just a regular contribution, the first singular contribution to the free energy comes in second order in U, and is

$$F_{\rm imp}^{(2)} = -\frac{U^2}{4\beta}\int_0^\beta d\tau_2 \int_0^\beta d\tau_1 G_\alpha^3(\tau_1-\tau_2){\rm sgn}(\tau_2-\tau_1)$$
$$= \frac{U^2}{2\beta^3}\sum_{\omega_1,\omega_2,\omega_3} G_\alpha(\omega_1)G_\alpha(\omega_2)G_0(\omega_3)G_\alpha(\omega_1+\omega_2+\omega_3). \quad (68)$$

The singular contribution from this term is

$$F_{\rm imp}^{(2)} \approx \frac{1}{4}\left(\frac{U}{\pi\Delta}\right)^2 \frac{(\pi T)^2}{\pi\Delta}\ln\left(\frac{\Delta}{T}\right),$$

which gives a singular correction to the impurity specific heat,

$$C_{\rm imp}^{(2)} \approx \frac{\pi^2}{2}\left(\frac{U}{\pi\Delta}\right)^2 \frac{T}{\pi\Delta}\ln\left(\frac{\Delta}{T}\right).$$

B. Self energies and electrical resistivity

From the general expressions for the self-energies, it can be seen that the lowest order self energy corrections are of second order. For the impurity scalar field, the perturbed self energy is

$$\Sigma_{\rm sc}^{(2)}(\omega_n) = -2i\frac{U^2}{\beta}\int_0^\beta d\tau_2 \int_0^{\tau_2} d\tau_1 G_\alpha^3(\tau_1-\tau_2)\sin[\omega_n(\tau_1-\tau_2)]$$
$$= -\frac{U^2}{2}\int_{-\beta}^{\beta} G_\alpha^3(\tau)e^{i\omega_n\tau}d\tau. \quad (69)$$

In the imaginary time space, this self energy is

$$\Sigma_{\rm sc}^{(2)}(\tau) = -U^2 G_\alpha^3(\tau), \quad (70)$$

which can be represented by a Feynman diagram shown in Fig.3a, and its Fourier transform is given by

$$\Sigma_{\rm sc}^{(2)}(\omega_n) = -\frac{U^2}{\beta^2}\sum_{\omega_1,\omega_2} G_\alpha(\omega_1)G_\alpha(\omega_2)G_\alpha(\omega_n-\omega_1-\omega_2). \quad (71)$$

The imaginary part of its retarded Fourier transform after integration is given by

$${\rm Im}\Sigma_{\rm sc}^{(2)}(\omega,T) \approx -\frac{\Delta}{2}\left(\frac{U}{\pi\Delta}\right)^2\left[\left(\frac{\omega}{\Delta}\right)^2+\left(\frac{\pi T}{\Delta}\right)^2\right]. \quad (72)$$

Here the analytical continuation has been performed, and according to the Kramers-Kronig relation the real part should be

$$\mathrm{Re}\Sigma^{(2)}_{\mathrm{sc}}(\omega,T) \approx -\left(\frac{U}{\pi\Delta}\right)^2 \omega. \tag{73}$$

Moreover, the renormalization factor of the wave function or quasiparticle weight ($z_{\mathrm{sc}}(\omega_n)$ taken at $\omega_n = 0$) to second order is given by

$$z^{(2)}_{\mathrm{sc}} = \left[1 - \frac{\partial \mathrm{Re}\Sigma^{(2)}_{\mathrm{sc}}}{\partial\omega}\right]^{-1}_{\omega=0} = \left[1 + \left(\frac{U}{\pi\Delta}\right)^2\right]^{-1}. \tag{74}$$

The scalar self energy exhibits the normal local FL behavior, and such a behavior ensures the fermionic zero mode in the unperturbed Hamiltonian is preserved in the presence of perturbations.

On the other hand, the vector perturbed self energy can be written down from the general expression,

$$\begin{aligned}\Sigma^{(2)}_{\mathrm{vec}}(\omega_n) &= -i\frac{U^2}{\beta}\int_0^\beta d\tau_2 \int_0^{\tau_2} d\tau_1 G^2_\alpha(\tau_1-\tau_2)\sin[\omega_n(\tau_1-\tau_2)] \\ &= \frac{U^2}{4}\int_{-\beta}^{\beta} d\tau G^2_\alpha(\tau)\mathrm{sgn}(\tau)e^{i\omega_n\tau}.\end{aligned} \tag{75}$$

The sign function appears when the double integrals are reduced to a single integral. In the same way, the self energy in imaginary time space becomes

$$\Sigma^{(2)}_{\mathrm{vec}}(\tau) = -U^2 G^2_\alpha(\tau) G_0(\tau), \tag{76}$$

which corresponds to the Feynman diagram in Fig.3b, and the Fourier transform is given by

$$\Sigma^{(2)}_{\mathrm{vec}}(\omega_n) = -\frac{U^2}{\beta^2}\sum_{\omega_1,\omega_2} G_\alpha(\omega_1)G_\alpha(\omega_2)G_0(\omega_n-\omega_1-\omega_2). \tag{77}$$

Since this self energy involves the fermionic zero mode, the calculations of its spectral function becomes subtle. The retarded imaginary part is given by

$$\mathrm{Im}\Sigma^{(2)}_{\mathrm{vec}}(\omega,T) = -\frac{\pi}{2}\left(\frac{U}{\pi\Delta}\right)^2 |\omega|\coth\left(\frac{|\omega|}{2T}\right) \sim \begin{cases} -(\frac{U}{\pi\Delta})^2(\pi T), & |\omega| \ll T \\ -\frac{\pi}{2}(\frac{U}{\pi\Delta})^2|\omega|, & \Delta \gg |\omega| \gg T. \end{cases} \tag{78}$$

The corresponding real part is also obtained as

$$\mathrm{Re}\Sigma^{(2)}_{\mathrm{vec}}(\omega,T) \approx \left(\frac{U}{\pi\Delta}\right)^2 \omega \ln\left(\frac{x}{\Delta}\right), \tag{79}$$

where $x = \max(|\omega|, T)$. Such a self energy is greatly different from the form given by the ordinary FL theory, in particular, the renormalization factor of the wave function or the quasiparticle weight logarithmically approaches to zero as $T \to 0$,

$$z^{(2)}_{\mathrm{vec}} = \left[1 - \frac{\partial \mathrm{Re}\Sigma^{(2)}_{\mathrm{vec}}}{\partial\omega}\right]^{-1}_{\omega=0} = \left[1 + \left(\frac{U}{\pi\Delta}\right)^2 \ln\left(\frac{\Delta}{T}\right)\right]^{-1}, \tag{80}$$

which implies that the impurity vector self-energy displays the marginal FL behavior.

There is a singular temperature-dependent contribution to $\mathrm{Im}\Sigma^{(2)}_{\mathrm{vec}}(0,T)$, and the conduction electron t-matrix determined by the Green function of the impurity vector propagator has been derived as follows

$$t(\omega,T) = \frac{V^2}{N} G_{\mathrm{vec}}(\omega,T). \tag{81}$$

Thus, the imaginary part of this t-matrix can be expressed in terms of the retarded self energy for the impurity vector field:

$$\begin{aligned}&\mathrm{Im}t(\omega,T)\\ &= -\frac{1}{\pi\rho N}\frac{\Delta^2}{(\omega - \mathrm{Re}\Sigma_{\mathrm{vec}})^2 + (\Delta - \mathrm{Im}\Sigma_{\mathrm{vec}})^2} + \frac{\Delta}{\pi\rho N}\frac{\mathrm{Im}\Sigma_{\mathrm{vec}}}{(\omega - \mathrm{Re}\Sigma_{\mathrm{vec}})^2 + (\Delta - \mathrm{Im}\Sigma_{\mathrm{vec}})^2},\end{aligned} \tag{82}$$

where the first term describes the elastic scatterings, while the second term describes the inelastic scatterings.

Assuming the conduction electrons incoherently scatter from the dilute magnetic impurities, the linear response theory allows the electrical conductivity to be expressed in terms of the following Kubo formula

$$\sigma(T) = -\frac{2}{3} e^2 v_f^2 \rho \int_{-\infty}^{\infty} \tau(\omega,T) \frac{\partial f}{\partial\omega} d\omega, \tag{83}$$

here $f(x)$ is the Fermi distribution function, v_f is the Fermi velocity of the conduction electrons with charge e and density of states ρ, and $\tau(\omega,T)$ is the electron relaxation time,

which is related to the t-matrix

$$\tau^{-1} = -2N_{\rm imp}{\rm Im}t(\omega, T)$$

where $N_{\rm imp}$ is the total number of the impurities, and $n_{\rm imp} = N_{\rm imp}/N$ is the impurity concentration and is supposed to be much less than unity. On substituting the second order self energy for the impurity vector field into equations (82) and (83), we obtain the electrical resistivity in the low temperature regime,

$$\rho(T) \approx \frac{3\pi n_{\rm imp}}{e^2}\left[1 + \left(\frac{U}{\pi\Delta}\right)^2\left(\frac{\pi T}{\Delta}\right)\right]. \tag{84}$$

Here we have just taken into account the contributions of order $(\frac{U}{\pi\Delta})^2$. Therefore, up to the second order of perturbation series, the electrical resistivity has a linear temperature dependence, which is a direct consequence of the anomalous behavior of the imaginary part of the impurity vector fermion self energy. *Such a resistivity makes the weak coupling fixed point of the present model differ from the strong coupling fixed point of the two-channel Kondo model.*

C. Vertex functions in the zero-frequency limit

As we said earlier, in the zero-frequency limit there is only one non-zero two-particle vertex function $\Gamma'_{0,1,2,3}$, the other two vertex functions vanish. Its general expression in terms of U has been given in section IV. In the first order perturbation, it is trivial to obtain as

$$\Gamma'^{(1)}_{0,1,2,3}(\omega, \omega', \omega'', \omega''') = -U\delta_{\omega+\omega'+\omega''+\omega''',0}, \tag{85}$$

the corresponding diagram is described in Fig.4a. To order U^3, it is given by

$$\Gamma'^{(3)}_{0,1,2,3}(0,0,0,0) = -\frac{U^3}{12\beta}\int_0^\beta d\tau_3\int_0^\beta d\tau_2\int_0^\beta d\tau_1\left[G_\alpha(\tau_1-\tau_2) - G_\alpha(\tau_1-\tau_3) + G_\alpha(\tau_2-\tau_3)\right]_l^3 \\ {\rm sgn}(\tau_2-\tau_1){\rm sgn}(\tau_3-\tau_2){\rm sgn}(\tau_3-\tau_1). \tag{86}$$

Using the diagrammatic method, we can clarify that four basic Feynman diagrams have contributions to this zero-frequency vertex function, and then re-group these terms to get the following expression,

$$\begin{aligned}\Gamma'^{(3)}_{0,1,2,3}(0,0,0,0) = \frac{U^3}{2\beta}\int_0^\beta d\tau_3\int_0^\beta d\tau_2\int_0^\beta d\tau_1\{&3G_\alpha^2(\tau_1-\tau_2)G_\alpha(\tau_1-\tau_3)\mathrm{sgn}(\tau_1-\tau_3)\\ &-G_\alpha^2(\tau_1-\tau_2)G_\alpha(\tau_1-\tau_3)\mathrm{sgn}(\tau_1-\tau_2)-G_\alpha^2(\tau_1-\tau_2)G_\alpha(\tau_1-\tau_3)\mathrm{sgn}(\tau_2-\tau_3)\\ &-G_\alpha^2(\tau_1-\tau_2)G_\alpha(\tau_1-\tau_3)G_\alpha(\tau_2-\tau_3)\mathrm{sgn}(\tau_1-\tau_2)\},\end{aligned} \tag{87}$$

where each term is described by the respective Feynman diagram (b), (c), (d), (e) in Fig.4, and it is easily found that only the first term (Fig.4b) gives the leading singular contributions, while the remaining three terms can be neglected in the limit $T \to 0$. Thus, the leading third order perturbation correction to the vertex function is

$$\begin{aligned}\Gamma'^{(3)}_{0,1,2,3}(0,0,0,0) &= -3U^3\left[-\frac{1}{\beta}\sum_{\omega_n}G^2(\omega_n)\right]\left[-\frac{1}{\beta}\sum_{\omega_n}G(\omega_n)G_0(\omega_n)\right]\\ &\approx -3U\left(\frac{U}{\pi\Delta}\right)^2\ln\left(\frac{\Delta}{T}\right), \qquad \text{for} \quad T\ll\Delta,\end{aligned} \tag{88}$$

and a logarithmic singularity appears. Up to third order in U, the proper vertex function is calculated and exhibits the same singularity as well,

$$\Gamma_{0,1,2,3}(0,0,0,0) = z_{\mathrm{sc}}^{-1}z_{\mathrm{vec}}^{-3}\Gamma'_{0,1,2,3}(0,0,0,0) \approx -U\left\{1+\left(\frac{U}{\pi\Delta}\right)^2+6\left(\frac{U}{\pi\Delta}\right)^2\ln\left(\frac{\Delta}{T}\right)\right\}, \tag{89}$$

which implies that the higher order perturbative expansion terms will become as important as low-order contributions in determining the low-temperature behavior of the model. At least the leading order logarithmic terms have to be summed to obtain the correct low-temperature behavior. The summation of all the leading logarithmic contributions for the vertex function is the so-called parquet approximation. In section VII, we will use another equivalent approach: the multiplicative renormalization-group method, to find the final results.

D. Impurity static susceptibilities

We have already obtained general expressions for the impurity dynamical susceptibilities in terms of the impurity scalar and vector self energies and their two-particle vertex functions. Since these associated self energies and vertex functions have been obtained in the lowest order perturbation theory, we can use them to derive the even and odd static susceptibilities. In the zeroth order, there are two contributions,

$$\tilde{\chi}^{(0)}_{\rm even} = \left[-\frac{1}{\beta}\sum_{\omega_n} G^2_\alpha(\omega_n)\right] + \left[-\frac{1}{\beta}\sum_{\omega_n} G_\alpha(\omega_n)G_0(\omega_n)\right], \tag{90}$$

where the first term is a regular FL like contribution, while the second term shown in Fig.5a is singular, and can be expressed as

$$\left[-\frac{1}{\beta}\sum_{\omega_n} G_\alpha(\omega_n)G_0(\omega_n)\right] = \left(\frac{1}{\pi\Delta}\right)\left[\psi\left(\frac{1}{2}+\frac{\Delta}{2\pi T}\right) - \psi\left(\frac{1}{2}\right)\right], \tag{91}$$

where $\psi(x)$ is the digamma function, and when $T \ll \Delta$, $\left[\psi\left(\frac{1}{2}+\frac{\Delta}{2\pi T}\right) - \psi\left(\frac{1}{2}\right)\right] \approx \ln\left(\frac{\Delta}{T}\right)$, and $\left[-\frac{1}{\beta}\sum_{\omega_n} G^2_\alpha(\omega_n)\right] \approx \left(\frac{1}{\pi\Delta}\right)$. Thus, the singular contribution gives rise to a logarithmic divergence in the impurity static susceptibilities. In the first order in U, there is also one singular term corresponding to the diagram in Fig.5b. From the first-order vertex correction for $\Gamma'_{0,1,2,3}(\omega_1,\omega_2;-\omega_1,-\omega_2)$, we get

$$\tilde{\chi}^{(1)}_{\rm odd} = 2U\left[-\frac{1}{\beta}\sum_{\omega_n} G^2(\omega_n)\right]\left[-\frac{1}{\beta}\sum_{\omega_n} G(\omega_n)G_0(\omega_n)\right]. \tag{92}$$

While to second order in U, the perturbative corrections to the impurity static susceptibilities become complicated,

$$\begin{aligned}\tilde{\chi}^{(2)}_{\rm even} = {} & U^2\left[-\frac{1}{\beta}\sum_{\omega_n} G^2(\omega_n)\right]^2\left[-\frac{1}{\beta}\sum_{\omega_n} G(\omega_n)G_0(\omega_n)\right] \\ & + U^2\left[-\frac{1}{\beta}\sum_{\omega_n} G^2(\omega_n)\right]\left[-\frac{1}{\beta}\sum_{\omega_n} G(\omega_n)G_0(\omega_n)\right]^2 + \text{less singular terms},\end{aligned} \tag{93}$$

where we have used the second-order results for the impurity self-energies $\Sigma'^{(2)}_{\rm sc}(\omega_n)$ and $\Sigma'^{(2)}_{\rm vec}(\omega_n)$ and vertex functions $\Gamma'^{(2)}_{1,1,2,2}(\omega_1,\omega_2;-\omega_1,-\omega_2)$ and $\Gamma'^{(2)}_{0,0,3,3}(\omega_1,\omega_2;-\omega_1,-\omega_2)$. Note that apart from the logarithmic contributions represented by the diagram in Fig.5c, there

appears another singular term: a squared logarithmic term arising from diagram (d) in Fig.5. Therefore, up to the second order in the perturbation series, the singular parts of the impurity spin and charge static susceptibilities are given by

$$\chi_{\sigma}^{\rm imp} \approx \left(\frac{g\mu_B}{2}\right)^2 \left\{ \left[1 + 2\left(\frac{U}{\pi\Delta}\right) + \left(\frac{U}{\pi\Delta}\right)^2 \right] \left(\frac{1}{\pi\Delta}\right) \ln\left(\frac{\Delta}{T}\right) + \left(\frac{U}{\pi\Delta}\right)^2 \left(\frac{1}{\pi\Delta}\right) \ln^2\left(\frac{\Delta}{T}\right) \right\},$$
$$\chi_{c}^{\rm imp} \approx \frac{1}{4} \left\{ \left[1 - 2\left(\frac{U}{\pi\Delta}\right) + \left(\frac{U}{\pi\Delta}\right)^2 \right] \left(\frac{1}{\pi\Delta}\right) \ln\left(\frac{\Delta}{T}\right) + \left(\frac{U}{\pi\Delta}\right)^2 \left(\frac{1}{\pi\Delta}\right) \ln^2\left(\frac{\Delta}{T}\right) \right\}. \quad (94)$$

E. Conduction electron local static susceptibilities and singlet superconducting pairing susceptibility

For the local static susceptibilities of the conduction electrons, it is straight forward to obtain results for the local static susceptibilities of the conduction electrons by taking the zero-frequency limit for the dynamical susceptibilities. As we have stated before, the unperturbed charge and spin static susceptibilities of the conduction electrons at the impurity site are regular. Then, after subtracting the contribution of the free conduction electrons ($V = 0$), it is expressed as

$$\delta\chi_{\rho,\sigma}^{\rm con.(0)} \approx \frac{1}{4}(\pi\rho V)^4 \left[-\frac{1}{\beta} \sum_{\omega_n} G_{\alpha}^2(\omega_n) \right] \approx \frac{1}{4}(\pi\rho V)^4 \left(\frac{1}{\pi\Delta}\right). \quad (95)$$

Unlike the impurity static susceptibilities, the first perturbative correction comes from second order in U. Since the vertex correction $\Gamma'^{(2)}_{1,1,2,2}(\omega_1, \omega_2; \omega_n - \omega_1, -\omega_n - \omega_2)$ involves a singular contribution analogous to the diagram Fig.5c, the singular part of the second-order correction is

$$\delta\chi_{\rho,\sigma}^{\rm con.(2)} = -\frac{U^2}{4}(\pi\rho V)^4 \left[-\frac{1}{\beta} \sum_{\omega_n} G_{\alpha}^2(\omega_n) \right]^2 \left[-\frac{1}{\beta} \sum_{\omega_n} G_{\alpha}(\omega_n) G_0(\omega_n) \right]. \quad (96)$$

When $T \ll \Delta$, up to the second order in U, the static susceptibilities are given by

$$\delta\chi_{\rho,\sigma}^{\rm con} \approx \frac{1}{4}(\pi\rho V)^4 \left(\frac{1}{\pi\Delta}\right) - \frac{1}{4}(\pi\rho V)^4 \left(\frac{U}{\pi\Delta}\right)^2 \left(\frac{1}{\pi\Delta}\right) \ln\left(\frac{\Delta}{T}\right), \quad (97)$$

where there appears a logarithmically temperature dependent contribution. The possible reason is that the impurity vector field transfers its low-energy singularity into the local

conduction electrons through the hybridization, so there is also a logarithmic temperature dependence of the impurity static susceptibilities to second order in U.

Another important result is the second order correction to the singlet superconducting pairing correlation function at the impurity site, its static susceptibility can also be obtained through the relation $\chi_{\rm sup}^{\rm con} = 2\chi_{\rho,\sigma}^{\rm imp}$,

$$\delta\chi_{\rm sup}^{\rm con} \approx \frac{1}{2}(\pi\rho V)^4 \left(\frac{1}{\pi\Delta}\right) - \frac{1}{2}(\pi\rho V)^4 \left(\frac{U}{\pi\Delta}\right)^2 \left(\frac{1}{\pi\Delta}\right) \ln\left(\frac{\Delta}{T}\right), \tag{98}$$

which indicates that the conduction electrons form a singlet pairing resonant state around the local impurity site, different from a single-particle resonant state in the single-channel Kondo problem. It should be pointed out that this pairing resonance is pinned at the Fermi level (chemical potential).

VII. MULTIPLICATIVE RENORMALIZATION-GROUP ANALYSIS

Since the perturbation theory on the present single-impurity model gives logarithmic contributions to the impurity vertex function and vector field propagator, the summation of the leading order logarithmic terms should lead to some form of scaling behavior. One way to sum these terms is by using the usual parquet approximation. Here we use another equivalent method to extend our second order perturbational results for the impurity self energies and vertex function: the multiplicative renormalization-group (RG) [16].

The usual multiplicative RG is a simple transformation procedure in which the Green functions, vertices and coupling constants are multiplied by real, frequency independent factors. The requirement that the two-particle interaction form defined in the model Hamiltonian be satisfied by both the original and the transformed quantities gives a relation between them. The classical formulation was used to obtain scaling laws for the X-ray edge problem and the single-channel Kondo problem [17,18]. For the logarithmic problem, generally, it is important to give a proper definition of the invariant coupling, the temperature dependence of which characterizes the behavior of the system. On the physical ground, we

take the hybridization width Δ as a natural scaling parameter. The perturbed impurity Green functions and the important two-particle vertex function are written in the form,

$$\begin{aligned} &G_{\rm sc}(\omega) = z_{\rm sc} G_0(\omega), && G_{\rm vec}(\omega) = z_{\rm vec} G_\alpha(\omega); \\ &\bar{U}\tilde{\Gamma}_{0,1,2,3} \equiv -\left(\frac{1}{\pi\Delta}\right)\Gamma_{0,1,2,3}(0,0,0,0), && \bar{U} \equiv \left(\frac{U}{\pi\Delta}\right), \end{aligned} \tag{99}$$

where G_0 and G_α are the unperturbed Green functions and $\bar{U}$ is the dimensionless bare vertex. For simplicity, in the functions of $z_{\rm sc}$, $z_{\rm vec}$, and $\tilde{\Gamma}_{0,1,2,3}$, the frequency variables are fixed at the Fermi level and only the temperature variables are retained. If the interaction is cut off at the energy Δ, the Green functions and vertex depend, as a rule, on the relative energies Δ/T.

Multiplicative RG is usually formulated as the transformation induced by a change of the cut-off parameter from Δ to Δ', *i.e.*.

$$\begin{aligned} &z_{\rm sc}\left(\frac{\Delta'}{T}, \bar{U}'\right) = z_1 z_{\rm sc}\left(\frac{\Delta}{T}, \bar{U}\right), \\ &z_{\rm vec}\left(\frac{\Delta'}{T}, \bar{U}'\right) = z_2 z_{\rm vec}\left(\frac{\Delta}{T}, \bar{U}\right), \\ &\tilde{\Gamma}_{0,1,2,3}\left(\frac{\Delta'}{T}, \bar{U}'\right) = z_3^{-1}\tilde{\Gamma}_{0,1,2,3}\left(\frac{\Delta}{T}, \bar{U}\right), \\ &\bar{U}'\left(\frac{\Delta'}{\Delta}\right) = z_1^{-1/2} z_2^{-3/2} z_3 \bar{U}, \end{aligned} \tag{100}$$

where z_1, z_2, and z_3 are independent of the temperature variables. Such a transformation can keep the four-legged vertex unchanged, which corresponds to the two-particle interaction of the model Hamiltonian. Since $z_{\rm sc}^{(2)} = [1 + \bar{U}^2]^{-1}$ does not depends on the cut-off factor at all, it can not induce any essential renormalization. For simplicity, we choose $z_1 = 1$. Therefore, when the above relations are obeyed, the cut-off dependent $\bar{U}'$, a self-consistent solution to these equations is given by

$$\bar{U}'\left(\frac{\Delta'}{\Delta}\right)\tilde{\Gamma}_{0,1,2,3}\left(\frac{\Delta'}{T}, \bar{U}'\right)\left[z_{\rm vec}\left(\frac{\Delta'}{T}, \bar{U}'\right)\right]^{3/2} = \bar{U}\tilde{\Gamma}_{0,1,2,3}\left(\frac{\Delta}{T}, \bar{U}\right)\left[z_{\rm vec}\left(\frac{\Delta}{T}, \bar{U}\right)\right]^{3/2}, \tag{101}$$

or

$$\bar{U}'\left(\frac{\Delta'}{\Delta}\right) = \bar{U}\frac{\tilde{\Gamma}_{0,1,2,3}(\frac{\Delta}{T},\bar{U})}{\tilde{\Gamma}_{0,1,2,3}(\frac{\Delta'}{T},\bar{U}')}\frac{\left[z_{\rm vec}(\frac{\Delta}{T},\bar{U})\right]^{3/2}}{\left[z_{\rm vec}(\frac{\Delta'}{T},\bar{U}')\right]^{3/2}}, \tag{102}$$

which is the invariant coupling constant of this model. When the denominator is normalized to unity at $T = \Delta'$, a simple definition of the invariant coupling constant is

$$\bar{U}_{\rm inv} = \bar{U}\tilde{\Gamma}_{0,1,2,3}\left(\frac{\Delta}{\Delta'},\bar{U}\right)\left[z_{\rm vec}\left(\frac{\Delta}{\Delta'},\bar{U}\right)\right]^{3/2}. \tag{103}$$

Up to the second order in $\bar{U}$, the impurity renormalization factors of the wave functions and the vertex function have already been obtained as

$$\begin{aligned} z^{(2)}_{\rm vec}\left(\frac{\Delta}{T},\bar{U}\right) &= \frac{1}{1+\bar{U}^2\ln\left(\frac{\Delta}{T}\right)}, \\ \tilde{\Gamma}_{0,1,2,3}\left(\frac{\Delta}{T},\bar{U}\right) &= 1+\bar{U}^2+6\bar{U}^2\ln\left(\frac{\Delta}{T}\right). \end{aligned} \tag{104}$$

Actually, the non-logarithmic parts are not essential in the present RG approach. From the above second-order results, the multiplicative factors z_2 and z_3 can be derived

$$z_2 = \frac{1}{1+\bar{U}^2\ln\left(\frac{\Delta'}{\Delta}\right)}, \quad z_3 = \frac{1}{1+6\bar{U}^2\ln\left(\frac{\Delta'}{\Delta}\right)}, \tag{105}$$

and then the dimensionless invariant coupling is given by

$$\bar{U}' = \bar{U}\left[1-\frac{9}{2}\bar{U}^2\ln\left(\frac{\Delta'}{\Delta}\right)+.....\right], \tag{106}$$

from which the basic RG scaling equation can be cast into a differential form,

$$\frac{d\ln\bar{U}}{d\ln\Delta} = -\frac{9}{2}\bar{U}^2. \tag{107}$$

The characteristic feature of this scaling equation is that the dimensionless coupling constant $\bar{U} = U/(\pi\Delta)$ always increases as the high-energy scale Δ is reduced. Integrating this differential equation gives the relation,

$$\bar{U}'^2 = \frac{\bar{U}^2}{1+9\bar{U}^2\ln\left(\frac{\Delta'}{\Delta}\right)}. \tag{108}$$

The scaling relation can be used to extend the second-order perturbational results, and effectively sum the leading order logarithmic terms. The scaling trajectory can be traversed

from the initial parameters Δ and U, to an effective Δ' of order T, and an effective coupling constant $\bar{U}'$, which becomes temperature dependent,

$$\bar{U}^2(T) = \frac{\bar{U}^2}{1 + 9\bar{U}^2 \ln\left(\frac{T}{\Delta}\right)}.$$

When we substitute this new temperature dependent coupling constant into the second-order perturbational results, we will get the results which are equivalent to a summation of the leading order logarithmic terms. For example, the electrical resistivity in the second order perturbation theory was calculated to be

$$\rho^{(2)}(T) = \frac{3\pi n_{\rm imp}}{e^2}\left[1 + \bar{U}^2\left(\frac{\pi T}{\Delta}\right)\right]. \tag{109}$$

After substituting the new coupling parameter, it becomes

$$\rho(T) = \frac{3\pi n_{\rm imp}}{e^2}\left[1 + \frac{\bar{U}^2}{1 + 9\bar{U}^2 \ln\left(\frac{T}{\Delta}\right)}\left(\frac{\pi T}{\Delta}\right)\right]. \tag{110}$$

The $\ln T$ term is in the denominator due to the summation of the leading order logarithmic series. As $T \to 0$, the difficulties arising from the $\ln T$ term become more severe as there is a divergence at finite temperature,

$$T_c = \Delta \exp\left[-\frac{1}{9}\left(\frac{\pi\Delta}{U}\right)^2\right], \tag{111}$$

which is a new weak-coupling low-temperature energy scale. When $T > T_c$, the perturbational scaling can be extended down to an effective hybridization width Δ' and effective coupling constant $\bar{U}'$, while $T < T_c$, the electrical resistivity derived from perturbation scaling diverges.

In the conventional RG treatments for the single-impurity Kondo problems [19], the conduction electron band-width D is usually chosen as the high energy cut-off factor, and the coupling parameters are renormalized as the bandwidth D is decreased. However, for the Anderson-type impurity model, the perturbation expansions in the interaction parameter U by Yamada and Yosida [14,15] and the numerical RG calculations [20] have shown that the model provides its own *intrinsic* high-energy cut-off of the order of the hybridization width

Δ ($\Delta > U$), inconsistent with using the conduction electron bandwidth as the effective cut-off factor [21]. Similarly, the present perturbation results of the compactified Anderson impurity model have also proved that the parameter Δ plays the role of the high-energy cut-off factor. In fact, the behavior of the present model is independent of the conduction electron bandwidth, which can be taken to the infinite bandwidth limit without lost of generality. It is noteworthy to point out that the scaling equation (107) is a for the dimensionless coupling parameter $\bar{U}$. Due to $\bar{U} = U/(\pi\Delta)$, there is another form of the RG scaling in terms of the dimensional coupling parameter U, $\frac{dU}{d\ln\Delta} = U\left[1 - \frac{9}{2}\left(\frac{U}{\pi\Delta}\right)^2\right]$. However, this form of the scaling equation is not useful for discussing the scaling behavior of the interaction because the parameter U is not the RG invariant coupling parameter. The RG scaling of the model has explicitly displayed in Eq.(107).

What have we learned from these scaling arguments? First of all, the hybridization width Δ can be reduced dramatically for the calculation of thermodynamic behavior from its initial value down to the thermal energy scale and still be described by a model of the same form but with a renormalized coupling constant. Second, there is a subtle point in the perturbation treatments. Basically the model has been described by two independent coupling parameters Δ and U, and the perturbation expansion is based on the following two-particle vertex function,

$$\Gamma_{0,1,2,3}(0,0,0,0) = -U\tilde{\Gamma}_{0,1,2,3}(\bar{U}), \tag{112}$$

where $\tilde{\Gamma}_{0,1,2,3}$ is the corresponding dimensionless vertex function and only depends on the dimensionless coupling constant $\bar{U} = \frac{U}{\pi\Delta}$. The important point is that dimensionless coupling parameter $|\bar{U}|$ increases as the scaling parameter is reduced. $\bar{U}$ becomes infinite when the scaling parameter approaches the characteristic low-energy scale T_c, and the dimensionless vertex function diverges as well. Thus, the perturbation theory begins to break down, and there is one weak-coupling low-energy scale T_c depending on the dimensionless coupling parameter $\bar{U}$ only.

VIII. DISCUSSIONS AND CONCLUSIONS

From the previous perturbational calculations and the multiplicative RG analysis, we have found a non-FL behavior in the weak-coupling regime ($\bar{U} \ll 1$), and that the invariant coupling parameter $\bar{U}$ increases as the high-energy scale Δ is reduced. So our perturbation results are not valid when the coupling constant becomes larger and larger, or the energy scale is decreased to the low-temperature regime ($T = T_c$).

However, in the large-$\bar{U}$ limit, the Schrieffer-Wolff canonical transformation can also be applied to the present model, and a s-d type of model (so-called compactified two-channel Kondo model [9]) is obtained

$$H = it \sum_{n} \sum_{\alpha=0}^{3} \Psi_\alpha(n+1)\Psi_\alpha(n) + J\,[\vec{\sigma}(0) + \vec{\tau}(0)] \cdot \vec{S}_d, \tag{113}$$

where $J = 2V^2/U$ and $\rho J = \frac{2}{\pi^2}\left(\frac{\pi\Delta}{U}\right)$. So the weak-coupling limit $\rho J \ll 1$ of this s-d model corresponds to the large-$\bar{U}$ limit, and the multiplicative RG analysis can be also used to this model as the usual treatments for the ordinary Kondo model [17,18]. The scaling equation has been derived as

$$\frac{d(\rho J)}{d \ln D} = -[2(\rho J)^2 - (\rho J)^3], \tag{114}$$

where the conduction electron bandwidth D plays the role of the high-energy cut-off. This scaling equation explicitly shows that the dimensionless coupling parameter ρJ grows stronger and stronger under the RG transformations. For the corresponding compactified Anderson model in the large $\bar{U}$ regime, this is equivalent to a decrease in the dimensionless coupling parameter $\bar{U}$ ($\rho J = 2/(\pi^2 \bar{U})$) as the low-energy scale is reduced. Using the relevant cut-off factors in the small and large $\bar{U}$ regime, we arrive at a schematic flow diagram as shown in Fig.6, and we conjecture a stable fixed point of the model Hamiltonian in the intermediate coupling regime at $\bar{U} = \bar{U}_c$. This flow diagram is analogous to that of the isotropic two-channel Kondo model [5]. Obviously, there are significant differences between the present compactified Anderson impurity model and the two-channel Kondo model.

In conclusion, we have developed a systematic perturbation theory for the compactified Anderson impurity model in the Majorana fermion representation, where the unperturbed Hamiltonian has a degenerate ground state. We have calculated the leading perturbational corrections in the weak coupling limit. We have also derived some general relations for the impurity susceptibilities and self energies in terms of the vertex functions. The main results of the paper are that a linear temperature dependence of the electrical resistivity is obtained from the second order theory, and some non-FL thermodynamic properties have been calculated as well. The conduction electron singlet superconducting pairing correlation function at the impurity site is found singular in second order in U, indicating the formation of the singlet conduction electron pairing resonance at the Fermi level. In the third order in U, the vertex function $\Gamma_{0,1,2,3}(0,0,0,0)$ has logarithmic corrections, and the summation of the leading order logarithmic terms results in a new weak-coupling low-temperature energy scale $T_c = \Delta \exp\left[-\frac{1}{9}\left(\frac{\pi\Delta}{U}\right)^2\right]$, below which the perturbational approach begins to break down. The behavior of the low-energy excitations below T_c is still open for the future investigations.

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FIGURES

FIG. 1. The Dyson's equations of the impurity self energies in terms of the impurity vertex function $\Gamma_{0,1,2,3}$. (a) is for $\Sigma_{\rm sc}$, and (b) is for $\Sigma_{\rm vec}$. The solid lines denote the perturbed propagators $G_{\rm vec}$ and the dotted lines are $G_{\rm sc}$.

FIG. 2. The even and odd parts of the impurity spin and charge dynamical susceptibilities in terms of the impurity propagators and vertex functions. The thick solid lines denote the perturbed propagators $G_{\rm vec}$ and the thick dotted lines are $G_{\rm sc}$, while the thin solid lines correspond to the unperturbed propagators G_α and the thin dotted lines to G_0.

FIG. 3. The second-order perturbative corrections to the impurity self energies. (a) is for the scalar field, and (b) for the vector field.

FIG. 4. The first order and third order corrections to the impurity vertex function $\Gamma'_{0,1,2,3}(0,0,0,0)$. (a) is the first order correction. (b), (c), (d), and (e) are the third order contributions. But only the diagram (b) gives the logarithmically temperature dependent correction.

FIG. 5. The perturbative corrections to the even and odd part of the impurity spin and charge static susceptibilities in the low order perturbations. (a) is the singular diagram of the even part in the zero order in U. (b) is the odd part in the first order perturbations. (c) is the $\ln T$ contribution to the even part in the second-order of U. (d) is the diagram giving rise to the $\ln^2 T$ terms of the even part in the second-order perturbations.

FIG. 6. The schematic RG flow diagram in terms of the dimensionless coupling parameter $\bar{U} = U/(\pi\Delta)$. The arrows denote the direction of decreasing the relevant high-energy scale. $\bar{U}_c$ corresponds to the stable fixed point of the model.