Hamilton paths in Cayley digraphs of metacyclic groups

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Abstract

We obtain a characterization of all Hamilton paths in the Cayley digraph of a metacyclic group G with generating set {x, y} where \langle xy^{-1} \rangle \triangleleft G. The abundance of these Hamilton paths allows us to show that Hamilton paths occur in groups of at least two.

1. Introduction

The study of Hamilton paths and circuits in Cayley digraphs has had a long history. Rankin's [4] pioneering work yielded necessary and sufficient conditions for the existence of a Hamilton circuit in the Cayley digraph of a metacyclic group G with generating set \{x, y\} where \langle xy^{-1} \rangle \triangleleft G. A finite group G is metacyclic if it has a normal cyclic subgroup whose factor group is cyclic. Curran and Witte [1] gave a complete characterization of all Hamilton paths in the Cayley digraph of an abelian group G with a generating set which consists of two elements. This characterization gave a one to one correspondence between the standard Hamilton paths in these Cayley digraphs and the collection of visible lattice points in triangles in the plane. In their paper they show that the number of standard Hamilton paths in Cay(x, y: G), where G is abelian, is asymptotic to \((3/\pi^2)|G|\). See Witte and Gallian [5] for a further list of results on Hamilton paths in Cayley digraphs.

In this paper we obtain a characterization of all Hamilton paths in the Cayley digraph of a metacyclic group G with generating set \{x, y\} where \langle xy^{-1} \rangle \triangleleft G. Thus we obtain a useful extension of both Rankin's result and Curran and
Witte's result. We also show that Hamilton paths occur in groups of at least two in these Cayley digraphs. This is a consequence of the abundance of Hamilton paths in these digraphs.

2. Preliminaries

Definition 2.1. The Cayley digraph of a group $G$ with generating set $S$, denoted $\text{Cay}(S: G)$, is the digraph whose vertex set is $G$ and whose arc set consists of an arc from $g$ to $gs$ whenever $g \in G$ and $s \in S$. We let $\text{Cay}(x, y: G)$ denote the Cayley digraph of $G$ with generating set $\{x, y\}$.

In this section we obtain a necessary condition on the structure that a Hamilton path in $\text{Cay}(x, y: G)$ must have. This allows us to study Hamilton paths in $\text{Cay}(x, y: G)$ by studying a collection of spanning subdigraphs of $\text{Cay}(x, y: G)$. The results in this section were first proved by Housman in a preliminary version of [3]. These results were proved by Curran and Witte [1, Section 6] in the case when $G$ is abelian.

Definition 2.2. We call the subgroup $\langle yx^{-1} \rangle$ the arc forcing subgroup. We call the left coset $x^{-1} \langle yx^{-1} \rangle$ the special coset. All other left cosets of $\langle yx^{-1} \rangle$ are said to be regular. Let $H$ be a spanning subdigraph of $\text{Cay}(x, y: G)$. We say that a vertex $v$ in $\text{Cay}(x, y: G)$ travels by $x$ in $H$ if the arc from $v$ to $vx$ is in $H$ and the arc from $v$ to $vy$ is not. We say that a set of vertices $V$ in $G$ travels by $x$ in $H$ if every vertex $v \in V$ travels by $x$.

Notation 2.3. We always assume the initial vertex of a Hamilton path in $\text{Cay}(S: G)$ is the identity element of $G$. There is no loss in generality in assuming this because Cayley digraphs are vertex transitive. Given a path in $\text{Cay}(S: G)$ we say that $g \in G$ travels $S$ if the arc from $g$ to $gs$ belongs to the path. We will list a path in $\text{Cay}(S: G)$ by listing the arcs one travels by in succession. Thus $(a_i: 1 \leq i \leq n)$ is the path whose list of vertices is $1, a_1, a_1a_2, \ldots, a_1a_2 \cdots a_n$.

Theorem 2.4. Let $G$ be a finite group with generating set $\{x, y\}$. Suppose $P$ is a Hamilton path in $\text{Cay}(x, y: G)$ with initial vertex 1. Then:

1. The terminal vertex of $P$ occurs in the special coset and there is a unique integer $0 < d < \text{ord}(yx^{-1})$ such that $x^{-1}(yx^{-1})^d$ is the terminal vertex of $P$.
2. The vertex $x^{-1}(yx^{-1})^i$ travels by $y$ if $0 \leq i < d$ and travels by $x$ if $d < i < \text{ord}(yx^{-1})$.
3. A regular coset travels by $x$ or travels by $y$.

Proof. Let $v$ be a vertex in a coset of $\langle yx^{-1} \rangle$ that does not contain the terminal vertex of $P$. Suppose $v$ travels by $y$. Either $v(yx^{-1})$ travels by $x$ or it travels by $y$. Now $vy$
follows \( v \) in \( P \). Thus \( v(yx^{-1}) \) cannot travel by \( x \). Otherwise \( vy = v(yx^{-1})x \) follows both \( v \) and \( v(yx^{-1}) \) in \( P \) and occurs twice in the Hamilton path. Hence \( v(yx^{-1}) \) travels by \( y \).

An inductive argument on \( v(yx^{-1}) \) shows that every vertex in the coset \( v(yx^{-1}) \) travels by \( y \). If \( v \) is a vertex in a coset of \( \langle yx^{-1} \rangle \) that does not contain the terminal vertex of \( P \) and \( v \) travels by \( x \), then a similar argument shows that every vertex of this coset travels by \( x \).

Both \( x^{-1} \) and \( y^{-1} \) are elements of the special coset. However, \( x^{-1} \) must travel by \( y \) and \( y^{-1} \) must travel by \( x \). Hence, not all elements of the special coset travel by the same generator. Thus the terminal vertex of \( P \) occurs in the special coset. Let \( x^{-1}(yx^{-1})d \) be the terminal vertex of \( P \) where \( 0 \leq d < \text{ord}(yx^{-1}) \). Since \( x^{-1} \) travels by \( y \), \( x^{-1}(yx^{-1})d \) must travel by \( y \). Otherwise \( x^{-1}y = x^{-1}(yx^{-1})d \) occurs twice in \( P \). An inductive argument on \( i \) shows that \( x^{-1}(yx^{-1})i \) travels by \( y \) for \( 0 \leq i < d \). Let \( m = \text{ord}(yx^{-1}) \). Since \( y^{-1} = x^{-1}(yx^{-1})m-1 \) travels by \( x \), \( y^{-1}(xy^{-1}) = x^{-1}(yx^{-1})m-2 \) travels by \( x \). Otherwise \( y^{-1}x = y^{-1}(xy^{-1})y \) occurs twice in \( P \). A backwards induction on \( i \) starting at \( \text{ord}(yx^{-1})-1 \) shows that \( x^{-1}(yx^{-1})i \) travels by \( x \) for \( d < i < \text{ord}(yx^{-1}) \).

Now the terminal vertex does not occur in a regular coset. By the argument at the beginning of the proof, a regular coset must travel by \( x \) or travel by \( y \).

**Notation 2.5.** Let \( m = \text{ord}(yx^{-1}) \) and \( n = |G: \langle yx^{-1} \rangle| \). Let \( g_0, g_1, \ldots, g_{n-1} \) be a list of left coset representatives of \( \langle yx^{-1} \rangle \). We will always choose \( g_{n-1} = x^{-1} \). Let \( (e_0, e_1, \ldots, e_{n-2}) \in \{0, 1\}^{n-2} \) and let \( 0 \leq d < m \). Let \( H(e_0, e_1, \ldots, e_{n-2}, d) \) be the following spanning subdigraph of \( \text{Cay}(x, y; G) \) such that every vertex, except \( x^{-1}(yx^{-1})d \), travels by one arc. The regular coset \( g_i(yx^{-1}) \) travels by \( (yx^{-1})e_i, \) for \( 0 \leq i \leq n-2 \). The vertex \( x^{-1}(yx^{-1})i \) in the special coset travels by \( y \), for \( 0 \leq i < d \), and travels by \( x \), for \( d < j < m \).

Translating Theorem 2.4 into Notation 2.5 yields the following corollary.

**Corollary 2.6.** A Hamilton path in \( \text{Cay}(x, y; G) \) with initial vertex \( 1 \) has the form \( H(e_0, e_1, \ldots, e_{n-2}, d) \) for some choice of \( e_i \in \{0, 1\} \) for \( 0 \leq i \leq n-2 \) and for some \( 0 \leq d < m \).

### 3. Metacyclic groups where \( \langle yx^{-1} \rangle \triangleleft G \)

We now consider Cayley digraphs \( \text{Cay}(x, y; G) \) in which \( \langle yx^{-1} \rangle \triangleleft G \). Then \( G \) is metacyclic because both \( \langle yx^{-1} \rangle \) and \( G/\langle yx^{-1} \rangle \) are cyclic. We will assume that \( \langle yx^{-1} \rangle \triangleleft G \) for the rest of the paper.

**Notation 3.1.** Let \( z = yx^{-1} \), \( m = \text{ord}(z) \), and \( n = |G: \langle z \rangle| \). Let \( F = \langle z \rangle \). Then by [3, Section 3], there exist integers \( r \) and \( k \) such that \( 0 < r < m \), \( 0 < k < m \), \( x^{-1}zx = z^r \), \( x^n = z^k \), \( \gcd(r, m) = 1 \), \( r^m = 1 (\text{mod } m) \), and \( kr = k (\text{mod } m) \). We always choose left coset representatives to be \( g_i = x^i \), for \( 0 \leq i \leq n-2 \), and \( g_{n-1} = x^{-1} \).
Because $x = y \pmod{\langle z \rangle}$, Hamilton paths in $\text{Cay}(x, y : G)$ have a regular pattern. The following lemma is due to Housman in a preliminary version of [3].

**Lemma 3.2.** Suppose $P = (a_i: 1 \leq i < mn)$ is a Hamilton path in $\text{Cay}(x, y : G)$ associated with the subdigraph $H(e_0, e_1, \ldots, e_{n-2}, d)$. Then $P = (R, a_n, R, a_{2n}, R, a_{3n}, \ldots, R, a_{(m-1)n}, R)$ where $R = (z^{e_0}x, z^{e_1}x, \ldots, z^{e_{n-1}}x)$.

**Proof.** Suppose $i \neq 0 \pmod{n}$. Now $a_j = x = y \pmod{\langle z \rangle}$ for all $1 \leq j < mn$. Thus $1a_1a_2 \cdots a_{i-1} = x^{i-1} \pmod{\langle z \rangle}$. Hence $a_1a_2 \cdots a_{i-1}$ belongs to the coset $x^{i-1} \langle z \rangle$. So $a_1a_2 \cdots a_{i-1}$ must travel by $z^{e_{i-1}}x$ in $P$ if $i-1 \neq -1 \pmod{n}$. Here we take the indices $i-1$ of $e_{i-1} \pmod{n}$. Therefore, $a_i = z^{e_{i-1}}x$ for $i \neq 0 \pmod{n}$. Thus $(a_{n+1}, a_{n+2}, \ldots, a_{n+m-1}) = (z^{e_0}x, z^{e_1}x, \ldots, z^{e_{n-1}}x)$ for $0 \leq j < m-1$. 

**Notation 3.3.** Let $X = x(z^{e_0}x)(z^{e_1}x) \cdots (z^{e_{n-2}}x)$ and $Y = y(z^{e_0}x)(z^{e_1}x) \cdots (z^{e_{n-2}}x)$. Let $p = k + \sum_{i=0}^{n-2} e_i^m i^t$. Then $X = z^p$ and $Y = z^{p+1}$. Let $b = \gcd(m, p)$, $a = m/b$, and let $0 \leq e < a$ such that $Y^b = X^e$.

**Definition 3.4.** We say that two digraphs are *homeomorphic as digraphs* if they can be obtained from the same digraph by inserting new vertices of inner semidegree one and outer semidegree one into its arcs such that the two new arcs have the same direction as the one they replace.

We would like to obtain a characterization of the Hamilton paths in $\text{Cay}(x, y : G)$. We consider the vertices of the special coset in $H(e_0, e_1, \ldots, e_{n-2}, d)$ and replace the paths $(x, R)$ and $(y, R)$ emanating from these vertices with the arcs $X$ and $Y$. This allows us to consider the subdigraph $H(e_0, e_1, \ldots, e_{n-2}, d)$ of $\text{Cay}(x, y : G)$ as a digraph homeomorphic to a subdigraph of a Cayley digraph of the cyclic group $F$ with generating set $\{X, Y\}$. A characterization of these subdigraphs for abelian groups has been obtained by Curran and Witte [1]. We assume the reader is familiar with their result. We state their result for the Cayley digraph $\text{Cay}(X, Y : F)$. The appropriate subdigraphs they consider is given in the notation below.

**Notation 3.5.** We let $H_0(d)$ be the spanning subdigraph of $\text{Cay}(X, Y : F)$ in which the vertex $X^{-1}x^i$ travels by $Y$, for $0 \leq i < d$, and travels by $X$, for $d < i < m$.

**Theorem 3.6** [1, Theorem 7.1]. The subdigraph $H_0(d)$ in $\text{Cay}(X, Y : F)$ is a Hamilton path iff $B_0(d) = (0, 0)$.

The points $B_0(d)$ are lattice points in the triangle whose vertices are $(0, 0), (b, 0)$ and $(e, a)$. See [1, Notation 3.14] for a complete definition of these points.

**Notation 3.7.** Let $\hat{H}(e_0, e_1, \ldots, e_{n-2}, d)$ be the subdigraph of $H(e_0, e_1, \ldots, e_{n-2}, d)$ which is obtained by deleting the vertices $1, z^{e_0}x, z^{e_0}xz^{e_1}x, \ldots, z^{e_0}xz^{e_1}x \cdots z^{e_{n-2}}x$. 

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*S.J. Curran*
Lemma 3.8. The digraph \( H(e_0, e_1, \ldots, e_{n-2}, d) \) is a Hamilton path in \( \text{Cay}(x, y: G) \) iff \( H'(e_0, e_1, \ldots, e_{n-2}, d) \) is a Hamilton path in the subdigraph of \( \text{Cay}(x, y: G) \) with the vertices \( 1, z^{e_0}x, z^{e_0}xz^{e_1}x, \ldots, z^{e_0}xz^{e_1}x \cdots z^{e_{n-3}}x \) removed.

Proof. (\( \Rightarrow \)) Suppose \( H(e_0, e_1, \ldots, e_{n-2}, d) \) is a Hamilton path in \( \text{Cay}(x, y: G) \). Since \( 1, z^{e_0}x, z^{e_0}xz^{e_1}x, \ldots, z^{e_0}xz^{e_1}x \cdots z^{e_{n-3}}x \) are the first \( n-1 \) vertices in the Hamilton path \( H(e_0, e_1, \ldots, e_{n-2}, d) \), it is a Hamilton path in the subdigraph of \( \text{Cay}(x, y: G) \) with the vertices \( 1, z^{e_0}x, z^{e_0}xz^{e_1}x, \ldots, z^{e_0}xz^{e_1}x \cdots z^{e_{n-3}}x \) removed.

(\( \Leftarrow \)) By replacing the vertices \( 1, z^{e_0}x, z^{e_0}xz^{e_1}x, \ldots, z^{e_0}xz^{e_1}x \cdots z^{e_{n-3}}x \) into \( H'(e_0, e_1, \ldots, e_{n-2}, d) \) and allowing the vertex \( \prod_{i=0}^{i-1} z^{e_i}x \) to travel by \( z^{e_i}x \) for \( 1 \leq i \leq n-2 \) and the vertex \( 1 \) to travel by \( z^{e_0}x \), we obtain the Hamilton path \( H(e_0, e_1, \ldots, e_{n-2}, d) \) in \( \text{Cay}(x, y: G) \).

Lemma 3.9. The digraph \( H'(e_0, e_1, \ldots, e_{n-2}, d) \) is homeomorphic as a digraph to \( H_0(d) \).

Proof. First relabel the vertex \( X^{-1}z^i \) by \( x^{-1}z^i \) for \( 0 \leq i < m \) in \( H_0(d) \). On each arc \( Y \) emanating from \( x^{-1}z^i \), for \( 0 \leq i < d \), the place the vertices \( x^{-1}z^i, x^{-1}z^{i+1}z^{e_0}x, \ldots, x^{-1}z^{i+d}z^{e_0}x \cdots z^{e_{n-3}}x \) in succession. On the arc \( X \) emanating from \( x^{-1}z^i, \) for \( d < i < m \), place the vertices \( x^{-1}z^i, x^{-1}z^{i+1}z^{e_0}x, \ldots, x^{-1}z^{i+d}z^{e_0}x \cdots z^{e_{n-3}}x \) in succession. We obtain a subdigraph of \( \text{Cay}(x, y: G) \) with the vertices \( 1, z^{e_0}x, z^{e_0}xz^{e_1}x, \ldots, z^{e_0}xz^{e_1}x \cdots z^{e_{n-3}}x \) deleted in which vertices in the regular coset \( x'\langle z \rangle \) travel by \( z^{e_0}x \) for \( 0 \leq i \leq n-2 \), and the vertex \( x^{-1}z^j \) travels by \( y \), for \( 0 \leq j < d \), and travels by \( x \), for \( d < j < m \). This is exactly the digraph \( H'(e_0, e_1, \ldots, e_{n-2}, d) \).

Theorem 3.10. The subdigraph \( H(e_0, e_1, \ldots, e_{n-2}, d) \) is a Hamilton path in \( \text{Cay}(x, y: G) \) iff \( B_0(d) = (0, 0) \).

Proof. Combine Lemma 3.8, Lemma 3.9, and Theorem 3.6 together.

As an application of this theorem, we let \( N(e, a, b) \) be the number of visible lattice points in the interior of the triangle whose vertices are \((0, 0), (b, 0), \) and \((e, a)\). A lattice point \((x, y)\) is visible if \( \gcd(x, y) = 1 \). We view \( a, b, \) and \( e \) as functions of \( (e_0, e_1, \ldots, e_{n-2}) \in \{0, 1 \}^{n-1} \) as in Notation 3.3.

Corollary 3.11. The number of Hamilton paths in \( \text{Cay}(x, y: G) \) is \( \sum(N(e, a, b)+1) \), where the sum is taken over all \((e_0, e_1, \ldots, e_{n-2}) \in \{0, 1 \}^{n-1} \).

Proof. The number of visible lattice points in the interior of the triangle whose vertices are \((0, 0), (b, 0), \) and \((e, a)\) is one fewer than the number of values of \( 0 < d < m \) for which \( B_0(d) = (0, 0) \) by [1, Proposition 3.15]. Thus the corollary follows from Theorem 3.10.
4. Hamilton paths occur in groups of at least two

We show that Hamilton paths occur in groups of at least two in $\text{Cay}(x, y; G)$. By Theorem 3.10, we need only consider Hamilton paths in $\text{Cay}(X, Y; F)$ where $F = \langle z \rangle$, $X = z^p$, and $Y = z^{p+1}$. We have $z^{p+1} = Y^b = X^e = z^p$. See Notation 3.3 for the definitions of $p, a, b, c,$ and $e$. Let $p_0 = p/b$. Then $(p + 1)b = pe \pmod{ab}$ implies that $p + 1 = p_0 e \pmod{a}$. Hence $e - b = p_0^{-1} \pmod{a}$. Thus $\gcd(e - b, a) = 1$ since $\gcd(p_0, a) = 1$. So it suffices to consider Cayley digraphs $\text{Cay}(X, Y; F)$ of cyclic groups $F = \langle X, Y | X^a = 1, Y^b = X^e, XY = YX \rangle$ where $\gcd(e - b, a) = 1$.

**Theorem 4.1.** Suppose $H_0(d)$ is a Hamilton path in $\text{Cay}(X, Y; F)$, $\gcd(e - b, a) = 1$, and $d \neq b - 1, ab - \gcd(e, a)$. Then either $H_0(d - 2)$ or $H_0(d + 2)$ is a Hamilton path.

**Proof.** By Theorem 3.6, we know that $B_0(d) = (0, 0)$. Thus $B_0(d - 1) = (x_1, y_1)$ and $B_0(d + 1) = (x_2, y_2)$ where $(x_1, y_1)$ are nonzero lattice points such that $\gcd(x_1, y_1) = 1$. The triangle $T$ whose vertices $(0, 0), (x_1, y_1),$ and $(x_2, y_2)$ is primitive. A triangle whose vertices are lattice points is primitive if the only lattice points contained either in the triangle or on the boundary of the triangle are the vertices themselves. A primitive triangle has area $\frac{1}{2}$ [2]. Thus $\text{area}(T) = \frac{1}{2}(x_1y_2 - x_2y_1) = \frac{1}{2}$. Hence $x_1y_2 - x_2y_1 = 1$. Since $d \neq b - 1, ab - \gcd(e, a)$, there are integers $d_1 < d < d_2$ such that $B_0(d_1) = B_0(d_2) = (0, 0)$; namely the integers $b - 1$ and $ab - \gcd(e, a)$ themselves.

Note that $(x_2, y_2)$ is an integral solution to

$$x_1y - xy_1 = 1$$

satisfying

$$ax + (b - e)y \leq ab, \quad ey \leq ax, \quad \text{and} \quad y \geq 0. \quad (\ast)$$

Consider the point $(x_1 + x_2, y_1 + y_2)$. This point is an integral solution to $x_1y - xy_1 = 1$ and the slope of the ray emanating from the origin passing through $(x_1 + x_2, y_1 + y_2)$ satisfies

$$\frac{y_1}{x_1} < \frac{y_1 + y_2}{x_1 + x_2} < \frac{y_2}{x_2}.$$

Let $T'$ be the triangle with vertices $(0, 0), (b, 0),$ and $(e, a)$. By the definition of the points $B_0(d)$, the point $B_0(d + 1) = (x_2, y_2)$ is the unique visible lattice point in $T'$ whose slope $y_2/x_2$ is the least among all the visible lattice points having slope strictly greater than $y_1/x_1$. Hence, $(x_1 + x_2, y_1 + y_2)$ lies outside of $T'$. Thus $a(x_1 + x_2) + (b - e)(y_1 + y_2) > ab$ because $(x_1 + x_2, y_1 + y_2)$ satisfies the conditions $ey \leq ax$ and $y \geq 0$ in $(\ast)$. Therefore, either $a(2x_1) + (b - e)(2y_1) > ab$ or $a(2x_2) + (b - e)(2y_2) > ab$. This implies that either the lattice point $(2x_1, 2y_1)$ or $(2x_2, 2y_2)$ lies outside of $T'$. Hence, either $B_0(d - 2) = (0, 0)$ or $B_0(d + 2) = (0, 0)$. By Theorem 3.6, either $H_0(d - 2)$ or $H_0(d + 2)$ is a Hamilton path. \[\square\]
Now it may be the case that the only Hamilton paths in \( \text{Cay}(X, Y; F) \) are \( H_0(b - 1) \) and \( H_0(ab - \gcd(e, a)) \). By imposing slight restrictions on \( a, b, \) and \( e \), it can be assured that this is not the case.

**Proposition 4.2.** Suppose \( a \geq 2, \ b \geq 3, \) and \( \gcd(e - b, a) = 1 \). Then there exists \( b - 1 < d < ab - \gcd(e, a) \) such that \( H_0(d) \) is a Hamilton path in \( \text{Cay}(X, Y; F) \).

**Proof.** Let \( c \) be the number of lattice points on the boundary of the triangle \( T' \) whose vertices are \((0, 0), (b, 0)\), and \((e, a)\). Then \( c = b + \gcd(e, a) + 1 \leq a + b \). Let \( f \) be the number of lattice points in the interior of \( T' \). By Pick’s Theorem [2], we have

\[
c + 2f = 2 \cdot \text{area}(T') = ab,
\]

Thus

\[
2f = ab - c + 2 \geq (a - 1)(b - 1) + 1 \geq 2(a - 1) + 1.
\]

Hence, \( f \geq a - \frac{1}{2} \). Since there are at most \( a - 1 \) lattice points contained in the interior of \( T' \) that lie on the same ray emanating from the origin, there are at least two distinct rays emanating from the origin passing through a lattice point in the interior of \( T' \). Hence, there exists \( b - 1 < d < ab - \gcd(e, a) \) such that \( B_0(d) = (0, 0) \). By Theorem 3.6, \( H_0(d) \) is a Hamilton path.

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**References**