Convergence of compressible Euler–Poisson equations to incompressible type Euler equations

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Abstract. In this paper, we study the convergence of time-dependent Euler–Poisson equations to incompressible type Euler equations via the quasi-neutral limit. The local existence of smooth solutions to the limit equations is proved by an iterative scheme. The method of asymptotic expansion and the symmetric hyperbolic property of the systems are used to justify the convergence of the limit.

Keywords: Euler–Poisson equations, incompressible Euler equations, quasi-neutral limit, asymptotic expansion and justification

1. Introduction

In mathematical modeling for plasmas and semiconductor devices, the hydrodynamic model like Euler–Poisson system is widely used\cite{4,9}. However, the plasma physics is usually concerned with large scale structures with respect to the Debye length. For such scales, the plasma is electrically neutral, i.e., there is no charge separation or electric field. Since this limit is widely used in practice\cite{2}, it is important to give its mathematical justification.

Several special cases of the quasi-neutral limit for plasmas have been studied in literature. The quasi-neutral limit in one-dimensional steady Euler–Poisson system was performed by Slemrod and Sternberg in\cite{14} for prepared boundary data and by Peng in\cite{10} for general boundary data. The steady problem in several space variables for a potential flow without the formation of boundary layers was studied by Peng in\cite{11}. The case with boundary layers was investigated recently by the authors in\cite{13}. In that paper, the convergence of the steady state compressible Euler–Poisson equations for potential flows to the incompressible Euler equations was proved in the subsonic region. In\cite{5}, Cordier and Grenier studied, by using pseudo-differential techniques, the quasi-neutral limit for local smooth solutions of an one-dimensional and isothermal model for plasmas in which the electron density is described by the Maxwell–Boltzmann relation with the electric potential. Let us also mention the paper of Brenier\cite{1} where the convergence of the Vlasov–Poisson system to the incompressible Euler equations was studied.

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The purpose of this paper is to study the Debye length limit by the method of asymptotic expansion to the Cauchy problem for the multi-dimensional Euler–Poisson equations for plasmas with the ion density \( b \) being given. Under the assumption that the initial densities satisfy certain compatibility conditions, which guarantee no initial layer formed, we first formally derive that as the Debye length vanishes the limits of the electron velocity and the electric potential satisfy an incompressible type Euler equations of fluids. In particular, when the ion density is a constant, the limits of the electron velocity and the electric potential satisfy the classical incompressible Euler equations. Furthermore, we rigorously justify the asymptotic expansion up to any order for the case that our problem is confined on a torus and the mean value of the electric potential on the torus vanishes. The uniform error estimates are given with respect to the Debye length for each variable. It is noticed that the initial value of the electron density could not be given arbitrarily, and it should be determined by the initial data of the electron velocity and the electric potential.

The remainder of this paper is arranged as follows: in Section 2, by formal analysis we derive that the leading profiles of the electron velocity and the electric potential with respect to the Debye length satisfy an incompressible type Euler equations, and their next order profiles satisfy the corresponding linearized equations. The Cauchy problem for this type Euler equations is solved in Section 3 by a way similar to that given by Chemin in [3]. In Section 4, we rigorously justify the asymptotic expansion developed in Section 2, and obtain the existence of solutions to the multi-dimensional Euler–Poisson system in a time interval independent of the Debye length.

2. Asymptotic analysis

Let \( n(t, x) > 0 \) and \( u(t, x) \) be the density and the velocity vector of the electronic particles in a plasma, \( p = p(n) \) be the pressure, \( \Psi(t, x) \) be the electric potential, \( b(t, x) > 0 \) be the ion density, and \( \lambda > 0 \) be the Debye length, which is a small quantity. We assume that \( p(n) \) is smooth and strictly increasing for \( n > 0 \), and there is a constant \( b_0 > 0 \) such that \( b(t, x) \geq b_0 \) for all \((t, x) \in \mathbb{R}_t \times \mathbb{R}^d_x\).

Based on the conservation law of density, and the conservation of momentum, the evolution of the electronic density, velocity and potential obey the following Euler–Poisson system in \( \mathbb{R}_t \times \mathbb{R}^d_x \):

\[
\begin{aligned}
\partial_t n + \text{div}(nu) &= 0, \\
\partial_t (nu) + \text{div}(nu \otimes u) + \nabla p(n) + n \nabla \Psi + nu &= 0, \\
\lambda^2 \Delta \Psi &= b(t, x) - n.
\end{aligned}
\]  

(2.1)

For the smooth solutions of (2.1), the second equation in (2.1) is equivalent to

\[
\partial_t u + (u \cdot \nabla) u + \nabla (h(n) + \Psi) + u = 0,
\]

(2.2)

where the enthalpy \( h(n) \) is defined as

\[
h(n) = \int_0^n \frac{p'(s)}{s} \, ds.
\]

Impose the initial data for (2.1) at \( t = 0 \) as

\[
n|_{t=0} = n_0^\lambda(x), \quad u|_{t=0} = u_0^\lambda(x),
\]

(2.3)
where
\[
\begin{align*}
n_0^\lambda(x) &= \sum_{j=0}^{m} \lambda^{2j} n_j(x) + \lambda^{2(m+1)} n_{m+1}^\lambda(x), \\
u_0^\lambda(x) &= \sum_{j=0}^{m} \lambda^{2j} u_j(x) + \lambda^{2(m+1)} u_{m+1}^\lambda(x),
\end{align*}
\]
with \( \{ n_j \}_{0 \leq j \leq m} = (n_0, n_1, \ldots, n_m) \) being determined by \( \{ u_j \}_{0 \leq j \leq m} \) and \( b(t, x) \) (see Remark 2.1), and \( \{ u_j \}_{0 \leq j \leq m} \) satisfying certain compatibility conditions, which will be given later.

Denote by \((n^\lambda, u^\lambda, \Psi^\lambda)\) the classical solutions to the Cauchy problem (2.1)–(2.3). In this section, we are going to study the formal expansions of \((n^\lambda, u^\lambda, \Psi^\lambda)\) when \( \lambda \to 0 \) under the assumption
\[
n_0(x) = b(0, x), \tag{2.4}
\]
which is a compatibility condition to guarantee no initial layer appeared when \( \lambda \to 0 \).

Take the following ansatz:
\[
\begin{align*}
n^\lambda(t, x) &= \sum_{j \geq 0} \lambda^{2j} n_j(t, x), \\
u^\lambda(t, x) &= \sum_{j \geq 0} \lambda^{2j} u_j(t, x), \\
\Psi^\lambda(t, x) &= \sum_{j \geq 0} \lambda^{2j} \psi_j(t, x)
\end{align*}
\]
in terms of \( \lambda \) for the solutions to the problem (2.1)–(2.3). Plugging the expansions (2.5) into the system (2.1), and using the initial conditions (2.3), we know that:

1. The leading profiles \((n^0, u^0, \psi^0)\) satisfy the following problem:
\[
\begin{align*}
\partial_t u^0 + (u^0 \cdot \nabla) u^0 + \nabla (h(b) + \psi^0) + u^0 &= 0, \\
\text{div}(bu^0) &= -\partial_t b, \\
 u^0(t)|_{t=0} &= u_0(x)
\end{align*}
\]
and
\[
n^0 = b(t, x). \tag{2.7}
\]

In particular, if the ion density \( b(t, x) \) is a constant, say \( b(t, x) = 1 \) for simplicity, then from (2.6) we know that \( u^0(t, x) \) satisfies a problem for the incompressible Euler equations.
(2) For any $j \geq 1$, the profiles $(n^j, u^j, \psi^j)$ satisfy the following problem for linearized Euler equations:

\[
\begin{aligned}
&\partial_t u^j + \sum_{k=0}^{j} (u^k \cdot \nabla) u^{j-k} + \nabla (h'(b)n^j + h^{j-1}(\{n^k\}_{k \leq j-1}) + \psi^j) + u^j = 0, \\
&\text{div}(bu^j) = -\partial_t n^j - \sum_{k=1}^{j} \text{div}(n^k u^{j-k}), \\
&n^j = -\Delta \psi^{j-1}, \\
&u^j|_{t=0} = u_j(x),
\end{aligned}
\]

(2.8)

where $h^0 = 0$ and for $j \geq 2$, $h^{j-1}(\{n^k\}_{k \leq j-1})$ is defined by

\[
h^{j-1}(\{n^k\}_{k \leq j-1}) = \frac{1}{(2j)!} \frac{d^{2j}}{d \lambda^{2j}} h \left( b + \sum_{k \geq 1} \lambda^2 n^k \right) \bigg|_{\lambda=0} - h'(b)n^j.
\]

The fact that $h^{j-1}$ depends only on $\{n^k\}_{k \leq j-1}$ can be obtained from the following relation:

\[
h^{j-1}(\{n^k\}_{k \leq j-1}) = \frac{1}{(2j)!} \frac{d^{2j}}{d \lambda^{2j}} h \left( b + \sum_{k \geq 1} \lambda^2 n^k \right) \bigg|_{\lambda=0} - h'(b)n^j.
\]

Remark 2.1. From (2.7) and (2.8), we see that each order profile of $n^\lambda$ is given by profiles of $\psi^\lambda$ explicitly, and each order profile of the initial data $n^\lambda_0$ given in (2.3) should be determined by $u^\lambda_0(x)$ and $b(t, x)$ completely when certain boundary conditions are imposed for $\psi$ (refer to the assumption (H1) given in Section 3 for example).

Indeed, from (3.6) we immediately deduce that $\psi^0|_{t=0}$ is given by $u^0(x)$ and $b(t, x)$, which gives

\[
n^1|_{t=0} = -\Delta (\psi^0|_{t=0})
\]

by (2.8) for $j = 1$. From (2.8) for $j = 2$, we get

\[
n^2|_{t=0} = -\Delta (\psi^1|_{t=0}),
\]

(2.10)

and $\psi^1$ satisfies

\[
\text{div}(bu^1) = \partial_t n^1 + \partial_t \text{div}(u^0) + \text{div}([\partial_t b - b] u^1 - b(u^1 \cdot \nabla) u^1 - b(u^1 \cdot \nabla) u^0 - b \nabla (h'(b)n^1)]).
\]

(2.11)

By substituting $\psi^0|_{t=0}$ given by (3.6) into (2.6), it gives $(\partial_t u^0)|_{t=0}$. From (3.6), we obtain $(\partial_t (\psi^0))|_{t=0}$, which gives $(\partial_t^2 u^0)|_{t=0}$ from (2.6) again. Substituting $(\partial_t^2 u^0)|_{t=0}$ ($k = 0, 1, 2$) into (3.6), it follows $(\partial_t^3 \psi^0)|_{t=0}$. Thus, from the relation $n^1 = -\Delta \psi^0$, we obtain $(\partial_t^3 u^1)|_{t=0}$ and $(\partial_t^3 u^1)|_{t=0}$, which depend on $n^0$ and $b(t, x)$. From (2.11) we determine $\psi^1|_{t=0}$ from $(u_0, u_1)$ and $b(t, x)$ by imposing certain boundary conditions on $\psi^1$, which also gives $n^2|_{t=0}$ from (2.10) immediately.
Successively, we conclude that for any \( j \geq 3 \), \( n^j_{t=0} \) can be represented by \((u_0(x), \ldots, u_j(x))\) and \( b(t, x) \). This is a kind of compatibility conditions on the initial data for Eqs (2.1) in order to study the quasi-neutral limit through the ansatz (2.5). It is an interesting problem to study the case when this compatibility condition does not hold, which will be investigated in a forthcoming paper.

3. Determinacy of the profiles \( \{u^j, \psi^j\}_{j \geq 0} \)

From (2.8), we know that once \((u^0, \psi^0)\) are solved from the problem (2.6), \((u^1, \psi^1)\) are solutions to the following problem for a linearized Euler type equations:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\partial_t u^1 + (u^0 \cdot \nabla) u^1 + (u^1 \cdot \nabla) u^0 + \nabla \psi^1 + u^1 = -\nabla (h'(b)n^1), \\
\text{div}(bu^1) = -\partial_t n^1 - \text{div}(n^1 u^0), \\
\end{array} \right. \\
&u^1|_{t=0} = u_1(x),
\end{aligned}
\]

(3.1)

where

\[
n^1 = -\Delta \psi^0,
\]

(3.2)

and \( u_1(x) \) satisfies the compatibility condition:

\[
\text{div}(bu_1) = \partial_t \Delta \psi^0 + \text{div}(\Delta \psi^0 u_0) \quad \text{at } t = 0.
\]

(3.3)

Inductively, suppose that \( \{n^j, u^j, \psi^j\}_{j \leq k-1} \) are solved already for any \( k \geq 2 \), from (2.8) we know that \((u^k, \psi^k)\) satisfy the following linear problem:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\partial_t u^k + (u^0 \cdot \nabla) u^k + (u^k \cdot \nabla) u^0 + \nabla \psi^k + u^k \\
= - \sum_{j=1}^{k-1} (u^j \cdot \nabla) u^{k-j} - \nabla (h'(b)n^k + h^{k-1}(\{n^j\}_{j \leq k-1})), \\
\text{div}(bu^k) = -\partial_t n^k - \sum_{j=1}^{k} \text{div}(n^j u^{k-j}), \\
u^k|_{t=0} = u_k(x),
\end{array} \right. \\
\end{aligned}
\]

(3.4)

where

\[
n^k = -\Delta \psi^{k-1}
\]

(3.5)

and \( u_k(x) \) satisfies the compatibility condition:

\[
\text{div}(bu_k) = \partial_t \Delta \psi^{k-1} + \sum_{j=1}^{k} \text{div}(\Delta \psi^{k-j-1} u_{k-j}) \quad \text{at } t = 0.
\]

(3.6)

Thus, the crucial step to determine the profiles of \((n^\lambda, u^\lambda, \psi^\lambda)\) is to solve the nonlinear problem (2.6) for \((u^0, \psi^0)\).
From now on, we suppose that

(H1) the problem (2.1)–(2.3) is defined for \( x \) on the torus \( T^d = (\mathbb{R}/2\pi)^d \), and the mean value of \( \Psi^\lambda(t, x) \) to (2.1) vanishes, which implies

\[
\mathbf{m}(\psi^j) = \frac{1}{(2\pi)^d} \int_{T^d} \psi^j(t, x) \, dx = 0,
\]

for any order profile \( \psi^j \) of \( \Psi^\lambda \);

(H2) for any fixed \( s > d/2 + 1 \), the ion density \( b(t, x) \) satisfies the following conditions:

\[
b \in \bigcap_{j=0}^{2} W^{j, \infty}((0, T_0), H^{s+2-j}(T^d)), \quad b \geq b_0 > 0
\]

for a fixed constant \( b_0 > 0 \).

Now let us study the nonlinear problem (2.6) in detail. Obviously, first we should impose a compatibility condition on \( u_0(x) \) as follows

\[
(\partial_t b)(0, x) + \text{div}(b(0, x)u_0(x)) = 0.
\]

From the equations in (2.6), we deduce

\[
\begin{cases}
\text{div}(b \nabla \psi^0) = \partial_t^2 b + \partial_t b - \Delta p(b) - \text{div}(\text{div}(bu^0 \otimes u^0)), \quad x \in T^d, \\
\mathbf{m}(\psi^0) = 0
\end{cases}
\]

which is an elliptic problem for \( \psi^0 \). Denote by

\[
\psi^0 = L^{-1}\left( \partial_t^2 b + \partial_t b - \Delta p(b) - \text{div}(\text{div}(bu^0 \otimes u^0)) \right)
\]

the unique solution to the problem (3.9). Then the problem (2.6) of \( u^0 \) can be reformulated as

\[
\begin{cases}
\partial_t u^0 + (u^0 \cdot \nabla)u^0 + u^0 - \nabla L^{-1} \text{div}(\text{div}(bu^0 \otimes u^0)) = f(b), \\
u^0|_{t=0} = u_0(x),
\end{cases}
\]

where

\[
f(b) = -\nabla h(b) - \nabla L^{-1}(\partial_t^2 b + \partial_t b - \Delta p(b)).
\]

To verify the equivalence between the problem (2.6) and (3.9), (3.10), we should have

**Lemma 3.1.** Let \( u^0(t, x) \) be the solution to the problem (3.10) for \( 0 < t < T \), and the initial data \( u_0(x) \) satisfy the assumption (3.8). Then, we have

\[
\partial_t b + \text{div}(bu^0) = 0
\]

for any \( t \in (0, T) \).
Proof. Obviously, the equation of \( u^0 \) in (3.10) is equivalent to

\[
\begin{align*}
&b \partial_t u^0 + b (u^0 \cdot \nabla) u^0 + b u^0 - b \nabla L^{-1} \div (\div (bu^0 \otimes u^0)) \\
&= -\nabla p(b) - b \nabla L^{-1} (\partial^2 b + \partial_t b - \Delta p(b)).
\end{align*}
\]  

(3.12)

Acting the divergence operator \( \div \) on Eq. (3.12), and using (3.9), it follows

\[
\partial_t \psi + \partial_t b + \div (bu^0) + \div [b (\partial_t u^0 + (u^0 \cdot \nabla) u^0)] - \div [\div (bu^0 \otimes u^0)] = 0,
\]

which can be simplified as

\[
\partial_t [\partial_t b + \div (bu^0)] - (u^0 \cdot \nabla) [\partial_t b + \div (bu^0)] + (1 - \div u^0) [\partial_t b + \div (bu^0)] = 0,
\]

yielding the conclusion (3.11) immediately by using the assumption (3.8).

Thus, to solve the problem (2.6), it suffices to study the nonlinear problem (3.10). First, let us study the elliptic problem (3.9). By repeatedly using the identity (3.11), we obtain

\[
\div (\div (bu^0 \otimes u^0)) = b \sum_{i,j=1}^{d} \frac{\partial (bu^0)}{\partial x_i} \frac{\partial (bu^0)}{\partial x_j} - 2 (u^0 \cdot \nabla) \partial_t b + \frac{\partial_t b}{b} (u^0 \cdot \nabla) b + \frac{(\partial_t b)^2}{b} \tag{3.13}
\]

with \( u^0_i \) being the \( i \)-th component of \( u^0 \).

Substituting (3.13) into (3.9), we deduce that \( \psi^0 \) satisfies the following problem:

\[
\begin{cases}
\div (b \nabla \psi^0) = -b \sum_{i,j=1}^{d} \frac{\partial (bu^0)}{\partial x_i} \frac{\partial (bu^0)}{\partial x_j} + 2 (u^0 \cdot \nabla) \partial_t b - \frac{\partial_t b}{b} (u^0 \cdot \nabla) b + \partial^2_t b + \partial_t b - \Delta p(b), & x \in T^d, \\
\m(\psi^0) = 0.
\end{cases}
\]  

(3.14)

Decompose the solution \( \psi^0 \) of (3.14) into three parts as follows:

\[
\psi^0 = \psi^0_1 + \psi^0_2 + \psi^0_3, \tag{3.15}
\]

where \( \psi^0_1, \psi^0_2 \) and \( \psi^0_3 \) satisfy the following problems

\[
\begin{cases}
\div (b \nabla \psi^0_1) = -b \sum_{i,j=1}^{d} \frac{\partial (bu^0)}{\partial x_i} \frac{\partial (bu^0)}{\partial x_j}, & x \in T^d, \\
\m(\psi^0_1) = 0,
\end{cases}
\]  

(3.16)

\[
\begin{cases}
\div (b \nabla \psi^0_2) = 2 (u^0 \cdot \nabla) \partial_t b - \frac{\partial_t b}{b} (u^0 \cdot \nabla) b, & x \in T^d, \\
\m(\psi^0_2) = 0.
\end{cases}
\]  

(3.17)
\begin{equation}
\begin{cases}
\text{div}(b\nabla \psi_3^0) = -\frac{(\partial_t b)^2}{b} + \partial_t^2 b + \partial_t b - \Delta p(b), \quad x \in T^d, \\
m(\psi_3^0) = 0,
\end{cases}
\tag{3.18}
\end{equation}

respectively.

Denote by $F(u^0, \nabla u^0)$ the right-hand side of equation in (3.16). Multiplying $\psi_1^0$ on both sides of equation in (3.16) and using (3.7), we get for any $0 \leq t \leq T_0$,

$$\|\nabla \psi_1^0(t)\|_{L^2(T^d)}^2 \leq C \|\psi_1^0(t)\|_{H^1(T^d)} \|F(u^0, \nabla u^0)(t)\|_{H^{-1}(T^d)},$$

which implies

$$\|\psi_1^0(t)\|_{H^{k+1}(T^d)} \leq C_k \|F(u^0, \nabla u^0)(t)\|_{H^{-k-1}(T^d)},$$

(3.19)

by using the Poincaré inequality due to $m(\psi_1^0) = 0$. Differentiating (3.16) with respect to $x \in T^d$, and by induction on $k$ we deduce

$$\|\psi_1^0(t)\|_{H^{k+1}(T^d)} \leq C_k \|u^0(t)\|_{H^{k-1}(T^d)},$$

(3.20)

for any $0 \leq k \leq s + 1$.

Similarly, from the problems (3.17) and (3.18) we have

$$\|\psi_2^0(t)\|_{H^{k+1}(T^d)} \leq C_k \|u^0(t)\|_{H^{k-1}(T^d)},$$

(3.21)

and

$$\|\psi_3^0(t)\|_{H^{k+1}(T^d)} \leq C_k,$$

(3.22)

for any $0 \leq k \leq s + 1$, where $C_k$ are constants depending only upon $k$ and the bound of $b$ in the norm of the space given in (3.7).

Denote by $\psi_1^0 = \psi_1(u^0)$ and $\psi_2^0 = \psi_2(u^0)$ the unique solutions to (3.16) and (3.17) respectively. Substituting the decomposition (3.15) into (2.6), we see that $u^0$ satisfies the following problem:

$$\begin{cases}
\partial_t u^0 + (u^0 \cdot \nabla) u^0 + u^0 + \nabla (\psi_1(u^0) + \psi_2(u^0)) = g, \\
\partial_t u^0_{|t=0} = u_0(x),
\end{cases},$$

(3.23)

where $g = -\nabla (h(b) + \psi_3^0)$.

We use the following iteration scheme for the nonlinear Cauchy problem (3.23):

$$\begin{cases}
\partial_t u^0_{l+1} + (u^0_{l} \cdot \nabla) u^0_{l+1} + u^0_{l+1} = g - \nabla (\psi_1(u^0_l) + \psi_2(u^0_l)), \\
\partial_t u^0_{l+1} |_{t=0} = u_0(x),
\end{cases},$$

(3.24)

where we set $u^0_0(t, x) \equiv u_0(x)$. 
Lemma 3.2. Suppose that there are positive constants \( G > 0 \) and \( \alpha > 0 \) such that \( L > 0 \), we deduce that \( \partial_x^\alpha u_{i+1}^0 \) satisfies the following problem:

\[
\begin{align*}
\partial_t (\partial_x^\alpha u_{i+1}^0) + (u_i^0 \cdot \nabla) (\partial_x^\alpha u_{i+1}^0) + \partial_x^\alpha u_{i+1}^0 &= \partial_x^\alpha (g - \nabla (\psi_1(u_i^0) + \psi_2(u_i^0))) - f_{\alpha, j}, \\
\partial_x^\alpha u_{i+1}^0 |_{t=0} &= \partial_x^\alpha u_0,
\end{align*}
\tag{3.25}
\]

where \( f_{\alpha, j} = \partial_x^\alpha [(u_i^0 \cdot \nabla) u_{i+1}^0] - (u_i^0 \cdot \nabla) \partial_x^\alpha u_{i+1}^0 \). Multiply \( \partial_x^\alpha u_{i+1}^0 \) on the equation in (3.25) and sum for all \(|\alpha| \leq s\). By integrating the resulting equation with respect to \( x \) over \( T^d \), it follows

\[
\max_{0 \leq t \leq T} \| u_{i+1}^0(t) \|_{H^s}^2 \leq C_0 \left( \| u_0 \|_{H^s}^2 + \int_0^T (\| g(t) \|_{H^s}^2 + \| \psi_1(u_i^0(t)) + \psi_2(u_i^0(t)) \|_{H^{s+1}}^2) \, dt \\
+ \sum_{|\alpha| \leq s} \int_0^T \| f_{\alpha, j}(t) \|_{L^2}^2 \, dt \right).
\tag{3.26}
\]

For the last term on the right-hand side of (3.26), by using the Moser-type inequality (refer to Lemma A.1 of [7]), we deduce

\[
\sum_{|\alpha| \leq s} \| f_{\alpha, j}(t) \|_{L^2}^2 \leq C \| u_i^0(t) \|_{H^s} \| u_{i+1}^0(t) \|_{H^s}.
\tag{3.27}
\]

Finally, substituting (3.27) into (3.26), and using (3.20) and (3.21), we immediately obtain

\[
\max_{0 \leq t \leq T} \| u_{i+1}^0(t) \|_{H^s}^2 \leq C_0 \left( \| u_0 \|_{H^s}^2 + \int_0^T \| g(t) \|_{H^s}^2 \, dt + \max_{0 \leq t \leq T} \| u_i^0(t) \|_{H^s}^2 \int_0^T \| u_{i+1}^0(t) \|_{H^s}^2 \, dt \\
+ \left( M \left( \max_{0 \leq t \leq T} \| u_i^0(t) \|_{H^s} \right) + 1 \right) \int_0^T \| u_{i+1}^0(t) \|_{H^s}^2 \, dt \right),
\tag{3.28}
\]

where \( M(\cdot) \) is a nondecreasing, continuous function with \( M(0) = 0 \).

To estimate (3.28), we have the following lemma:

**Lemma 3.2.** Suppose that there are positive constants \( C_1, C_2 \) and \( C_3 \), and a nonnegative continuous function \( g(t) \) such that the nonnegative continuous function sequence \( \{ a_i(t) \} \) satisfies

\[
\begin{align*}
0 \leq a_0(t) &\leq 2C_1 + 2C_2 \int_0^t g(\tau) \, d\tau, \\
0 \leq a_{i+1}(t) &\leq C_1 + C_2 \int_0^t g(\tau) \, d\tau + M(a_i(t)) \int_0^t a_i(\tau) \, d\tau + C_3 \int_0^t (a_i(\tau)a_{i+1}(\tau) + a_i(\tau)) \, d\tau,
\end{align*}
\tag{3.29}
\]

for any \( n \in \mathbb{N} \), where \( M(\cdot) \) is a nondecreasing, continuous function with \( M(0) = 0 \). Let \( T > 0 \) and \( G > 0 \) be such that

\[
\begin{align*}
C_1 e^{3(C_3 + M(G))T} + C_2 \int_0^T g(\tau) e^{2(C_3 + M(G))(T-\tau)} \, d\tau &\leq G/2, \\
C_1 (e^{3(C_3 + M(G))T} - 1) + C_2 \int_0^T g(\tau)(e^{2(C_3 + M(G))(T-\tau)} - 1) \, d\tau &\leq \frac{C_3 + M(G)}{C_3}.
\end{align*}
\tag{3.30}
\]
Then the estimate
\[ a_l(t) \leq 2C_1 e^{2(C_3 + M(G))t} + 2C_2 \int_0^t g(\tau) e^{2(C_3 + M(G))(t-\tau)} d\tau, \quad t \in [0, T], \] (3.31)
holds for any \( l \in \mathbb{N} \).

**Proof.** Obviously, the estimate (3.31) hold for \( l = 0 \). Suppose that (3.31) is true for some \( l \in \mathbb{N} \), let us estimate \( a_{l+1}(t) \). Denote by
\[ b(t) = C_1 + C_2 \int_0^t g(\tau) d\tau + (M(a_l(t)) + C_3) \int_0^t a_l(\tau) d\tau. \]

From (3.30) and the assumption, we have
\[ a_l(t) \leq G \quad \text{for any} \quad t \in [0, T]. \] (3.32)

By using (3.32) and a simple computation, we deduce
\[ b(t) \leq C_1 e^{2(C_3 + M(G))t} + C_2 \int_0^t e^{2(C_3 + M(G))(t-\tau)} g(\tau) d\tau. \] (3.33)

It is obvious that (3.29) can be written as
\[ a_{l+1}(t) \leq C_3 a_l(t) \int_0^t a_{l+1}(\tau) d\tau + b(t), \]
which implies
\[ a_{l+1}(t) \leq C_3 a_l(t) \int_0^t e^{\int_0^\tau C_3 a_l(s) ds} b(\tau) d\tau + b(t) \leq C_3 a_l(t) \int_0^t e^{\int_0^t C_3 a_l(s) ds} b(\tau) d\tau + b(t). \] (3.34)

Using the estimate (3.31) of \( a_l(t) \) and (3.30)\(_2\), it follows
\[ \int_0^t C_3 a_l(s) ds \leq 1 \quad \text{for any} \quad t \in [0, T]. \]

Thus, from (3.34) we conclude
\[ a_{l+1}(t) \leq b(t) + \frac{1}{2} a_l(t) \leq 2C_1 e^{2(C_3 + M(G))t} + 2C_2 \int_0^t g(\tau) e^{2(C_3 + M(G))(t-\tau)} d\tau, \]
which means (3.31) being held for the case \( l + 1 \) as well. \( \square \)

By applying Lemma 3.2 in (3.28), we obtain there are \( T_1 \in (0, T_0] \) and \( G > 0 \) depending only on \( b \) and \( u_0 \) such that
\[ \max_{0 \leq t \leq T_1} \| u^0(t) \|_{H^s}^2 \leq 2C_0 e^{2(C_0 + M(G)T_1)} \| u_0 \|_{H^s}^2 + 2C_0 \int_0^{T_1} \| g(\tau) \|_{H^s}^2 e^{2(C_0 + M(G))(T_1-\tau)} d\tau, \] (3.35)
for any \( l \in \mathbb{N} \), which shows that the solution sequence \( \{u_0^l(t, x)\}_{l \in \mathbb{N}} \) is bounded in \( C([0, T_1], H^s(T^d)) \). From the equation in (3.24), we further obtain that \( \{u_0^l(t, x)\}_{l \in \mathbb{N}} \) is bounded in \( C^1([0, T_1], H^{s-1}(T^d)) \).

To prove the convergence of \( \{u_0^l(t, x)\}_{l \in \mathbb{N}} \), by setting \( w_0^l = u_0^l + u_0^l \), then from (3.24) we see that \( w_0^l \) satisfies the following problem:

\[
\begin{aligned}
&\partial_t u_0^l + (u_0^l \cdot \nabla) u_0^l + \nabla (\psi_1(u_{l-1}^0) - \psi_1(u_0^l) + \psi_2(u_{l-1}^0) - \psi_2(u_0^l)) = f_l,
&w_0^l|_{t=0} = 0,
\end{aligned}
\]

where

\[
f_l = - (u_0^l \cdot \nabla) u_0^l + \nabla (\psi_1(u_{l-1}^0) - \psi_1(u_0^l) + \psi_2(u_{l-1}^0) - \psi_2(u_0^l)).
\]

Applying the classical theory of transport equations in (3.36), and using the boundedness of \( \{u_0^l\}_{l \geq 0} \), it follows

\[
\max_{0 \leq t \leq T} \|w_0^l(t)\|_{L^2}^2 \leq C \int_0^T \|w_0^{l-1}(\tau)\|_{L^2}^2 \, d\tau,
\]

for any \( T \in (0, T_1] \), which implies there is \( T_2 \in (0, T_1] \) such that \( \{u_0^l\}_{l \geq 0} \) is a Cauchy sequence in \( C([0, T_2], L^2(T^d)) \).

By using the classical interpolation theorem, we conclude that \( \{u_0^l\}_{l \geq 0} \) constructed by (3.24) is convergent in \( C([0, T_2], H^{s-\varepsilon}(T^d)) \cap C^1([0, T_2], H^{s-1-\varepsilon}(T^d)) \) for any \( \varepsilon > 0 \), and its limit \( u^0 \), belonging to \( C([0, T_2], H^s(T^d)) \cap C^1([0, T_2], H^{s-1}(T^d)) \), is the unique solution to the problem (3.23).

Therefore, we have established the following result:

**Theorem 3.3.** For the nonlinear Cauchy problem (2.6), given any

\[
b \geq \sum_{j=0} \begin{cases} C^j([0, T_0], H^{s+2-j}(T^d)) \end{cases}
\]

and \( u_0 \in H^s(T^d) \) satisfying the compatibility condition

\[
\partial_t b(0, x) + \text{div}(b(0, x)u_0(x)) = 0
\]

and \( b(t, x) \geq b_0 > 0 \) for a constant \( b_0 \) and \( s > d/2 + 1 \), there is \( T_s \in (0, T_0] \) depending only on \( b \) and \( u_0 \) such that the problem (2.6) admits a unique solution

\[
\begin{cases}
&u^0 \in C([0, T_s], H^s(T^d)) \cap C^1([0, T_s], H^{s-1}(T^d)), \\
&\psi^0 \in C([0, T_s], H^{s+1}(T^d))
\end{cases}
\]

in the class \( m(\psi^0) = 0 \).

The regularity of \( \psi^0 \) stated above is easily obtained from problems (3.16), (3.17) and (3.18).
Now, let us briefly describe the solvability of \( \{ u^j, \psi^j \} \) for any \( j \geq 1 \) from the problem (2.8) provided that we have known \( \{ u^k, \psi^k \}_{k \leq j - 1} \) already. Obviously, from (2.8) we know that \( (u^j, \psi^j) \) satisfy the following problem for a linearized Euler system:

\[
\begin{cases}
\partial_t u^j + (u^0 \cdot \nabla) u^j + (u^j \cdot \nabla) u^0 + u^j + \nabla \psi^j = g^j, \\
\text{div}(bu^j) = \Delta \psi^j - 1 + \text{div}(\Delta \psi^j - 1 u^0) - \sum_{k=1}^{j-1} \text{div}(n^k u^{j-k}), \\
u^j|_{t=0} = u_j(x), \\
m(\psi^j) = 0,
\end{cases}
\]

where

\[
g^j = \nabla (h'(b)\Delta \psi^j - 1 - h^j - 1 (b, n^j, \ldots, n^{j-1})) - \sum_{k=1}^{j-1} (u^k \cdot \nabla) u^{j-k}.
\]

From (3.37), we easily deduce that \( \psi^j \) satisfies the following problem

\[
\begin{cases}
\text{div}(b \nabla \psi^j) = F(u^j, \nabla u^j) + G^j, \\
m(\psi^j) = 0,
\end{cases}
\]

where

\[
F(u^j, \nabla u^j) = \text{div}(u^j (u^0 \cdot \nabla) b + u^0 (u^j \cdot \nabla) b + u^j (\partial_t b - b)) \\
- \sum_{i,k=1}^{d} \left( \frac{\partial u^0_k}{\partial x_i} \frac{\partial (bu^j)}{\partial x_k} + \frac{\partial u^j_k}{\partial x_i} \frac{\partial (bu^0)}{\partial x_k} \right) + (u^j \cdot \nabla) (\partial_t b)
\]

and

\[
G^j = \text{div}(bg^j) - (\partial_t + (u^0 \cdot \nabla)) \left( \Delta \psi^j - 1 + \text{div}(\Delta \psi^j - 1 u^0) - \sum_{k=1}^{j-1} \text{div}(n^k u^{j-k}) \right).
\]

Decompose the solution \( \psi^j \) of (3.38) into two parts

\[
\psi^j = \psi^j_1 + \psi^j_2,
\]

where \( \psi^j_1 \) satisfies

\[
\begin{cases}
\text{div}(b \nabla \psi^j_1) = F(u^j, \nabla u^j), \\
m(\psi^j_1) = 0,
\end{cases}
\]

(3.39)
and $\psi^j_2$ satisfies

\[
\begin{aligned}
\text{div}(b \nabla \psi^j_2) &= G^j, \\
\mathbf{m}(\psi^j_2) &= 0.
\end{aligned}
\tag{3.40}
\]

As in (3.20), from (3.39) and (3.40), we have the estimates

\[
\|\psi^j_1(t)\|_{H^{k+1}(T^d)} \leq C_k \|u^j(t)\|_{H^{k}(T^d)}
\tag{3.41}
\]

and

\[
\|\psi^j_2(t)\|_{H^{k+1}(T^d)} \leq C_k,
\tag{3.42}
\]

for any $0 \leq k \leq s$ provided that we have

\[
\begin{aligned}
\psi^{j-1} &\in C([0, T], H^{s+3}(T^d)) \cap C^1([0, T], H^{s+2}(T^d)) \cap C^2([0, T], H^{s+1}(T^d)), \\
(n^k, u^k) &\in C([0, T], H^{s+1}(T^d)) \cap C^1([0, T], H^s(T^d)),
\end{aligned}
\tag{3.43}
\]

for all $0 \leq k \leq j - 1$.

Using (3.41) and (3.42) in the problem (3.37), we deduce that under the condition (3.43) and $u_j \in H^s(T^d)$, we have

\[
u^j \in C([0, T], H^{s+1}(T^d)),
\]

which implies

\[
\psi^j \in C([0, T], H^{s+1}(T^d)),
\]

from (3.41) and (3.42).

In summary, we obtain:

\[\textbf{Theorem 3.4.}\] Let $T_\ast \in [0, T_0]$ be determined in Theorem 3.2, and $s > d/2$. For any $j \in \mathbb{N}$, suppose that

\[
\begin{aligned}
u^j &\in H^{s+2j-k}(T^d), \\
b &\in \bigcap_{k=0}^{2j+2} C^k([0, T_\ast], H^{s+2j+2-k}(T^d))
\end{aligned}
\]

and the compatibility conditions (3.8), (3.3) and (3.6) hold for any $0 \leq k \leq j$, then the problem (2.8) has a unique solution

\[
\begin{aligned}
u^j &\in C([0, T_\ast], H^s(T^d)) \cap C^1([0, T_\ast], H^{s-1}(T^d)), \\
\psi^j &\in C([0, T_\ast], H^{s+1}(T^d)),
\end{aligned}
\]

in the class $\mathbf{m}(\psi^j) = 0$.\]
4. Rigorous justification

In this section, we rigorously justify the asymptotic analysis of solutions \((n^\lambda, u^\lambda, \Psi^\lambda)\) to the Cauchy problem (2.1)–(2.3) developed in Section 2. As a consequence, we obtain the existence of exact solutions \((n^\lambda, u^\lambda, \Psi^\lambda)\) to (2.1)–(2.3) in a time interval independent of \(\lambda \in (0, \lambda_0]\), and the convergence of \((u^\lambda, \Psi^\lambda)\) to the solution \((u^0, \psi^0)\) of the incompressible type Euler equations (2.6) when the Debye length \(\lambda\) goes to zero.

For any fixed \(m \in \mathbb{N}\), denote by

\[
\begin{align*}
n^\lambda_{a,m} &= \sum_{j=0}^{m} \lambda^{2j} n^j(t, x), \\
u^\lambda_{a,m} &= \sum_{j=0}^{m} \lambda^{2j} \psi^j(t, x), \\
\psi^\lambda_{a,m} &= \sum_{j=0}^{m} \lambda^{2j} \phi^j(t, x),
\end{align*}
\]

with \((n^k, u^k, \psi^k)\) being given by Theorems 3.3 and 3.4. From the asymptotic analysis of Section 2, we know that \((n^\lambda_{a,m}, u^\lambda_{a,m}, \psi^\lambda_{a,m})\) satisfy the following problem:

\[
\begin{align*}
\begin{cases}
\partial_t n^\lambda_{a,m} + \text{div} \left( n^\lambda_{a,m} u^\lambda_{a,m} \right) &= R_n^\lambda, \\
\partial_t u^\lambda_{a,m} + (u^\lambda_{a,m} \cdot \nabla) u^\lambda_{a,m} + u^\lambda_{a,m} + \nabla (h(n^\lambda_{a,m}) + \psi^\lambda_{a,m}) &= R_u^\lambda, \\
\lambda^2 \Delta \psi^\lambda_{a,m} &= b(t, x) - n^\lambda_{a,m} + R_\psi^\lambda, \\
m(\psi^\lambda_{a,m}) &= 0,
\end{cases}
\end{align*}
\]

where the remainders \(R_n^\lambda, R_u^\lambda\) and \(R_\psi^\lambda\) satisfy

\[
\sup_{0 \leq t \leq T_*} \left( \| R_n^\lambda(t, \cdot) \|_{H^1(T^d)}, \| R_u^\lambda(t, \cdot) \|_{H^1(T^d)}, \| R_\psi^\lambda(t, \cdot) \|_{H^1(T^d)} \right) \leq C \lambda^{2(m+1)},
\]

for a constant \(C > 0\) for any \(0 \leq s \leq s - 1\).

Let \((n^\lambda, u^\lambda, \psi^\lambda)\) be the unknown solutions to the Cauchy problem (2.1)–(2.3), and denote by

\[
N^\lambda = n^\lambda - n^\lambda_{a,m}, \quad U^\lambda = u^\lambda - u^\lambda_{a,m}, \quad \Phi^\lambda = \psi^\lambda - \psi^\lambda_{a,m}.
\]

Obviously, \((N^\lambda, U^\lambda, \Phi^\lambda)\) satisfy the following problem:

\[
\begin{align*}
\begin{cases}
\partial_t N^\lambda + \text{div} \left( N^\lambda (U^\lambda + u^\lambda_{a,m}) + n^\lambda_{a,m} U^\lambda \right) &= - R_n^\lambda, \\
\partial_t U^\lambda + \left[ (U^\lambda + u^\lambda_{a,m}) \cdot \nabla \right] U^\lambda + (U^\lambda \cdot \nabla) u^\lambda_{a,m} + U^\lambda + \nabla (h(N^\lambda + n^\lambda_{a,m}) - h(n^\lambda_{a,m})) &= - R_u^\lambda, \\
\lambda^2 \Delta \Phi^\lambda &= - N^\lambda - R_\psi^\lambda, \\
m(\Phi^\lambda) &= 0,
\end{cases}
\end{align*}
\]

\[
N^\lambda|_{t=0} = n^\lambda_0 - \sum_{j=0}^{m} \lambda^{2j} n^j(0, x), \quad U^\lambda|_{t=0} = u^\lambda_0 - \sum_{j=0}^{m} \lambda^{2j} u^j(x).
\]
Denote by $\Phi^\lambda = \Phi^\lambda(N^\lambda)$ the unique solution of the third and fourth equations in (4.4). It is easy to deduce

$$
\| \nabla \Phi^\lambda(t) \|_{H^\tau} \leq \frac{C}{\lambda^\tau} (\| N^\lambda(t) \|_{H^{\tau-1}} + \| R^\lambda(t) \|_{H^{\tau-1}}),
$$

(4.5)

for any $\tau \geq 0$.

Set

$$
w^\lambda = \left( \begin{array}{c} N^\lambda \\ U^\lambda \end{array} \right), \quad w^\lambda_0 = \left( \begin{array}{c} n_0^\lambda - \sum_{j=0}^m \lambda^j n^j(0, x) \\ u_0^\lambda - \sum_{j=0}^m \lambda^j u_j(x) \end{array} \right),
$$

$$
B_j(w^\lambda) = (U^\lambda + u^\lambda_{a,m})_j I + \left( \begin{array}{cc} 0 & (N^\lambda + n^\lambda_{a,m})e_j^T \\ h'(N^\lambda + n^\lambda_{a,m})c_j & 0 \end{array} \right),
$$

$$
F(w^\lambda) = \left( U^\lambda \cdot \nabla \right) u_{a,m}^\lambda + (h'(N^\lambda + n^\lambda_{a,m}) - h'(n^\lambda_{a,m})) \nabla n_{a,m}^\lambda
$$

and

$$
R^\lambda = \left( \begin{array}{cc} -R^\lambda_n & -R^\lambda_u \\ -R^\lambda_n & -R^\lambda_u \end{array} \right), \quad \tilde{\Phi}^\lambda(w^\lambda) = \left( \begin{array}{c} 0 \\ \nabla \Phi^\lambda(N^\lambda) \end{array} \right).
$$

Then the problem (4.4) for the unknowns $w^\lambda$ can be rewritten as

$$
\begin{aligned}
\partial_t w^\lambda + \sum_{j=1}^d B_j(w^\lambda) \partial_{x_j} w^\lambda + F(w^\lambda) + \tilde{\Phi}(w^\lambda) &= R^\lambda, \\
\left. w^\lambda \right|_{t=0} &= w^\lambda_0.
\end{aligned}
$$

(4.6)

It is not difficult to see that the equations of $w^\lambda$ in (4.6) are symmetrizable hyperbolic, i.e., if we introduce

$$
A_0(w^\lambda) = \left( \begin{array}{cc} h'(N^\lambda + n^\lambda_{a,m}) & 0 \\ 0 & (N^\lambda + n^\lambda_{a,m}) I_{d \times d} \end{array} \right),
$$

which is positively definite when $N^\lambda + n^\lambda_{a,m} \geq C_0 > 0$ for $0 < \lambda \leq \lambda_0 \ll 1$, to be verified later, then $A_j(w^\lambda) = A_0(w^\lambda) B_j(w^\lambda)$ are symmetric for all $1 \leq j \leq d$.

As usual (cf. [8]), we solve the nonlinear problem (4.6) by the following iteration:

$$
\begin{aligned}
\partial_t w^\lambda_{k+1} + \sum_{j=1}^d B_j(w^\lambda_{k}) \partial_{x_j} w^\lambda_{k+1} + F(w^\lambda_{k}) + \tilde{\Phi}(w^\lambda_{k}) &= R^\lambda, \\
\left. w^\lambda_{k+1} \right|_{t=0} &= w^\lambda_0,
\end{aligned}
$$

(4.7)
with \( w^{\lambda,0}(t, x) \equiv 0 \). For studying the problem (4.7), we introduce the weighted norms

\[
\|w\|_{s, \lambda} = \left( \sum_{|\alpha| \leq s} \lambda^{d|\alpha|} \|\partial_x^\alpha w\|_{L^2(T^d)}^2 \right)^{1/2}, \quad \|w\|_{s, \lambda, T} = \sup_{0 \leq t \leq T} \|w(t)\|_{s, \lambda}.
\]

Obviously, we have

\[
\|w\|_{L^\infty(T^d)} \leq C \lambda^{-2s_0} \|w\|_{s_0, \lambda},
\]

for any fixed \( s_0 > d/2 \).

For any fixed \( s > d/2 + 2 \), \( \alpha \in \mathbb{N}^d \) with \( |\alpha| \leq s - 1 \), from (4.7) we know that \( V^{\lambda, k+1}_\alpha = \partial_x^{\alpha} w^{\lambda, k+1}(t, x) \) satisfies the following problem:

\[
\begin{aligned}
& A_0(w^{\lambda, k}) \partial_t V^{\lambda, k+1}_\alpha + \sum_{j=1}^d A_j(w^{\lambda, k}) \partial_{x_j} V^{\lambda, k+1}_\alpha = G^{\lambda, k}_\alpha, \\
& V^{\lambda, k+1}_\alpha|_{t=0} = \partial_x^{\alpha} w^{\lambda, 0}_\alpha.
\end{aligned}
\]  

(4.8)

Employing the classical energy estimate of hyperbolic problems (cf. [8]) for the problem (4.8), we deduce

\[
\sup_{0 \leq t \leq T} \left\| V^{\lambda, k+1}_\alpha(t) \right\|_{L^2} \leq C e^{M \| \text{div} A(w^{\lambda, k}) \|_{L^\infty} T} \left\{ \left\| \partial_x^{\alpha} w^{\lambda, 0}_\alpha \right\|_{L^2} + \int_0^T \left\| G^{\lambda, k}_\alpha(\tau) \right\|_{L^2} d\tau \right\},
\]

(4.9)

where \( \text{div} A(w^{\lambda, k}) = \partial_t A_0(w^{\lambda, k}) + \sum_{j=1}^d \partial_{x_j} A_j(w^{\lambda, k}) \), and \( M(\cdot) \) is a continuous nondecreasing function with respect to its argument.

Denote by \( \| \cdot \|_{L^\infty(0, T; L^2(T^d))} \) the norm \( \| \cdot \|_{L^\infty(0, T; L^2(T^d))} \) for simplicity. By using (4.5) and the classical Moser-type inequality (cf. [6,7]), we deduce

\[
\int_0^T \left\| G^{\lambda, k}_\alpha(\tau) \right\|_{L^2} d\tau \leq N_1(\|w^{\lambda, k}\|_{L^\infty(T^d)}) \left( \int_0^T (\|R^{\lambda, k}(\tau)\|_{H^{s_1}} + \|w^{\lambda, k}(\tau)\|_{H^{s_1}}) d\tau \right)
\]

\[
+ \frac{1}{\lambda^2} \int_0^T (\|R^{\lambda, k}_j(\tau)\|_{H^{s_1} + 1} + \|w^{\lambda, k}(\tau)\|_{H^{s_1} + 1}) d\tau
\]

\[
+ N_2(\|w^{\lambda, k}\|_{L^\infty(T^d)}, \|w^{\lambda, k+1}\|_{L^\infty(T^d)}) \int_0^T \|w^{\lambda, k}(\tau)\|_{H^{s_1}} d\tau,
\]

(4.10)
where \( N_j(\cdot) \ (j = 1, 2, 3) \) are continuous nondecreasing functions with respect to their arguments. Substituting (4.10) into (4.9), it follows

\[
\begin{align*}
\| w^{\lambda k+1} \|_{s-1, \lambda, T} & \leq C e^{M(\| w^{\lambda k} \|_{L^{\infty}(W^{1, \infty})}) T} \left\{ \| w^{\lambda}_0 \|_{s-1, \lambda} \\
& + N_1(\| w^{\lambda k} \|_{L^{\infty}(W^{1, \infty})}) \int_0^T \left( \| w^{\lambda k}(\tau) \|_{s-1, \lambda} + \| R^{\lambda}(\tau) \|_{s-1, \lambda} + \| R^{\lambda}_\psi(\tau) \|_{s-2, \lambda} \right) d\tau \\
& + N_2(\| w^{\lambda k} \|_{L^{\infty}(W^{1, \infty})}) \int_0^T \| w^{\lambda k+1}(\tau) \|_{s-1, \lambda} d\tau \\
& + N_3(\| w^{\lambda k} \|_{L^{\infty}(W^{1, \infty})}, \| w^{\lambda k+1} \|_{L^{\infty}(W^{1, \infty})}) \int_0^T \| w^{\lambda k}(\tau) \|_{s-1, \lambda} d\tau \right\}.
\end{align*}
\]

(4.11)

From (4.2), we have

\[
\int_0^T (\| R^{\lambda}(\tau) \|_{s-1, \lambda} + \| R^{\lambda}_\psi(\tau) \|_{s-2, \lambda}) d\tau \leq C \lambda^{2(m+1)} T,
\]

(4.12)

when \( s \geq d/2 + 2 \) in Theorems 3.3 and 3.4.

Thus, under the assumption

\[
\| w^{\lambda}_0 \|_{s-1, \lambda} \leq C \lambda^{2(m+1)},
\]

(4.13)

we have

\[
\begin{align*}
\| w^{\lambda k+1} \|_{s-1, \lambda, T} & \leq C e^{M(\lambda^{-2(s-1)} \| w^{\lambda k} \|_{s-1, \lambda, T}) T} \left\{ \lambda^{2(m+1)} \\
& + N_1(\lambda^{-2(s-2)} \| w^{\lambda k} \|_{s-2, \lambda, T}) \left( T \lambda^{2(m+1)} + \int_0^T \| w^{\lambda k}(\tau) \|_{s-1, \lambda} d\tau \right) \\
& + N_2(\lambda^{-2(s-1)} \| w^{\lambda k} \|_{s-1, \lambda, T}) \int_0^T \| w^{\lambda k+1}(\tau) \|_{s-1, \lambda} d\tau \\
& + N_3(\lambda^{-2(s-2)} \| w^{\lambda k} \|_{s-2, \lambda, T}, \lambda^{-2(s-1)} \| w^{\lambda k+1} \|_{s-1, \lambda, T}) \int_0^T \| w^{\lambda k}(\tau) \|_{s-1, \lambda} d\tau \right\},
\end{align*}
\]

(4.14)

for any fixed \( s > d/2 + 2 \), which implies by induction on \( k \) that there are \( T_* \in (0, T_*) \) with \( T_0 \) being given in Theorem 3.3, and a constant \( C > 0 \) such that

\[
\| w^{\lambda k} \|_{s-1, \lambda, T_*} \leq C \lambda^{2(m+1)},
\]

(4.15)

for any \( k \in \mathbb{N} \) when \( m \geq s - 2 \).
Set $v^{\lambda,k} = w^{\lambda,k+1} - w^{\lambda,k}$. From (4.7), we know that $v^{\lambda,k}$ satisfies the following problem

$$
\begin{aligned}
\partial_t v^{\lambda,k} + \sum_{j=1}^{d} B_j (w^{\lambda,k}) \partial_{x_j} v^{\lambda,k} &= F^{\lambda,k}, \\
v^{\lambda,k}|_{t=0} &= 0,
\end{aligned}
$$

(4.16)

where

$$
F^{\lambda,k} = F(w^{\lambda,k-1}) - F(w^{\lambda,k}) + \tilde{\Phi}^{\lambda}(w^{\lambda,k-1}) - \tilde{\Phi}^{\lambda}(w^{\lambda,k}) + \sum_{j=1}^{d} (B_j (w^{\lambda,k-1}) - B_j (w^{\lambda,k})) \partial_{x_j} w^{\lambda,k}
$$

satisfies

$$
\|F^{\lambda,k}(t)\|_{H^1(T^d)} \leq C \left( \|v^{\lambda,k-1}(t)\|_{H^1(T^d)} + \frac{1}{\lambda^2} \|v^{\lambda,k-1}(t)\|_{L^2(T^d)} \right),
$$

obviously for any $0 \leq t \leq T^*$ by using the boundedness of $\{w^{\lambda,k}\}_{k \geq 0}$ in $L^\infty(0,T^*; W^{1,\infty}(T^d))$.

Employing the classical theory of linear symmetrizable hyperbolic equations for the problem (4.16), we deduce

$$
\|v^{\lambda,k}\|_{1,\lambda,T^*} \leq C \int_0^{T^*} \|v^{\lambda,k-1}(\tau)\|_{1,\lambda} d\tau,
$$

which implies the convergence of $w^{\lambda,k}(t,x)$ to $w^\lambda(t,x)$ in $C([0,T^*], H^1(T^d))$ when $k \to +\infty$. Combining this with the boundedness result (4.15), it gives rise to that $w^{\lambda,k}(t,x)$ converges to $w^\lambda(t,x)$ in $C([0,T^*], H^{s+\varepsilon}(T^d))$ for any $\varepsilon > 0$, and $w^\lambda(t,x) \in C([0,T^*], H^s(T^d))$ is the unique solution to the problem (4.6). Rewriting $s$ as $s + 1$, returning to the problem (4.4) and the transform (4.3), we conclude

**Theorem 4.1.** For any fixed $s > d/2 + 1$ and integer $m \geq s - 1$, suppose that

$$
u_j \in H^{s+2(m-j)+1}(T^d), \quad b \in \bigcap_{k=0}^{2m+2} C^k([0,T_0], H^{s+2m+3-k}(T^d))
$$

(4.17)

satisfy the compatibility conditions (3.8), (3.3) and (3.6) for all $0 \leq j \leq m$, and

$$
\begin{aligned}
\left\| \frac{d^n}{dx^n} b(0,x) - \sum_{j=1}^{m} \lambda^j n^j(0,x) \right\|_{s,\lambda} &\leq C \lambda^{2(m+1)}, \\
\left\| \frac{d^n}{dx^n} u_j(x) \right\|_{s,\lambda} &\leq C \lambda^{2(m+1)},
\end{aligned}
$$

(4.18)

with $\{n^j(0,x)\}_{1 \leq j \leq m}$ being given as in Sections 2 and 3, then there is $T^* \in (0,T_0]$ such that the problem (2.1)–(2.3) has a unique solution

$$
n^\lambda, u^\lambda \in C([0,T^*], H^s(T^d)), \quad \Psi^\lambda \in C([0,T^*], H^{s+2}(T^d))
$$

(4.19)
satisfying

\[
\begin{align*}
\left\| n^{\lambda} - b(t, x) - \sum_{j=1}^{m} \lambda^{2j} n_{j} \right\|_{s, \lambda, T^*} &\leq \tilde{C} \lambda^{2(m+1)}, \\
\left\| u^{\lambda} - \sum_{j=0}^{m} \lambda^{2j} u_{j} \right\|_{s, \lambda, T^*} &\leq \tilde{C} \lambda^{2(m+1)}, \\
\left\| \psi^{\lambda} - \sum_{j=0}^{m} \lambda^{2j} \psi_{j} \right\|_{s+2, \lambda, T^*} &\leq \tilde{C} \lambda^{2(m+1)},
\end{align*}
\]

where \( \{n^{j}, u^{j}, \psi^{j}\}_{0 \leq j \leq m} \) are solutions to the problems (2.6)–(2.8).

**Proof.** From (4.2), we know that to have the estimate (4.20), one needs

\[
\begin{align*}
n^{\lambda}(t, x) &= b(t, x) + \sum_{j=1}^{m} \lambda^{2j} n_{j}(t, x) \in C([0, T], H^{s+1}(T^{d})), \\
u^{\lambda}(t, x) &= \sum_{j=0}^{m} \lambda^{2j} u_{j}(t, x) \in C([0, T], H^{s+1}(T^{d})), \\
\psi^{\lambda}(t, x) &= \sum_{j=0}^{m} \lambda^{2j} \psi_{j}(t, x) \in C([0, T], H^{s+2}(T^{d})),
\end{align*}
\]

which leads to the assumption (4.17) by using Theorems 3.3 and 3.4.

The existence of \((n^{\lambda}, u^{\lambda}) \in C([0, T^*], H^{s+1}(T^{d}))\) was established already with \(T^*\) being given in (4.15). From the equation of \(\psi^{\lambda}\) in (2.1), we immediately deduce

\[
\psi^{\lambda} \in C([0, T], H^{s+2}(T^{d})).
\]

Under the assumption (4.18), the first two asymptotic results in (4.20) follow from (4.15) immediately, and the last one in (4.20) can be easily derived from the estimate (4.5) of \(\Phi^{\lambda} = \psi^{\lambda} - \sum_{j=0}^{m} \lambda^{2j} \psi_{j}(t, x)\). \(\square\)

**Remark 4.2.** From (4.20) we immediately conclude

\[
\begin{align*}
n^{\lambda}(t, x) &= b(t, x) + O(\lambda^{2}), \\
u^{\lambda}(t, x) &= u^{0}(t, x) + O(\lambda^{2}), \\
\psi^{\lambda}(t, x) &= \psi^{0}(t, x) + O(\lambda^{2})
\end{align*}
\]

in \(C([0, T], L^{2}(T^{d}))\) and in \(L^{\infty}((0, T) \times T^{d})\) under the assumptions (4.17) and (4.18) for the case \(s = \lfloor d/2 \rfloor + 2\) and \(m = \lfloor d/2 \rfloor + 1\).
Note in the proof

The first version of this paper appeared as a preprint in [12]. Recently, we saw a publication [15] by S. Wang in which a similar result was proved on the convergence of the compressible Euler–Poisson equations to the incompressible Euler equations. Here we state the main differences between two papers. First, S. Wang only dealt with the case $b \equiv 1$ and we consider the problem for a general function $b = b(t, x)$ for which the existence of solutions to the limit problem has to be shown. Second, the convergence of the asymptotic expansion was proved in [15] only up to first order which is not sufficient to give an error estimate on the electric potential. Our asymptotic expansions are justified up to any order and provide the error estimate on this variable. Finally, S. Wang studied also the quasi-neutral limit in Navier–Stokes equations which is out of the goal of our paper.

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References