Undecidability and temporal logic: some landmarks from Turing to the present

Valentin Goranko
Technical University of Denmark

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Outline

- Introduction: a brief retrospective.
- Turing’s undecidability of the Halting Problem (HP) from temporal logics perspective.
- Undecidability of interval temporal logics by reduction from the HP
- Undecidability by reduction from tiling problems
- Undecidability of temporalized logics
- Undecidability of quantitative temporal logics
- Outlook and concluding remarks
The focus of this talk

- Undecidability of the *satisfiability/validity* problem; (almost) no model checking;

- Only *propositional* temporal (and modal) logics; (almost) no first-order logics;

- Details of some interesting cases and an overview of the rest.

No claim of completeness!
Introduction: a brief retrospective

- 1936: Turing proves the undecidability of the Halting problem.
- 1936: Church proves the undecidability of first-order logic.
- 1962: Büchi: decidability of the MSO of one successors
- 1969: Rabin: decidability of the MSO of two successors
- 1974: Burstall proposes the use of temporal logics in CS.
- 1977: Pnueli introduces LTL and proposes the use of temporal logics to specification and verification of reactive systems.
- Since early 1970s: many decidability results for propositional modal and temporal logics, using FMP or Büchi/Rabin results. Modal/temporal logics praised for their “robust decidability”.
- However, since the mid 1970s a variety of undecidability results in propositional temporal logics emerges, too.
- In retrospective: Turing’s undecidability of the Halting problem is the first such undecidability result.
Turing’s undecidability of the Halting problem from temporal logic perspective
The halting of a Turing machine as a temporal logic formula

- The configuration graph of a TM as a transition system $Conf(M)$:
  - States: configurations of the TM
  - Transitions: determined by the TM transition relation
  - Labels for initial and terminal states

- Temporal logic for Turing machines:
  - Atomic propositions $\text{init}$ for the initial states and $\text{term}$ for terminal states, plus temporal operators incl. $\mathcal{X}$ and $\mathcal{F}$.
  - Expressing the halting property (for deterministic TM):
    $$ \text{init} \rightarrow \mathcal{F} \text{term} $$
The Halting problem as a model checking problem

The Halting problem as a local model checking problem:

- The halting of a Turing machine $M$ on any given input is equivalent to the truth of $\mathcal{F}_{\text{term}}$ at the corresponding initial state in $\text{Conf}(M)$.
- Thus, the undecidability of the Halting problem translates into an undecidable local model checking problem in the class of transition systems of type $\text{Conf}(M)$.
- Applying this to the universal Turing machine $U$ yields an undecidable local model checking problem on $\text{Conf}(U)$ alone.

The Halting problem as a global model checking problem:

- The halting of $M$ on every given input is equivalent to the validity of $\text{init} \rightarrow \mathcal{F}_{\text{term}}$ in $\text{Conf}(M)$.
- The undecidability of the Halting problem implies that the problem whether a given TM always halts is undecidable.
The Halting problem as a validity problem

- Any Turing machine $M$ can be described by a temporal logic formula $\Phi(M)$ in a sufficiently expressive temporal language.

- The Halting problem for $M$ on a blank tape is equivalent to the validity of

$$ (\Phi(M) \land \text{init-blank}) \rightarrow \mathcal{F} \text{ term} $$
The first known to me undecidability result for a propositional temporal logic:

Steve Thomason
Reduction of Second-Order Logic to Modal Logic
Mathematical Logic Quarterly, vol 21, 1975, pp. 107-114

Reduction of the frame validity based logical consequence the MSO theory of a binary relation to a propositional tense logic $T_{15}$ with a set of Prior’s tense operators $H_i, G_i, P_i, F_i$ over each of 15 temporal orderings $\preceq_1, \ldots, \preceq_{15}$, satisfying special interrelations.

Further, the logical consequence in $T_{15}$ is reduced to logical consequence in plain modal logic.

NB: the reduction adds a special formula $\delta$ to the premises, so it does not reduce validity to validity.
Early undecidability results in propositional temporal logics cont’d

Stephen Isard
A Finitely Axiomatizable Undecidable Extension of K.

This is seemingly the first undecidability result in modal logic, using reduction from the Halting problem (of Minsky machines).

Valentin Shehtman
Undecidable Propositional Calculi.

These results refer to specially constructed, mostly artificial, logics. The first undecidability results on ‘natural’, purely temporal logics? The answer seems to lead to interval logics.
Undecidability of interval temporal logics
Moszkowski’s Propositional Interval Temporal Logic (PITL)

Joseph Halpern, Zohar Manna, and Ben Moszkowski
A Hardware Semantics Based on Temporal Intervals.

Ben Moszkowski
Reasoning about Digital Circuits.

PITL-formulae:

\[ \phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \Diamond \phi \mid \phi ; \psi. \]

Models of PITL: based on (finite) discrete linear orderings.

Formulae are evaluated on discrete intervals:
finite sequences of states \( \sigma = s_0, s_1, \ldots, s_n \), with \( n \geq 0 \).
PITL with locality: semantics and decidability

- Atomic propositions evaluated at states.
- **Locality principle**: the value of an atomic proposition over an interval is its value at the initial state of that interval.
- Semantics of ‘next’ operator $\bigcirc$
  
  $s_0, s_1, \ldots, s_n \models \bigcirc \varphi$, where $n > 0$, iff $s_1, \ldots, s_n \models \varphi$

- Semantics of ‘chop’ operator $;$
  
  $s_0, s_1, \ldots, s_n \models \phi; \psi$ iff there is $i$ where $0 \leq i \leq n$, such that $s_0, s_1, \ldots, s_i \models \phi$ and $s_{i+1}, \ldots, s_n \models \psi$.

**Theorem**[Halpern and Moszkowski, 1983]: The satisfiability problem for PITL with locality is decidable, though [Kozen’92] with nonelementary complexity.
PITL: undecidability

**Theorem** [Halpern and Moszkowski, 1983]: The satisfiability problem for PITL without locality is undecidable.


Uses undecidability of emptiness of the intersection of the languages of two context-free grammars (in Greibach normal form).

Given two context-free grammars $G_1$ and $G_2$, one can construct a PITL formula that is satisfiable iff the intersection of the languages generated by $G_1$ and $G_2$ is nonempty.

Proof details in:

Ben Moszkowski

A Hierarchical Completeness Proof for Propositional Interval Temporal Logic with Finite Time.

Undecidability of duration calculi

Duration calculus: extension of the PITL framework with the notion of a state and state duration.

Zhou Chaochen, C. A. R. Hoare and A.R. Ravn
A Calculus of Durations

Even very simple fragments of DC are undecidable:

Michael R. Hansen and Zhou Chaochen
Duration Calculus: Logical Foundations

Proof technique: reduction from the halting problem for 2-counter Minsky machines.
Allen’s interval relations

J. F. Allen

Maintaining knowledge about temporal intervals.


\[ \langle L \rangle \langle L \rangle \]
\[ \langle A \rangle \langle A \rangle \]
\[ \langle O \rangle \langle O \rangle \]
\[ \langle E \rangle \langle E \rangle \]
\[ \langle D \rangle \langle D \rangle \]
\[ \langle B \rangle \langle B \rangle \]
Halpern-Shoham’s interval logic

J. Halpern and Y. Shoham
A propositional modal logic of time intervals.

J. Halpern and Y. Shoham
A propositional modal logic of time intervals.

HS: a multimodal logic with modal operators associated with Allen’s interval relations.

In the case of non-strict semantics when point intervals are allowed, it suffices to choose as primitive the modalities $\langle B \rangle$, $\langle E \rangle$, $\langle \overline{B} \rangle$, $\langle \overline{E} \rangle$ corresponding to the relations begins, ends, and their inverses:

$$\phi ::= p \mid \neg \phi \mid \phi \land \psi \mid \langle B \rangle \phi \mid \langle E \rangle \phi \mid \langle \overline{B} \rangle \phi \mid \langle \overline{E} \rangle \phi.$$

In the case of strict semantics without point intervals, the right and left neighbourhood modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ must be added.
Undecidability in interval logics: the bad news

Hereafter we assume the non-strict semantics, but all results apply to the strict semantics, too.

**Theorem** [Halpern and Shoham’91]
The validity in HS over any class of ordered structures containing at least one with an infinitely ascending sequence is r.e.-hard.

Thus, in particular, HS is undecidable over the classes of all (non-strict) models, all linear models, all discrete linear models, all dense linear models, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, etc.

Proof idea: reduction from the *non-halting problem for Turing machines* to testing satisfiability in HS.
Undecidability in interval logics: can be worse...

**Theorem** [Halpern and Shoham] The validity in HS over any class of Dedekind complete ordered structures containing at least one with an infinitely ascending sequence is $\Pi^1_1$-hard.

In particular, the validity in HS over any of the orderings of the natural numbers, integers, or reals is not recursively axiomatizable.

Proof: by reduction to satisfiability in HS of the *recurrence problem for non-deterministic TM*, asking for existence of a computation of a given NTM entering the start state infinitely often.
Undecidability occurs even without existence of infinitely ascending sequences. A class of ordered structures has **unboundedly ascending sequences** if for every $n$ there is a structure in the class with an ascending sequence of length at least $n$.

**Theorem**[Halpern and Shoham] The validity problem in HS interpreted over any class of Dedekind complete ordered structures having unboundedly ascending sequences is co-r.e. hard.

In particular, satisfiability of HS formulae in the finite is r.e. hard.

Proof idea: reduction from the halting problem for Turing machines to testing satisfiability in HS.
Some details of Halpern-Shoham’s reduction
setting the stage

Fix a Turing machine $M = \langle \{0, 1\}, Q, q_0, q_f, \delta \rangle$.

Atomic propositions: $L = \{0, 1, \ast, \#, (q, 0), (q, 1), (q, B) : q \in Q\}$.

Truth in all future intervals: $[F] \phi := [A] \phi \land [L] \phi$.

A special propositional constant $\pi$, true at all point intervals.

Truth at the beginning/end of the current interval:

$[[BP]] \phi := [B](\pi \rightarrow \phi); \quad [[EP]] \phi := [E](\pi \rightarrow \phi)$.

Every cell on the tape represented by an interval satisfying

$\text{cell}(p) := [[BP]] \# \land [[EP]] \# \land [D]p \land \langle D\rangle p$. 
Some details of Halpern-Shoham’s reduction
IDs and configurations

\(ID\): a sequence of cells, represented by an interval satisfying

\[ID := \langle B \rangle \text{cell}(\ast) \land \langle E \rangle \text{cell}(\ast) \land \langle D \rangle \lor \text{cell}(l) \land \neg \langle D \rangle \text{cell}(\ast)\]

\[\forall l \in L, l \neq \#\]

Starting configurations:

\[\text{startID} := ID \land \langle D \rangle (\text{cell}((q_0, 0)) \lor \text{cell}((q_0, 1)) \lor \text{cell}((q_0, b))).\]

Final configuration:

\[\text{finalID} := ID \land \langle D \rangle (\text{cell}((q_f, 0)) \lor \text{cell}((q_f, 1)) \lor \text{cell}((q_f, b))).\]
Some details of Halpern-Shoham’s reduction
encoding computations

Computations of $M$ are encoded as sequences of configurations:

$\ast ID1 \ast \ast ID2 \ast \ast ID3 \ast \ast \ldots$

To ensure matching the transition relation $\delta$, a special atomic
proposition $corr$ is used, saying that an interval start and ends
with cells that are corresponding in two consecutive IDs.
Describing $corr$ is the most ingenious part of the reduction.

In the long run, the formula $\text{computation}$ is defined, which is true
of an interval iff it encodes a legitimate computation of $M$.

Now, non-halting is expressed by

$$\text{NoHalt} := \text{computation} \land [F] \neg \text{finalID}.$$  

Hence, the reduction from non-halting of $M$ to SAT($\text{NoHalt}$).

For the satisfiability of $\text{NoHalt}$, any interval structure with an
infinite ascending chain suffices.
Reduction from the halting problem

Note, that halting cannot be expressed by

$$\text{computation} \land F \ \text{finalID}.$$  

because there may be non-standard models, e.g. on dense orders.  
Such non-standard models can be eliminated on Dedekind complete orders by using the formula

$$\text{NoTelescope} := \neg \langle B \rangle [E] \langle D \rangle \text{cell}.$$  

Eventually, the halting problem for $M$ is reduced to satisfiability of

$$\text{Halt} := \text{computation} \land \text{standard} \land \langle B \rangle \text{startID} \land \langle E \rangle \text{finalID},$$

where $\text{standard}$ is a formula ensuring that any interval starting and ending with IDs can be subdivided into a finite number of IDs.  
On Dedekind complete structures one can also express the property of a computation to visit infinitely often its starting state.
Every subset of the 12 Allen’s relations (excl. the equality) defines a fragment of HS.

Thus 4096 fragments arise; of them over 1000 expressively distinct.

([D. Della Monica, A. Montanari, VG., G. Sciavicco, IJCAI’ 2011]: in strict semantics there are 1347 expressively distinct fragments.)

We denote fragments by listing the letters representing the occurring modalities, e.g. $BE$, $OA\bar{A}$, etc.
Sharpening the undecidability: early results

An inspection of the formulae in the constriction shows that any of the fragments $ABE$ and $\overline{ABE}$ suffices for these reductions.

By refining Halpern and Shoham’s reduction, Lodaya proved in

Kamal Lodaya
Sharpening the Undecidability of Interval Temporal Logic.

the following:

\textbf{Theorem} The $BE$-fragment of HS is undecidable over the classes of dense linear interval structures, and consequently, over all linear interval structures.

\textbf{Corollary} The interval logic with 'Chop' alone is undecidable over the classes of all (dense) linear interval structures.
Undecidability of temporal logics via tiling
The Integer Grid Tiling Problem (IGTP)

Tile: a 'square' with coloured sides: \( \langle c_{up}, c_{right}, c_{down}, c_{left} \rangle \).

The \( \mathbb{N} \times \mathbb{N} \) - tiling Problem:

*Given a finite set of tile types \( \mathcal{T} = \{ t_1, \ldots, t_k \} \) of unlimited supply, can it be applied to tile the integer plane \( \mathbb{N} \times \mathbb{N} \) by matching the respective colors of adjacent tiles?*

**Theorem** [Berger, 1966]
The Integer Grid Tiling Problem is undecidable.

The reason: there exist sets of tiles that can only tile the plane aperiodically.
Aperiodic tiling: example
Applications of tiling problems to logical undecidability/complexity

David Harel

Recurring Dominoes: Making the Highly Undecidable Highly Understandable


Reduction from the IGTP can be used to prove plain undecidability, i.e. non-recursiveness, but recursive enumerability.

Tiling can also be used to prove $\Sigma^1_1$-hardness, by reduction from the *recurrent Tiling problem*, asking for existence of tiling in which a given tile occurs infinitely often in the first row.

There are many decidable tiling problems. Polynomial reduction to them can be used to prove complexity results.
Generic proof of undecidability via tiling

Proving undecidability via reduction from the IGTP of a logic $L$:

1. Construct a formula GRID in $L$ setting the grid.
2. Construct formulae in $L$ describing the tiles in a given tile set.
3. Construct a formula in $L$ describing correct tiling.
4. Translate any tiling problem to satisfiability of a formula of $L$. 
Early undecidability via tiling results in temporal logic

Sample results in [Harel’85]: satisfiability in each of the following is \( \Sigma_1^1 \)-hard by reduction from the recurrent Tiling problem:

- Quantified LTL(X,F),
- 2-dimensional LTL(X,F),
- the temporal spatial logic combining LTL(X,F) with K4.

Edith Spaan
Complexity of modal logics

*Ph. D. Thesis. University of Amsterdam, 1993*

Proves undecidability and \( \Sigma_1^1 \)-hardness via tiling of the satisfiability of various modal logics; in particular, logics obtained from decidable ones by extending with universal modality.
An easy proof of undecidability via tiling

Consider the two-dimensional temporal logic $X^2$ with ”next” operators for each of the 4 directions $\langle \uparrow \rangle$, $\langle \rightarrow \rangle$, $\langle \downarrow \rangle$, $\langle \leftarrow \rangle$, as well as a global modality, interpreted over the integer grid. Then there is a straightforward encoding of IGTP, following

Mark Reynolds

Two-dimensional temporal logic


1. The formula GRID is not necessary, only needed to indicate the origin, by $\neg \langle \leftarrow \rangle \top \land \neg \langle \downarrow \rangle \top$.

2. Every tile $\tau$ is treated as an atomic proposition.

3. The formula describing correct tiling is:

$$[U] \left( \bigvee_{\tau \in T} \tau \land \bigwedge_{\tau \neq \tau'} \neg (\tau \land \tau') \land \bigwedge_{\text{up}(\tau) \neq \text{down}(\tau')} \neg (\tau \land \langle \uparrow \rangle \tau') \land \bigwedge_{\text{right}(\tau) \neq \text{left}(\tau')} \neg (\tau \land \langle \rightarrow \rangle \tau') \right)$$

4. The tiling problem readily translates into satisfiability of the conjunction of the above in $\mathbb{N} \times \mathbb{N}$. 
A difficult proof of undecidability via tiling of the Compass Logic

*Compass Logic* [Venema’90]: a two-dimensional modal logic interpreted on products of two linear orders, with modal operators for each coordinate direction. NB: no ’next time’ operators. Yet:

**Theorem**[Marx and Reynolds’1997] The satisfiability in the compass logic is undecidable.

Maarten Marx and Mark Reynolds

Undecidability of the compass logic

*Journal of Logic and Computation, 9(6), 1997, pp. 897-914.*

Proof by elaborated encoding of the tiling problem.

NB: high undecidability on $\mathbb{N} \times \mathbb{N}$ was proved earlier by Spaan, by reduction from the recurrence problem for NTM, in:

Edith Spaan

Nexttime is not necessary

*Proc of TARK’1990, 241-256*
Undecidability of hybrid logic with binders via tiling
Hybrid modal/temporal logics bring useful features of first-order logic into modal logic, thus boosting the expressiveness of ML without affecting its good computational properties.

Historical origins: Prior and Bull, in tense logic.

Explicitly developed since the early 1980’s.
Main hybrid logic features

- **Nominals**: referring to single worlds in the model. Intuition: time stamps, ‘clock variables’. Formally:
  \[(W, R, V), u \models i \iff V(i) = u\]

- **Universal/global modality**: referring to all worlds in the model:
  \[M, u \models [\U] \phi \iff M, w \models \phi \text{ for every } w \in M.\]

- **Satisfaction operators**: refer to the truth at a named world:
  \[(W, R, V), u \models @i \phi \iff (W, R, V), V(i) \models \phi.\]

- **State variables**: like nominals, but with no fixed interpretation. Assigned values by a separate variable assignment and used for reference to earlier stored possible worlds.

- **Reference pointers/binders**: refer to the current world. \(\downarrow_s \phi\) means: '\(\phi\) is true if the current world is assigned to \(s\)'. Formally:
  \[M, g, u \models \downarrow_s \phi \iff M, g[s \leftarrow u], u \models \phi, \text{ where } g[s \leftarrow u] \text{ is the assignment } g, \text{ modified by assigning } u \text{ to } g(s).\]
Examples on the expressiveness with binders

• The difference modality is definable in $\mathcal{H}([U], \downarrow)$:
  
  $$[D]\varphi = \downarrow_s [U](\neg s \rightarrow \varphi).$$

• Nominals can be modelled $\mathcal{H}([U], \downarrow)$:
  
  $$NOM(\varphi) = \langle U \rangle (\varphi \land [D] \neg \varphi)$$

• $Until$ and $Since$ are definable in the tense hybrid logic $\mathcal{H}_t(\downarrow)$:
  
  $$p \mathcal{U} q = \downarrow_s (Fq \land (H(Ps \rightarrow p))),$$
  
  and likewise for $Since$.

• $Until$ is definable even in $\mathcal{H}(\circ, \downarrow)$:
  
  $$p \mathcal{U} q = \downarrow_s \diamond \downarrow_t (q \land \circ_s \Box (\diamond t \rightarrow p)).$$
Undecidability of $\mathcal{H}(\{U\}, \downarrow)$ via tiling

Theorem [G., 1994] The satisfiability in $\mathcal{H}(\{U\}, \downarrow)$ is undecidable, by reduction from the Integer Grid Tiling Problem.

Valentin Goranko

Temporal Logic with Reference Pointers,

Later, strengthened [Areces, Blackburn, and Marx, 1999] to undecidability of $\mathcal{H}(\emptyset, \downarrow)$.

The encoding is not straightforward, but is quite intuitive:

*the formula GRID($p, q$) says that every point of the model has exactly two successors: at one of them the value of $p$ changes and the value of $q$ remains the same (the move ”to the right”), while at the other (the move ”upwards”) the opposite happens. Moreover, the routes ”right;up” and ”up;right” converge.*
Setting the Grid in \(\mathcal{H}([U], \downarrow)\)

\[
\varphi_1 = [U]((p \land q \rightarrow F(p \land \neg q) \land F(\neg p \land q) \land G((p \land \neg q) \lor (\neg p \land q))) \land \\
(p \land \neg q \rightarrow F(p \land q) \land F(\neg p \land \neg q) \land G((p \land q) \lor (\neg p \land \neg q))) \land \\
(\neg p \land q \rightarrow F(\neg p \land \neg q) \land F(p \land q) \land G((\neg p \land \neg q) \lor (p \land q))) \land \\
(\neg p \land \neg q \rightarrow F(\neg p \land q) \land F(p \land \neg q) \land G((\neg p \land q) \lor (p \land \neg q))))),
\]

\[
\varphi_2 = [U] \downarrow_s ((p \land q \rightarrow [U](Fs \rightarrow G(p \land q \rightarrow s))) \land \\
(p \land \neg q \rightarrow [U](Fs \rightarrow G(p \land \neg q \rightarrow s))) \land \\
(\neg p \land q \rightarrow [U](Fs \rightarrow G(\neg p \land q \rightarrow s))) \land \\
(\neg p \land \neg q \rightarrow [U](Fs \rightarrow G(\neg p \land \neg q \rightarrow s)))),
\]

\[
\varphi_3 = [U] \downarrow_s ((p \land q \rightarrow [U]((\neg p \land \neg q \land FFs) \rightarrow GG(p \land q \rightarrow s))) \land \\
(p \land \neg q \rightarrow [U]((\neg p \land q \land FFs) \rightarrow GG(p \land \neg q \rightarrow s))) \land \\
(\neg p \land q \rightarrow [U]((p \land \neg q \land FFs) \rightarrow GG(\neg p \land q \rightarrow s))) \land \\
(\neg p \land \neg q \rightarrow [U]((p \land q \land FFs) \rightarrow GG(\neg p \land \neg q \rightarrow s))))).
\]

\[
GRID(p, q) = p \land q \land \varphi_1 \land \varphi_2 \land \varphi_3
\]
Describing the tiles in PL

Consider a tiling problem with a set of tiles \( T = \{ t_1, \ldots, t_m \} \) with colours \( C = \{ c_1, \ldots, c_k \} \).

Every tile has four sides: "up", "down", "left" and "right", each coloured in one of the colours from \( C \).

To every colour \( c_i \) we assign four propositional variables \( u_i \) ("up"), \( d_i \) ("down"), \( l_i \) ("left"), and \( r_i \) ("right").

Each tile \( t \) with sides "up", "down", "left" and "right" coloured respectively in \( c_{i_1}, c_{i_2}, c_{i_3}, \) and \( c_{i_4} \), we represent by the formula

\[
\theta_t = (u_{i_1} \land \bigwedge_{j \neq i_1} \neg u_j) \land (d_{i_2} \land \bigwedge_{j \neq i_2} \neg d_j) \land (l_{i_3} \land \bigwedge_{j \neq i_3} \neg l_j) \land (r_{i_4} \land \bigwedge_{j \neq i_4} \neg r_j).
\]
Describing the tiling in $\mathcal{H}([U], G)$

Now we define the formulae:

$$\text{COVER}_T = [U](\bigvee_{i=1}^{m} \theta_i)$$

which says that the model is properly tiled, i.e. every point in the model is covered by exactly one tile.

$$\text{MATCHUP} = [U](\bigwedge_{i=1}^{k} (u_i \rightarrow (p \land q \rightarrow G(p \land \neg q \rightarrow d_i)) \land (p \land \neg q \rightarrow G(p \land q \rightarrow d_i)) \land (\neg p \land q \rightarrow G(\neg p \land \neg q \rightarrow d_i)) \land (\neg p \land \neg q \rightarrow G(\neg p \land q \rightarrow d_i))))$$

which says that the colour ”up” of each tile of the cover matches the colour ”down” of the one above it;

Likewise, MATCHRIGHT is defined.
Translating the tiling problem in $\mathcal{H}([U], \downarrow)$

Finally, we put

$$\Phi_T := GRID \land COVER_T \land MATCHUP \land MATCHCHRIGHT$$

**Theorem**

$\Phi_T$ is satisfiable if and only if $\mathbb{N} \times \mathbb{N}$ can be properly tiled by $T$. 

More undecidability via tiling

The undecidability result uses the relative strength of the language $\mathcal{H}([U], \downarrow)$ but no special properties of the models.

A number of similar results were established in the 1990s.

For instance [Spaan, 1996]: “there is a uni-modal, decidable, finitely axiomatizable, and canonical logic for which adding the universal modality causes undecidability and for which adding the reflexive transitive closure modality causes high undecidability.”

Edith (Spaan) Hemaspaandra
The Price of Universality,
Notre Dame J. Formal Logic Volume 37, Number 2 (1996), 174-203.

See also Spaan’s PhD thesis, as well as many more undecidable polymodal logics in:

Marcus Kracht
Highway to the Danger Zone,
Undecidability of interval temporal logics by reduction from the Octant Tiling Problem
The Octant Tiling Problem

The 2nd octant of $\mathbb{Z} \times \mathbb{Z}$:

$$O = \{(i, j) : i, j \in \mathbb{N} \land 0 \leq i \leq j\}$$

A natural interpretation of intervals on $\mathbb{N}$ into $O$.

The Octant Tiling Problem: can a given finite set of tile types $T = \{t_1, \ldots, t_k\}$ tile $O$ while respecting the color constraints?

**Theorem** The Octant Tiling Problem is undecidable.

Proof: by reduction from the tiling problem for $\mathbb{N} \times \mathbb{N}$, using König’s Lemma.
Undecidability of interval logics via tiling: generic construction

Given a finite set of tiles, we consider a signature containing, inter alia, special propositional letters $u$, tile, $\text{Id}$, $t_1, \ldots, t_k$, cbb, cbe, ceb, corr, and possibly others. The letters $t_i$ represent the tiles.

The tiling framework is set by forcing the existence of a (usually unique) infinite chain of unit-intervals ($u$-intervals) on the linear order, which covers an initial segment of the interval model.

Unit intervals are used to place tiles and delimiting symbols.

Then, ID-intervals are introduced to represent the layers of tiles.
Undecidability of the interval logics via tiling: generic construction cont’d
Undecidability of the interval logics via tiling: generic construction cont’d

Each ID-interval must have the right number of tiles, and they must match horizontally: the Right-Neightbour relation.

The most challenging part usually is to ensure that the consecutive ID-intervals match vertically: the Above-Neightbour relation.

For that, we use several auxiliary propositional letters to refine and implement the idea of corr: cbb for matching the beginning point of a tile to the beginning point of the corresponding tile above; cbe, for matching beginning point with ending point above, and ceb for matching ending point with a beginning point above.
Undecidability of the interval logics via tiling: generic construction completed

Eventually, we encode the given Octant tiling problem by specifying the matching conditions between adjacent tiles.

The specific part of the construction is to use the given fragment of HS to set the chain of unit intervals and to express all necessary properties of IDs, the propositional letters for correspondence intervals, and the tile matching conditions.

For instance, using the After modality $A$ the matching conditions can be expressed as follows, where $[F]p := [A]p \land [A][A]p$:

$$
[F](\text{tile} \land \langle A \rangle \text{tile}) \rightarrow \bigvee_{right(t_i)=left(t_j)} (t_i \land \langle A \rangle t_j),
$$

$$
[F](\langle A \rangle \text{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\langle A \rangle t_i \land \langle A \rangle (cbb \land \langle A \rangle t_j))).
$$
A sample result using the Octant Tiling problem: undecidability of the logic $O$ over (discrete) linear orderings

Semantics of the Overlap operator $O$:

$M, [a, b] \models \langle O \rangle \phi$ iff

there exist $c, d$ such that $a < c < b < d$ and $M, [c, d] \models \phi$. 

\[ \begin{array}{c}
\text{a} \\
| \text{c} \\
\text{b} \\
| \text{d}
\end{array} \]
Encoding the Octant
Encoding the Octant
u- and k-intervals of length 2

$\begin{align*}
\text{u-intervals} & \rightarrow \\
\text{begin}_u\text{-intervals} & \rightarrow
\end{align*}$

$\begin{align*}
\text{begin}_u\text{-intervals} & \text{ cannot overlap begin}_u\text{-intervals starting inside the same u-interval}
\end{align*}$
Encoding the Above-Neighbour Relation

\[ \text{up}_\text{rel}^\text{b} \rightarrow \neg \langle O \rangle \text{up}_\text{rel}^\text{b} \]

\[ \text{up}_\text{rel}^\text{f} \rightarrow \neg \langle O \rangle \text{up}_\text{rel}^\text{f} \]
Undecidability of the logic $O$ over discrete linear orderings

In the long run, for every finite set of tiles $\mathcal{T}$ we build a formula $\phi_\mathcal{T} \in O$ such that

$$\phi_\mathcal{T} \text{ is satisfiable in a discrete linear ordering iff }$$

$$\mathcal{T} \text{ can tile the 2nd octant.}$$

**Theorem** [Bresolin, Della Monica, G., Montanari, Sciavicco, 2009]

The satisfiability problem for the logic $O$ is undecidable over any class of discrete linear orderings that contains at least one linear ordering with an infinite ascending sequence.

Likewise for $\bar{O}$, on classes having infinite descending sequences.
More recent results on undecidability of interval logics

Using variations of the Octant Tiling Problem encoding:

**Theorem** The satisfiability problem for each of the HS fragments $O, \overline{O}, AD, \overline{AD}, \overline{AD}, BE, \overline{BE}, \overline{BE}, \overline{BE}$, is undecidable in any class of linear orders that contains at least one linear order with length greater than $n$, for each $n > 0$.

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco

The dark side of Interval Temporal Logic: sharpening the undecidability border

*Proc. of TIME'2011*

$O$ and $\overline{O}$ were the first uni-modal fragments of HS proved undecidable over the class of discrete orderings.

A recent last blow by Marcinkowski and Michal(iszyn (LICS’2011): undecidability of $D$ and $\overline{D}$ over discrete linear orderings.
More recent results on undecidability of interval logics

D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco

Interval Temporal Logics over Finite Linear Orders: the Complete Picture

Proc. of ECAI’2012

Of the 1347 expressively different fragments of HS, only the following 35 and their symmetric versions are decidable over the class of finite linear orders:

Complexity class:

1: Non primitive recursive
2: EXPSPACE-complete
3: NEXPTIME-complete
4: NP-complete
Undecidability of temporalized logics
Temporalizing logics

Temporalization: combination of temporal and other logics, e.g.: products, fusions, etc.

Marcelo Finger and Dov Gabbay
Combining Temporal Logic Systems

Temporalization often leads to undecidability.

Three important case studies:

- Products of modal and temporal logics
- Temporal epistemic logics
- Temporal description logics
Undecidability of products of logics

Dov Gabbay and Valentin Shehtman
Products of Modal Logics, Part I
*L. J. of the IGPL, Vol. 6, No. 1, 1998, pp. 73-146*

Frank Wolter
The decision problem for combined (modal) logics
*Habilitationsschrift, Univ. of Leipzig, 1999*

Many-dimensional modal logics: theory and applications
*Elsevier, 2003.*

Products of modal logics are massively undecidable. For instance:

- $ML(\mathbb{N} \times \mathbb{N})$ [Spaan, 1993]; $K[U] \times K[U]$ [Marx, 1999];
- $K4.3 \times K4.3$ [Reynolds and Zakharyaschev 1999];
- Almost all three-dimensional modal/temporal logics. Related to the undecidability of $FO^3$. 
Temporal-epistemic logics

Combine temporal and multi-agent epistemic logics.

An important earlier work with detailed proof of undecidability by reduction from the Halting Problem for Turing machines:

Richard Ladner and John Reif:
The Logic of Distributed Protocols
Proc of TARK’1986: 207-222

Various other developments during the 1980s. Unifying study in:

Joseph Halpern and Moshe Vardi
The complexity of reasoning about knowledge and time I: Lower bounds

Joseph Halpern and Moshe Vardi
The complexity of reasoning about knowledge and time: Synchronous systems
IBM Research Report, 1989
A variety of temporal-epistemic logics

Semantics based on so called *interpreted systems*: sets of runs in a transition system with epistemic indistinguishability relations on the state space for each agents.

A variety of 96 logics, based on six parameters:

- **number of agents** (one or many),
- **the language** (with or without common knowledge, linear or branching time, etc.),
- **recall abilities** (no recall, bounded recall, perfect recall),
- **learning abilities** (learning or no learning),
- **synchrony** (synchronous or asynchronous),
- **unique initial state**.
Complexity of the validity in temporal-epistemic logics

Both linear and branching time logics involving more than one agents become highly undecidable ($\Pi_1^1$-complete) under some combined assumptions, e.g., of both unbounded memory and common knowledge.

<table>
<thead>
<tr>
<th>Logic</th>
<th>$CKL(m) / CKB(m)$, $m \geq 2$</th>
<th>$KL(m) / KB(m)$, $m \geq 2$</th>
<th>$KL(1) / KB(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(nf)$, $C(nf,syncc)$, $C(nf,usis)$, $C(nf,usis,us)$</td>
<td>$\Pi_1^1$</td>
<td>nonelementary (time $ex(ad(\varphi) + 1, c</td>
<td>\varphi</td>
</tr>
<tr>
<td>$C(nf,usis)$, $C(nf,syncc)$</td>
<td>$\Pi_1^1$</td>
<td>nonelementary (space $ex(ad(\varphi), c</td>
<td>\varphi</td>
</tr>
<tr>
<td>$C(nf,usis)$, $C(nf,syncc)$, $C(nf,usis,us)$</td>
<td>$\Pi_1^1$</td>
<td></td>
<td>EXPSPACE</td>
</tr>
<tr>
<td>$C(nf,usis)$, $C(nf,usis)$, $C(nf,usis,us)$</td>
<td>$\Pi_1^1$</td>
<td></td>
<td>EXPSPACE</td>
</tr>
<tr>
<td>$C(nf,usis)$, $C(nf,usis,us)$</td>
<td>$\Pi_1^1$</td>
<td></td>
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<tr>
<td>$C(nf,usis)$</td>
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<td></td>
<td>EXPSPACE</td>
</tr>
<tr>
<td>$C(nf,usis)$, $C(nf,usis,us)$</td>
<td>EXPSPACE</td>
<td>EXPSPACE</td>
<td>EXPSPACE</td>
</tr>
<tr>
<td>$C(nf,usis)$, $C(nf,usis,us)$</td>
<td>EXPSPACE</td>
<td>PSPACE for $KL(m)$, EXPTIME for $KB(m)$</td>
<td>EXPSPACE for $KL(m)$, EXPTIME for $KB(m)$</td>
</tr>
</tbody>
</table>

Figure 1: The complexity of the validity problem for logics of knowledge and time
Sharpening the undecidability of temporal epistemic logics

Spaan showed that neither Nexttime nor Until are needed for most of these results, but the knowledge operator $K$ and the temporal operator $G$ suffice:

Edith Spaan
Nexttime is not necessary

*Proc of TARK’1990, 241-256*
Undecidability of temporal description logics

Description logics: very close to modal logics.
Involve concepts (unary predicates) and roles (binary predicates).

TBoxes: finite sets of concept inclusions.

Description logics can be temporalized in various ways:

- **Alessandro Artale and Enrico Franconi**
  A survey of temporal extensions of description logics,

Many undecidability consequences from Halpern-Shoham results.

Many more undecidability results for temporal description logics in:

- **Frank Wolter**
  The decision problem for combined (modal) logics
  *Habilitationsschrift, Univ. of Leipzig, 1999*
Undecidability of temporal description logics, cont’d

More recent undecidability results for quite weak fragments in:

Carsten Lutz, Frank Wolter and Michael Zakharyaschev
Temporal Description Logics: A Survey,
Proc. of TIME’2008, pp.3-14

$\mathcal{ALC}$ is the basic propositionally closed description logic.

**Theorem** Concept satisfiability in $LTL_{\mathcal{ALC}}$ w.r.t. T-Boxes and with a single rigid (over time) role is $\Sigma_1^1$-hard.

Proof: by reduction from the *recurrent tiling problem*: given a set of tile types $T$, decide whether it can tile $\mathbb{N} \times \mathbb{N}$ so that a given tile $t$ appears infinitely often in the first row.

Also: concept satisfiability in $LTL_{SHIQ}$ with rigid roles and without TBoxes is undecidable.

Proof: by reduction from Post’s Correspondence Problem.
Undecidability of quantitative temporal logics
Real-time extensions of temporal logics

R. Alur and T. Henzinger
Logics and models of real-time: a survey

Real-time extensions of temporal logics:

- **time-bounded operators:**
  \[ G(p \rightarrow F_{=10} q) \]

- **freeze quantification:** very similar to hybrid binders: binds a variable \( x \) to the current time, e.g.:
  \[ Gx.(p \rightarrow Fy.(q \land y \leq x + 3)) \]

- **time variables and quantification,** e.g.:
  \[ \forall x G(p \land T = x \rightarrow F(q \land T \leq x + 3)) \]

Timed transition systems and timed automata.
Undecidability of metric temporal logics

Ron Koymans
Specifying real-time properties with metric temporal logic

MTL augments the LTL operators with time bounded operators.

R. Alur and T. Henzinger
Real-time logics: Complexity and expressiveness

Punctuality causes undecidability: e.g., on discrete orderings with addition and on dense orderings with constant increment operation. Proof: by reduction from repeated reachability in Minsky machines.

A relaxed decidable version: MITL with interval constraints:

\[ G(p \rightarrow F_{[2,10]} q) \]

J. Ouaknine and J. Worrell.
Some recent results in Metric Temporal Logic
Undecidability of real-time logics

Timed propositional temporal logic TPTL: like LTL, but interpreted on time sequences and extended with freeze quantifiers.

R. Alur and T. Henzinger
A really temporal logic

Basic version decidable, but various extensions, e.g., time addition or multiplication by 2, or dense time domain cause $\Pi_1^1$-hardness.
Summary: what causes undecidability in temporal logics?

Propositional temporal logics are generally decidable, but adding some syntactic or semantic features can make them explode.

Many important types of temporal logics are generally undecidable, even under very weak assumptions.

What are the typical causes of undecidability in temporal logics?

- Grid-like models, many-dimensional or temporalized systems,
- Interval-based semantics, where truth of formulae is defined on time intervals, with no locality assumptions.
- Temporal operators along multiple (at least two) time-lines. Products of simple temporal logics.
- Time reference mechanisms, such as freeze quantifiers and hybrid binders.
- Arithmetic features: time addition, exact time constraints, etc.
Conclusion: is there life beyond decidability?

Yes, of course. Classical first-order logic is a witness.

Undecidability is bad, but how bad? And, (how) should we care?

Possible ways out:

- Syntactic restrictions, identifying decidable fragments (e.g. FO², guarded fragments, etc.)
- Suitable parametric restrictions, e.g. on number of propositional variables, depth of nesting, etc.
- Semantic restrictions, ‘taming’ the semantics. E.g.: locality.
- Semi-decision procedures, e.g. resolution or tableaux.
- Using heuristics and human-computer interaction, etc.

Maurice Margenstern
Frontier between decidability and undecidability: a survey
PostScriptum: Terminator vs Turing

Terminator: research project at Microsoft Research, Cambridge, focused on the development of automatic methods for proving program termination and general liveness properties.


See article “Terminator Tackles an Impossible Task”

So, can the theoretically undecidable be practically decided?
The future will tell.
Maybe.

The end