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How Financial Theory Applies to Catastrophe-Linked Derivatives—An Empirical Test of Several Pricing Models.*

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Abstract

This paper discusses the PCS Catastrophe Insurance Option Contracts, providing empirical support on the level of correspondence between real quotes and standard financial theory. The highest possible precision is incorporated since the real quotes are perfectly synchronized and the bid-ask spread is always considered. A static setting is assumed and the main topics of arbitrage, hedging, and portfolio choice are involved in the analysis. Three significant conclusions are reached. First, the catastrophe derivatives may often be priced by arbitrage methods, and the paper provides some examples of practical strategies that were available in the market. Second, hedging arguments also yield adequate criteria to price the derivatives, and some real examples are provided as well. Third, in a variance aversion context many agents could be interested in selling derivatives to invest the money in stocks and bonds. These strategies show a suitable level in the variance for any desired expected return. Furthermore, the methodology here applied seems to be quite general and may be useful to price other derivative securities. Simple assumptions on the underlying asset behavior are the only required conditions.

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1. INTRODUCTION

New investment and financing opportunities and innovative risk management techniques involving derivatives have been developed to allow individuals and corporations to cost-effectively reallocate funds and transfer risks to other parties. A growing concern about catastrophe losses has particularly brought attention to catastrophe derivatives and their potential financing and risk sharing benefits for the insurance industry.

The PCS (Property Claim Services) Catastrophe Insurance Options Contracts, launched by the Chicago Board of Trade (CBOT) on September 29, 1995, are among the most significant catastrophe derivatives. These are standardized option contracts based on indices that track the insured losses, as estimated by PCS, resulting from catastrophic events that occur in a given area and period. Previously, the CBOT had traded catastrophe futures and options contracts on an index provided by Insurance Services Office (ISO). Moreover, the CBOT planned to list PCS Single-Event Catastrophe options in 1998 to broaden its product offering. In 1997 the Bermuda Commodities Exchange (BCOE) also began trading derivative securities based on the Guy Carpenter Catastrophe Index, an index of losses from climate events in the United States.

This article focuses on the CBOT's PCS options. Previous literature on these particular contracts and other related catastrophe derivatives can be roughly divided into two major categories, according to their main objective. The first group of articles concentrates on pricing issues. They view catastrophe derivatives as financial instruments and, accordingly, they take a financial approach to valuing (see Cummins and Geman (1995), Geman and Yor (1997), among others; Tomas (1998) suggests an actuarial approach). They theorize on the dynamic stochastic behavior of the relevant underlying variables in order to obtain the desired pricing result. From a theoretical point of view this line of research is extremely important and very promising. From a practical point of view, there are some difficulties due to market imperfections (bid-ask spread, other transaction costs, short-selling restrictions, illiquidity that makes continuous trading rather difficult, etc.) and some specific properties shown by the underlying indices (their stochastic behavior, the absence of any underlying security available for trading, etc.). This motivates the existence of a second group of articles devoted to describing the contracts and illustrating their most significant applications to both insurance and capital markets (e.g., D'Arcy and France (1992), Canter et al. (1996), Litzenberger et al. (1996), O'Brien (1997) and Jaffee and Russell (1997)). They explore the potential benefits of using catastrophe derivatives for the insurance industry, and they compare these securities to other competitive alternatives such as reinsurance and catastrophe-linked bonds. Some articles analyze from any investor's perspective the potentially attractive new investment opportunities provided by the catastrophe-linked assets. Most of all these articles stress the traders' need for an understandable and reliable complete pricing methodology for these innovative securities.

The present article may be included in the second group, but the standard static asset pricing models are applied. We consider real bid and ask prices of catastrophe-linked derivatives and we test their adequacy with static financial theory. Examining static valuation minimizes the impact of real market imperfections, and problems deriving from the nature of the underlying variables may be solved if one prices an arbitrary derivative by only bearing in mind the interest rates and the prices of other derivative securities. Thus, we can apply the main topics of asset pricing, arbitrage, hedging, and portfolio choice in a model where bonds and derivatives are the only marketed assets.[1]

In order to use static theory, we will consider a two-period model characterized by the current date, the derivatives' expiration date, their current bid and ask prices, and their final payoffs. The analysis is independently implemented once a day.

Once the context has been fixed, we will start by analyzing the existence of arbitrage portfolios. There are two different perspectives. First, we test the situation of an investor who incurs in the cost of the bid-ask spread, i.e., he/she sells at the bid and buys at the ask price. As expected, we will find it impossible to form any arbitrage portfolio. Second, we explore the position of any agent who posts one of the available prices (i.e., either he/she buys at the bid or sells at the ask for a given asset and incurs on the cost of the bid-ask spread for the rest of assets). If an arbitrage portfolio were available in this context, any other agent could offer a better price and still retain some of the arbitrage gains. Competition among traders willing to earn money without any risk should lead to a more reduced spread. As will be shown, arbitrage portfolios may be available from this second perspective and, consequently, some relative misspricings may be found, narrower spreads may be possible, and traders can sometimes improve real bid or ask quotes without any type of risk or, equivalently, they can price by arbitrage methods.

When a concrete derivative cannot be priced by arbitrage methods, we explore the existence of hedged portfolios containing this derivative. In particular, there may exist some hedged portfolios with a slightly lower guaranteed positive return than the risk-free return, but with a possible return far larger (under some conditions). Our results show that interesting portfolios of this type can actually be formed in some cases. Again, competition among traders trying to exploit the attractive benefits of these portfolios should lead to reductions of the spread.

Previous studies, Canter et al. (1996) and Litzenberger et al. (1996), utilized empirical evidence concerning the insignificant correlation of the PCS national index with the S&P 500 index (see also Litzenberger et al. (1996) and the references contained therein for more evidence in this regard). Canter et al. (1996) stress the diversification benefits open to investors participating in the new securitized insurance risk. Litzenberger et al. (1996) illustrate the attractiveness of including some hypothetical catastrophe bonds in diversified stock or bond portfolios in terms of the new risk/returns opportunities offered. This article follows the theoretical framework proposed by Fisher Black and Robert Litterman based on the Capital Asset Pricing Model (CAPM) including the calculation of some necessary parameters from historical data of insurance losses and premiums.

While our study on portfolio choice is closely related to these previous studies, it takes a different point of view. We focus on investment in the catastrophe insurance

[1] Stocks, whose returns show an insignificant correlation with the PCS indices (see Canter et al. (1996)), can be included, too.

options market. Suppose that an investor, possibly attracted by the accompanied diversification benefits, adds insurance risk to his/her traditional portfolio of stocks, bonds, and real estate. How should he or she efficiently combine PCS options in this insurance portfolio with the riskless asset in order to obtain the desired expected return with a minimum variance? We will try to answer this question with the assistance of two important theorems concerning static pricing theory.

We also need the real probability distribution for the underlying insured loss index and a linear pricing rule compatible with the real quotes. The probability distribution is obtained via simulation and historical catastrophe data. The linear pricing rule originates from a risk-neutral probability measure attained by applying a methodology proposed by Rubinstein (1994) and a number of others.

Once the probability measure and the pricing rule are established, we shall look for minimum variance portfolios. An interesting result seems to hold. For investors whose risk is not correlated with the PCS indices (i.e., investors that are not insurers), it may be very useful to sell catastrophe derivatives and to invest the money in other kinds of assets, like bonds or stocks.

Summarizing, our empirical results confirm the potential interest of catastrophe-linked derivatives. They are useful to insurers because, in some sense, they can be regarded as a special type of reinsurance. Besides, they may also be interesting for other types of financial institutions (banks, for instance) because, if arbitrage and hedging arguments lead to low bid-ask spreads, these institutions can adequately diversify their portfolios by selling derivatives. Consequently, the high level of risk due to catastrophic events may be appropriately diversified among large numbers of investors who trade "reinsurances" in a financial market.

Finally, the applied methodology highlights two interesting properties: first, its usefulness to traders in providing practical criteria and investment strategies; second, it may be quite general and can be implemented to analyze other kinds of securities. Very weak assumptions are then required. Arbitrage and hedging arguments will hold if one is able to identify the underlying uncertainty, i.e., the underlying variables if we are working with derivatives. Variance aversion and CAPM-type arguments will work well if the probability measure, affecting the underlying uncertainty, can be determined.[2]

The remainder of the article is organized as follows. Section 2 briefly reviews the main theoretical results we rely on to carry out our empirical analysis of PCS options quotes. Section 3 summarizes the foremost characteristics of PCS options and all our data. In Section 4 we present the concrete methodology we adopt in our empirical research of PCS options quotes and provide our results. The article ends with some concluding remarks in Section 5.

[2] When the usual CAPM is tested, it is not possible to describe the underlying probability space, and researchers have to obtain information about it by studying correlations within the set of available securities. Nevertheless, in this case, the underlying PCS indices' behavior has been directly analyzed and the derivatives' quotes and returns were not used for this purpose.

2. THEORETICAL BACKGROUND

Throughout this article we will consider a static setting to analyze how the theory of portfolio selection and different asset pricing models may be applied to PCS option contracts. Thus, first of all, we must summarize the general framework and the basic assumptions that lead to the most important theoretical results on asset pricing. A brief review of these topics is the main purpose of the present section. Later, we will provide the way this theory applies in this article to study the market of PCS option contracts.

We focus on the two-period approach characterized by the present date t_0, a future date t_1, n securities denoted by $S_1, S_2, \ldots, S_n$, their bid prices at t_0 denoted by $v_1, v_2, \ldots, v_n$, their ask prices $c_1, c_2, \ldots, c_n$, and the future prices (or final payoffs) at t_1 which depend on a finite number of states of the world $W_1, W_2, \ldots, W_k$ and are given by the matrix $\mathbf{A} = (a_{ij})$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$, being $a_{ij} \geq 0$ the price of S_j if the state W_i takes place.[3] The probability of W_i is denoted by $\mu_i > 0$, $i = 1, 2, \ldots, k$, and μ will denote the whole probability measure. The inequalities $c_j \geq v_j$ for $j = 1, 2, \ldots, n$ are clear, and we will accept the convention $v_j = 0$ ($c_j = \infty$) if there is no bid (ask) price available for S_j. The inequalities $c_j > 0$ and $v_j \geq 0$, $j = 1, 2, \ldots, n$, will also be assumed.

The first security S_1 will be a riskless asset (its final payoff is 1 and does not depend on the state of the world) and $c_1 = v_1 > 0$.[4] As usual, the riskless return is given by $R = 1 / v_1$.

The row matrix $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ will represent the portfolio composed of x_j units of S_j, $j = 1, 2, \ldots, n$, and $x_j \geq 0$ ($x_j \leq 0$) must hold if $v_j = 0$ ($c_j = \infty$). Its current (at t_0) price will be $P(x) = \sum_{j=1}^{n} p_j x_j$ being $p_j = c_j$ ($p_j = v_j$) if $x_j \geq 0$ ($x_j \leq 0$),[5] where we assume the convention $\infty \times 0 = 0$ if $c_j = \infty$ and $x_j = 0$. Its price at t_1 depends on the state of the world and is given by the column matrix $\mathbf{A}\mathbf{x}^{\mathrm{T}}$ where $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$.

For an arbitrary portfolio $\mathbf{x}$, we will consider the portfolios $\mathbf{x}^+ = [x_1^+, \ldots, x_n^+]$ and $\mathbf{x}^- = [x_1^-, \ldots, x_n^-]$ composed of the purchased and sold securities respectively. To be precise, $x_j^+ = \mathrm{Max}\{x_j, 0\}$ and $x_j^{\pm} = \mathrm{Max}\{-x_j, 0\}$ for $j = 1, 2, \ldots, n$.

The prices of the purchased and sold assets will be denoted by $C(x)$ and $V(x)$ respectively, and are given by

$$C(x) = \sum_{j=1}^{n} c_j \mathrm{x}_j^+ \text{ and } V(x) = \sum_{j=1}^{n} v_j \mathrm{x}_j^{\pm} .$$

The relationships $P(x) = C(x) - V(x)$ and $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ are clear.

[3] Almost all the results still hold for a matrix $\mathbf{A}$ whose elements are also negative, but the constraint $a_{ij} \geq 0$ makes things a little easier and is always fulfilled in our empirical analysis.

[4] Once again, this assumption can be avoided in a general framework, but it is useful and fulfilled in this article.

[5] That is, agents can buy or sell any security, but prices are larger if they buy.

We will follow the approach proposed for instance by Prisman (1986) or Ingersoll (1987) to introduce the concept of arbitrage.[6]

Definition 1. *The portfolio* $\mathbf{x}$ *is said to be an arbitrage portfolio of the second type, or a strong arbitrage portfolio, if* $P(x) < 0$ *and* $\mathbf{Ax}^{\mathrm{T}} \geq 0$, *or* $P(x) = 0$ *and* $\mathbf{Ax}^{\mathrm{T}} >> 0$.

The portfolio x is said to be an arbitrage portfolio of the first type, or a weak arbitrage portfolio, if $P(x) = 0$ *and* $\mathbf{Ax}^{\mathrm{T}} > 0$.

Let us consider a simple numerical example in order to illustrate what is meant by weak and strong arbitrage. Suppose that

$$A = \begin{bmatrix} 1, 1, 0, 1 \\ 1, 0, 1, 2 \end{bmatrix}$$

the bid prices are $v_1 = 1$, $v_2 = 0.7$, $v_3 = 0.4$ and $v_4 = 2.6$, and the ask prices are $c_1 = 1$, $c_2 = 0.8$, $c_3 = 0.5$ and $c_4 = 3$. Then, $x' = [1, -1, -1, 0]$ is an arbitrage portfolio of the second type because its current price is –0.1 and its final payoffs are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The portfolio $x' = [1.1, -1, -1, 0]$ is also a strong arbitrage portfolio because its current price and final payoffs are 0 and $\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$ respectively. Finally, $\mathbf{x}''' = [0, 2, 2, -1]$ is a weak arbitrage portfolio with current price equal to 0 and future payoffs equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Previous literature has characterized the absence of arbitrage by the existence of state prices or discount factors (see for instance Chamberlain and Rothschild (1983), Ingersoll (1987), or Hansen and Richard (1987)). The following result is a minor extension that incorporates the bid-ask spread and may be easily proved by readapting classical proofs (see also Jouini and Kallal (1995)).

Theorem 1. *There are no arbitrage opportunities if and only if there exists a vector* $\boldsymbol{d} = [d_1, d_2, \ldots, d_k]$ *of discount factors such that* $d_i > 0$, $i = 1, 2, \ldots, k$ *and*

$$v_j \leq \sum_{i=1}^{k} a_{ij} d_i \mu_i \leq c_j \tag{2.1}$$

for $j = 1, 2, \ldots, n$.

There are no arbitrage opportunities of the second type if and only if there exists a vector $\boldsymbol{d} = [d_1, d_2, \ldots, d_k]$ *of discount factors such that* $d_i \geq 0$, $i = 1, 2, \ldots, k$, and (2.1) holds.[7]

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[6] In what follows, $\mathbf{Ax}^{\mathrm{T}} \geq 0$ denotes that all the elements in this matrix are larger than or equal to 0. Analogously, $\mathbf{Ax}^{\mathrm{T}} >> 0$ denotes that all the elements are larger than 0, and $\mathbf{Ax}^{\mathrm{T}} > 0$ denotes that elements are larger than or equal to 0, but at least one element is strictly positive. Similar notations will appear in similar cases.

[7] An analogous result holds if the probability measure μ is not specified. In such a case, the discount factors $d_i' > 0 (\geq 0)$ must verifiy $v_j \leq \sum_{i=1}^{k} a_{ij} d_i' \leq c_j$. The proof is trivial since one can define $d_i' = d_i \mu_i$.

Let us remark that (2.1) leads to $1 / R = \sum_{i=1}^{k} d_i \mu_i$. If we set

$$\lambda_i = R d_i \mu_i \tag{2.2}$$

$i = 1, 2, \ldots, k$, then $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$, and, therefore, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ can be considered as a probability measure. Furthermore, (2.1) leads to

$$v_j \leq \frac{1}{R} E^{\lambda}(S_j) \leq c_j \tag{2.3}$$

for $j = 1, 2, \ldots, n$, being $E^{\lambda}(S_j)$ the expected value of S_j at t_1 computed with the probability measure λ instead of μ. This is the reason why λ is called a *Risk Neutral Probability Measure*, and Theorem 1 shows that its existence (and positiveness) is the necessary and sufficient condition to guarantee the absence of arbitrage of the second type (of any kind).

The latter theorem provides a very well-known and important condition to ensure the absence of arbitrage. Nevertheless, if the arbitrage occurs, it will be interesting to measure, in monetary terms, the degree of arbitrage. This measurement will be useful to improve bid or ask real prices for PCS option contracts, and will also permit us to analyze other type of market imperfections. For instance, owing to transaction costs, the presence of arbitrage could only be apparent but not real.

The following result summarizes some properties of the measures developed by Balbás and Muñoz (1998).

Theorem 2. *Suppose that the set X of arbitrage strategies of the second type is non void. Then, problems*

$$\text{Max} \frac{\pm P(x)}{V(x)} \quad \{x \in X$$

and

$$\text{Max} \frac{\pm P(x)}{C(x) + V(x)} \quad \{x \in X$$

achieve an optimal value at the same portfolio $\mathbf{x}^*$.

The ratios in the theorem above represent relative arbitrage profits ($-P(x)$) with respect to the price of the sold assets ($V(x)$) or the total volume of trade ($C(x) + V(x)$, the price of the purchased and sold assets without considering a negative sign for the sold assets). For instance, if we consider the strong arbitrage portfolio $x' = [1,-1,-1,0]$ introduced in the previous numerical example, the price of the sold assets is $V(x') =$ 1.1, the price of the purchased assets is $C(x') = 1$, the total volume of trade is $V(x') +$ $C(x') = 2.1$, and the price of $\mathbf{x}'$ is $P(x') = C(x') - V(x') = -0.1$. Consequently, the relative arbitrage profits with respect to $V(x')$ and $C(x') + V(x')$ are 0.1/1.1 and 0.1/2.1.

Consider the portfolio $\mathbf{x}^*$ whose existence is guaranteed by Theorem 2. The disagreement measures m and l are defined by $m = -P(x^*) / V(x^*)$, $l = -P(x^*) / [C(x^*) + V(x^*)]$, or zero if no arbitrage opportunities of the second type exist. Measures m and l vanish if and only if there are no arbitrage opportunities of the second type. When the arbitrage occurs, m (l) yields available relative arbitrage gains with respect to the value of the sold (interchanged) assets. The inequalities $0 \le l \le m \le 1$ may be proved, and the level of violation of the arbitrage absence grows as the measures move from 0 to 1. The relationship $l = m/(2 - m)$ holds, and thus since $[0,1] \ni z \to z/(2-z) \in [0,1]$ is an increasing one-to-one function, both measures provide equivalent information. Further details may be found in Balbás and Muñoz (1998) or Balbás et al. (1998).[8, 9]

Let us remark that Theorems 1 and 2 permit us to analyze and detect arbitrage portfolios without previously specifying the exact nature of the strategy to be used. For instance, many financial papers empirically test the existence of violations for the usual put-call parity or the relationship between spot and future prices. This type of test cannot be implemented when analyzing PCS options, in which case all the available securities must be simultaneously considered and the market globally tested.

Let us turn now to hedging strategies and arbitrage portfolios of the first type. Assume that the model does not permit arbitrage portfolios of the second type. Then, R is the highest return than can be guaranteed. However, an investor may be interested in a hedging portfolio whose guaranteed return is very close to R but provides larger returns in some states of the world.

Let us fix a concrete security S_{j_0} and consider the usual way of hedging the purchase of this security, i.e., solve the problem.[10]

$$\text{Min } P(x) \quad \begin{cases} x_{j_0} & = 1 \\ \mathbf{A}\mathbf{x}^{\mathrm{T}} & \ge 1 \end{cases} \tag{2.4}$$

If the solution is attained at $\hat{x}$, the absence of strong arbitrage ensures that $P(\hat{x}) > 0$, and $1/P(\hat{x})$ is the optimal (maximum) guaranteed return if a unit of S_{j_0} is bought.[11]

Notice that the solution $\hat{x}$ of (2.4) dominates the riskless security and, consequently, $\hat{x}$ is really a hedged portfolio. Moreover, $P(\hat{x}) \ge 1/R$ because, otherwise, investors could implement strong arbitrage (against our assumptions) by selling the riskless asset and buying $\hat{x}$. Furthermore, if $P(\hat{x}) = 1/R$, then either $\hat{x}$ replicates the riskless asset or the previous strategy is weak arbitrage. Thus, there are arbitrage portfolios of the first type such that $x_{j_0} = 1$ if $R = 1/P(\hat{x})$ and $\mathbf{A}\hat{\mathbf{x}}^{\mathrm{T}} > 1$.

An analogous analysis may be done to hedge the sale of S_{j_0}. Just write $x_{j_0} = -1$ in (2.4) instead of $x_{j_0} = 1$. Obviously, not only hedging portfolios, but also arbitrage of the first type, can be detected by computing all the hedging portfolios when j_0 moves from 1 to n.

The last part of this synopsis focuses on individual portfolio selection and variance

[8] Once again, these results are also verified in a model where μ is not specified.

[9] Another procedure, useful to detect arbitrage portfolios, may be found in Garman (1976).

[10] Recall that $\mathbf{A}\mathbf{x}^{\mathrm{T}} \ge 1$ means that all the elements in the column matrix $\mathbf{A}\mathbf{x}^{\mathrm{T}}$ are larger than or equal to 1.

[11] Clearly, the return $1/P(x)$ is maximum if and only if the price $P(x)$ is minimum.

aversion. Assume that there are no arbitrage opportunities (of any sort) in the model. Then, Theorem 1 shows that the arbitrage absence still holds for some concrete linear pricing rule π such that $v_j \leq \pi_j \leq c_j$, $j = 1, 2, \ldots, n$, being π_j the price of S_j provided by π.[12] Besides, π may be considered as a positive real valued linear operator over the space *span*(A), which is the span of the columns of $\mathbf{A}$.[13] Then, the *Riesz Representation Theorem* of linear operators in Hilbert spaces allows us to establish the following result (see Chamberlain and Rosthchild (1983)).

Theorem 3. *There exists a unique discount factor d such that*

$$\sum_{i=1}^{k} a_{ij} d_i \mu_i = \pi_j$$

for $j = 1, 2, \ldots, n$, *and* d^t *belongs to span*(A).

We will assume that d^t is not the payoff of a riskless asset. This hypothesis is not restrictive (it only affirms that the market is not risk-neutral and, consequently, $\lambda \neq \mu$) and will always hold in our empirical test.

In order to achieve an easier notation, denote by S_j the j^{th}-column of $\mathbf{A}$, $j = 1, 2, \ldots, n$, and let us identify each feasible portfolio $\mathbf{x}$ with its final payoff $\mathbf{A}\mathbf{x}^{\mathrm{T}} = y \in$ *span*(A). Denote $\pi\left(A\mathbf{x}^{\mathrm{T}}\right) = \pi(y) = \sum_{i=1}^{k} y_i d_i \mu_i$ by its current price provided by π. In particular, $\pi(d^t) = \sum_{i=1}^{k} d_i^2 \mu_i > 0$. Define by $R(y) = y / \pi(y)$ the return (provided by π) of any $y \in$ *span*(A) such that $\pi(y) > 0$, and consider its expected value $E^{\mu}(R(y))$ and standard deviation $\sigma^{\mu}(R(y))$.[14] Then, the statement below, whose proof is a consequence of the *Projection Lemma* of Hilbert spaces (see for instance Duffie (1988)), provides the optimal portfolios in a variance-averse model.

Theorem 4. *For any* $y \in$ *span*(A) *such that* $\pi(y) > 0$, *there exists a linear combination of* d^t *and the riskless asset,* $\varphi S_1 + \psi d^t$, *such that*

i) $\psi \leq 0$

ii) $\pi(y) = \pi(\varphi S_1 + \psi d^t)$

iii) $E^{\mu}(R(y)) = E^{\mu}(R(\varphi S_1 + \psi d^t))$

iv) $\sigma^{\mu}(R(y)) \geq \sigma^{\mu}(R(\varphi S_1 + \psi d^t))$

Hence, for a desired expected return, the minimum variance is attained by selling the portfolio $\bar{\mathbf{x}}$ such that $\mathbf{A}\bar{\mathbf{x}}^{\mathrm{T}} = d^t$ and investing the price of $\bar{\mathbf{x}}$, along with the investor's capital, in the riskless asset.[15]

[12] Take, for instance, the linear pricing rule provided by the second term of (2.3).

[13] That is, the space of $k \times 1$ column matrices that can be obtained by linear combinations of the columns of A.

[14] Recall that y and $R(y)$ may be considered random variables and, therefore, $E^{\mu}\left(R(y)\right) = \left(1 / \pi(y)\right)\sum_{i=1}^{k} y_i \mu_i$ and $\sigma^{\mu}\left(R(y)\right) = \left(1 / \pi(y)\right)\left(\sum_{i=1}^{k}\left[y_i - \pi(y)E^{\mu}\left(R(y)\right)\right]^2 \mu_i\right)^{1/2}$

[15] Under the usual CAPM assumptions, the considered securities are stocks and, under the suitable hypotheses on their stochastic behavior, the portfolio $\bar{\mathbf{x}}$ is composed of a long position in the riskless asset and short positions in the stocks. Since the coefficient ψ must be negative, the *Market Portfolio* consists of stocks in a long position.

Theorem 4 will play a crucial role in our analysis because it allows us to compute the minimum variance portfolios without deriving correlations within the set of available PCS options. In fact, these portfolios are given by the vector $\boldsymbol{d}$ of discount factors and, as pointed out by (2.2), $\boldsymbol{d}$ is the density between a risk neutral measure λ and the initial probability measure μ.

3. MARKETS AND DATA

The Chicago Board of Trade's (CBOT) PCS Catastrophe options are standardized contracts based on PCS indices that track the insured losses resulting from catastrophic events that occur in a given area and risk period, as estimated by PCS.

When PCS estimates that a natural or man-made event within the US is likely to cause more than $25 million in total insured property losses and determines that such effect is likely to affect a significant number of policyholders and property-casualty insurance companies, PCS identifies the event as a catastrophe and assigns it a catastrophe serial number. PCS provides nine loss indices daily to the CBOT: a national index, five regional indices, and three state indices (National, Eastern, Northeastern, Southeastern, Midwestern, Western, Florida, Texas, and California loss indices). Each PCS loss index represents the sum of current PCS estimates for insured catastrophic losses in the area and loss period covered, divided by $100 million, and rounded to the nearest first decimal point.

The loss period is the time during which a catastrophic event must occur in order for the resulting losses to be included in a particular index. Most PCS indices have quarterly loss periods, some of them (California and Western) have annual loss periods, and one of them (National) has both quarterly and annual risk periods. Following the loss period, there exists a development period (twelve months) during which PCS continues estimating and reestimating losses for catastrophes occurring during the loss period. The development period estimates affect PCS indices and determine the final settlement value of the indices.

Catastrophe options are available for trading until the end of the development period. They are European and cash-settled (each point equals $200 cash value). They can be traded as either "small-cap" or "large-cap" contracts. These caps limit the amount of losses that are included under each contract: insured losses from $0 to $20 billion for the small contracts and losses from $20 to $50 billion for large contracts. In practice, traders prefer negotiating call spreads, further limiting their associated payoffs. Sophisticated combinations traded as a package, which include several expiration dates and indices, are also available.[16] Catastrophe options bid and ask quotes, and the current value of the indices are provided daily by the CBOT. Premiums are quoted in (index) points and tenths of a point (each point equals $200). Strike values are listed in integral multiples of five points.

Our empirical work studies two periods: from February 25 to April 20, 1998, and from June 23 to July 30, 1998. The quotes used in the empirical analysis were provided by the CBOT and correspond to synchronized bid and ask quotes posted at the end of each day. For each of the days considered, we have also included a risk-free

[16] Henceforth, for short, we will merely say "derivative" or "option" to refer to a single option or a package of options.

asset. Its price was obtained from the coupon-only strips quotes reported by *The Wall Street Journal*.[17] Treasury bills could have been used instead, but strips maturities were much closer to the options expiration dates.[18]

In order to learn about the distributional properties of the catastrophe waiting times and their associated amount of insured losses, our data also includes a 25-year (1973-1997) catastrophe record provided by PCS. This record included all catastrophes that occurred in each state with indication of its serial number, beginning and ending dates, causes and PCS's estimates of insured losses. The monetary value of losses was converted into 1997 dollars by using the Producer Price Index reported by the Bureau of Labor Statistics (US Department of Labor). We restricted our observations of the value of the insured losses to the sample period from 1990 through 1997 for several reasons that will be analyzed in the fourth section.

4. Empirical Research: Methodology and Results

Henceforth, we will first specify the methodology employed in the empirical analysis and then present the obtained results.

Our analysis only targets those derivatives with a single underlying index and a unique expiration date. We group those derivatives with the same expiration date and the same underlying index that are available for trading. For a given day, we require a minimum of four assets in each set. For the first period, this filtering left us with derivatives associated with the following indices: National Annual-98 (36 valid days), California Annual-98 (36 days), Eastern September-98 (12 days) and Southeastern September-98 (10 days). For the second period we have the National Annual-98 (27 valid days), Eastern September-98 (10) and Southeastern September-98 (10) indices.[19] Table 1 summarizes the number of relevant derivatives satisfying the above criteria and the number of their quotes available in the two periods.[20] This constitutes our whole sample for Subsections 4.1 and 4.2.

In Subsections 4.3 and 4.4, a large amount of data on waiting times and their associated amount of losses is required to perform reliable simulations and, therefore, the characteristics of our historical data set compel us to concentrate exclusively on the National Annual-98 Index.

4.1. Pricing by Strong Arbitrage Methods

The price of the PCS derivatives will be analyzed once a day during each tested period. Hence, under the notations of the second section, the date t_0 will always be

[17] Exact values were obtained through linear interpolation of midpoints of the bid-ask prices by using the closest maturities to the option expiration dates.

[18] Some of our computations were also implemented with T-bill returns. Our main results remained unchanged.

[19] The expiration dates for the Annual-98 contracts and the September-98 contracts are December 31, 1999, and September 30, 1999, respectively.

[20] We have also detected that it was possible to synthetically produce some other Eastern September-98 options based on their corresponding Northeastern and Southeastern options for some days in our sample. To be consistent, as the latter two negotiated independently, we decided not to include the synthesized Eastern options as other derivatives available for trading in our sample.

Table 1

Overview of the PCS Catastrophe Options Sample

This table describes our sample of derivatives. For each specific index, we require a minimum of four tradable securities in order to include a given day in the analysis. The first column gives cumulative figures corresponding to each of the entire periods, and the subsequent total columns summarize the daily number of derivatives and quotes.

	NAT ANN 98					EST SEP 98					SE SEP 98					CAL ANN 98				
	Total	Daily Statistics				Total	Daily Statistics				Total	Daily Statistics				Total	Daily Statistics			
Number of		Mean	Min.	Med.	Max.		Mean	Min.	Med.	Max.		Mean	Min.	Med.	Max.		Mean	Min.	Med.	Max.
Panel A: First Period																				
Days	35					12					10					36				
Derivatives	316	8.77	8	9	9	53	4.42	4	4	5	40	4	4	4	4	144	4	4	4	4
Bid Quotes	253	7.3	7	7	8	48	4	4	4	4	35	3.5	3	3.5	4	112	3.11	1	4	4
Ask Quotes	260	7.22	5	7	9	53	4.42	4	4	5	40	4	4	4	4	120	3.33	3	3	4
Panel B: Second Period																				
Days	27					10					10									
Derivatives	283	10.48	8	11	13	40	4	4	4	4	40	4	4	4	4					
Bid Quotes	187	6.93	5	7	8	40	4	4	4	4	18	1.8	1	2	2					
Ask Quotes	232	8.59	5	10	11	27	2.7	2	2	4	36	3.6	2	4	4					

the corresponding day, while securities $S_2, S_3, \ldots, S_n$ will be PCS option contracts (call or put spreads, butterflies, etc.) available this day, and with the same underlying index W and expiration date t_1.[21] Their bid and ask prices are perfectly synchronized and are provided by CBOT. Security S_1 will be a pure discount bond available at t_0 such that its maturity is as close to t_1 as possible. Of course, all the data and parameters (dates t_0 or t_1, securities, prices, etc.) depend on the concrete day under revision.

Let W_1 be the current value of the index and denote by $W_2, \ldots, W_r$ the strike prices corresponding to S_j, $j = 2, 3, \ldots, n$. The future state of the world will be determined by the final (at t_1) value of W (any real number greater than W_1 and rounded to the nearest first decimal point), and the matrix $\mathbf{A}$ of final payoffs may be easily computed. In fact, all the elements in matrix $\mathbf{A}$'s first column (payoffs of the riskless asset) are equal to 1, and the rest of the columns are given by the usual differences between W and W_i, $i = 1, 2, \ldots, r$. It is obvious that, for an arbitrary strategy x, its final payoffs verify the constraints $\mathbf{A}\mathbf{x}^{\mathrm{T}} \geq 0$, $\mathbf{A}\mathbf{x}^{\mathrm{T}} > 0$, or $\mathbf{A}\mathbf{x}^{\mathrm{T}} >> 0$ if and only if these constraints are fulfilled when the settlement value of W belongs to the set $\{W_1, W_2, \ldots, W_r\}$.[22] So, the absence or existence of arbitrage may be tested under the assumption that these elements are the only possible states of the world. Furthermore, this simplification neither modifies the value of the disagreement measures m and l, nor affects the results when hedging or weak arbitrage portfolios are being computed. Consequently, we will allow W to attain all the feasible values only when testing portfolio choice models.

Once the available derivatives, their real bid-ask prices provided by CBOT, the r states of the world, and the matrix $\mathbf{A}$ are fixed, we can compute the measure m and the portfolio $\mathbf{x}^*$ introduced in Theorem 2. If $m \neq 0$, there are arbitrage opportunities. This case never appeared within the period tested.

Next, we fix an arbitrary option S_{j_0}, $j_0 = 2, 3, \ldots, n$, and consider an agent who can buy this derivative by paying the price v_{j_0}.[23] If the new values for m and $\mathbf{x}^*$ show the presence of arbitrage and the profits represented by m are high enough to overcome the market frictions, it may be concluded that the market allows us to price S_{j_0} by arbitrage methods. An agent can offer a new bid price v'_{j_0} (such that $v_{j_0} \leq v'_{j_0} \leq c_{j_0}$ and, therefore, better than the current bid price v_{j_0}) without any kind of risk. The position will be hedged by implementing the arbitrage portfolio $\mathbf{x}^*$ if a new investor accepts the new bid price.

Analogously, one can analyze if the ask price c_{j_0} may be improved. Just consider that c_{j_0} equals both, the bid and the ask price and compute the new solutions for m and $\mathbf{x}^*$.[24]

The above procedure can be applied for all the available securities (for $j_0 = 2, 3, \ldots, n$) in order to test how often the market allows us to price by strong arbitrage methods. The empirical results are confined to Table 2.

[21] That is, the underlying index and the loss and development periods coincide for all the considered derivatives.

[22] Strategies 1 and 2 below illustrate this fact. Notice that the same property holds if one writes 1 instead of 0 in the right side of the above inequalities.

[23] That is, the bid-ask spread vanishes for the j_0th security. The rest of the prices are not modified. Of course, this analysis has not been implemented in cases where $v_{j_0} = 0$.

[24] This analysis has not been implemented in cases where $c_{j_0} = \infty$.

TABLE 2
Second Type Arbitrage Opportunities

The bid-ask spread has been removed for each option at a time, and its price was set equal alternatively to the bid quote and the ask quote when possible. The table summarizes the resulting arbitrage opportunities of the second type and their associated optimal gains as quantified by the m measure. The first two columns show the number of days for which there are some arbitrage opportunities. Subsequent columns provide statistics computed over those days with arbitrage opportunities ($m \neq 0$).

			Detected Second Type Arbitrage Opport.								
	Days		Daily Number of Opport.				Daily Maximum m (percent)				
Index	No.	Percent	Mean	Min.	Med.	Max.	Mean	Min.	Med.	Max.	Mode
Panel A: First Period											
NAT ANN 98	7	19.44	2	1	2	3	10.23	9.09	9.09	13.98	9.09
EST SEP 98	5	41.67	1	1	1	1	6.82	6.67	6.82	6.98	6.70
SE SEP 98	0	0.00									
CAL ANN 98	26	72.22	1	1	1	1	15.46	11.76	16.67	21.05	16.67
Panel B: Second Period											
NAT ANN 98	27	100.00	2.22	1	2	3	19.05	2.50	9.09	37.50	37.50
EST SEP 98	9	90.00	1	1	1	1	10.13	9.96	10.17	10.23	10.17
SE SEP 98	0	0.00									

This particular type of arbitrage is quite often detected. It should be noted that these results seem to reveal that the price setting process might be improved. Hedging (with arbitrage portfolios) would be feasible. The arbitrage profits are quite large and this should be used by investors to offer new prices. For the National Annual-98 index, arbitrage opportunities appear in 7 out of 36 days for the first period (see Table 2) and in up to three different cases. The maximum value of m is equal to .1398 (this corresponds to an l value of .0752). For the second period and the same index, arbitrage is feasible every day for up to three different available premium quotes. This time the maximum value of m is .375 (l = .2308). This reflects a riskless benefit that amounts to a 37.5 percent of the total monetary value of the sold assets (or a 23.08 percent of the total monetary value of all traded assets). With respect to other indices, California and Eastern include a unique position that allows for arbitrage hedging in the first period (the maximum value of m is .2105 and .0698, respectively) and the same may be said about the Eastern index in the second period (maximum m = .1023). No mispricings were found for the Southeastern index. In any event, the number of available positions was notably low for these last three indices (see Table 1). Thus, for a significant percentage of days, agents could analyze the bid-ask spread and offer more efficient prices in some cases without assuming any kind of risk. This fact should lead to smaller spreads.

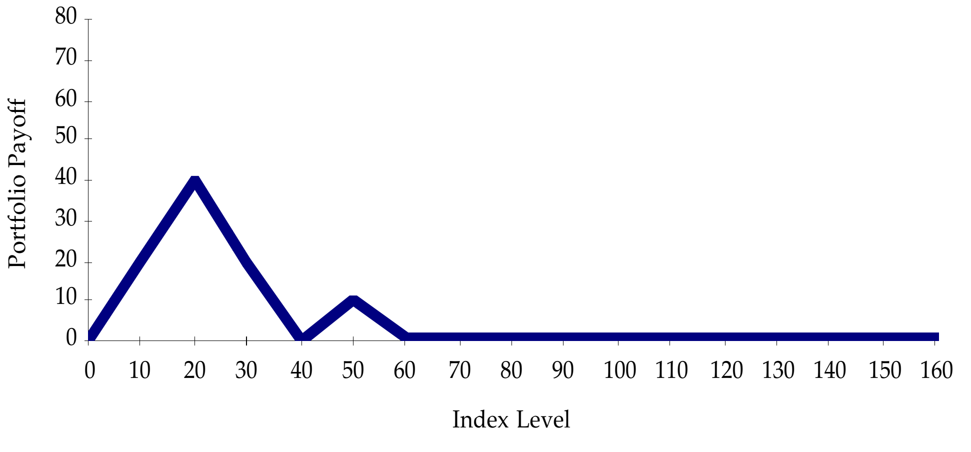
For illustration purposes, we show in Table 3 the optimal (maximum m value) second type arbitrage portfolio detected on date July 24, 1998 for the Eastern September-98 Index (Strategy 1). Figure 1 plots the portfolio payoffs pattern for different levels of the final index value.

TABLE 3
Optimal Second Type Arbitrage on July 24, 1998, for the Eastern September-98 Index

This table shows the optimal second type arbitrage opportunity corresponding to July 24, 1998 and the Eastern September-98 Index. CA 20 40 and PU 50 stand for a call spread and a put with relevant exercise prices as indicated, respectively. All derivatives available for trading together with their bid and ask prices are reported. The price for a zero-coupon bond (riskless asset) with a maturity value of one point is also given. All prices are expressed in points, each with a value of $200. Bold face is used to indicate those assets involved in the detected arbitrage portfolio, and the number of bought or sold units is given in parentheses beside the affected price (a negative sign indicates a sale). The last two rows give the portfolio price and the *m* value. This arbitrage was detected when the bid quote was set equal to the ask for the PU 50 derivative. The same arbitrage strategy was detected for nine days.

Asset	Bid	Ask
Bond	0.93867188	**0.93867188 (100)**
CA 20 40	**3 (–4)**	n.a.
CA 40 60	**2.5 (–1)**	3.5
CA 150 200	2	n.a.
PU 50	30	**45 (–2)**
Portfolio Price	–10.6328	
Value of *m*	0.1017	

FIGURE 1
Final Payoffs of Strategy 1



Strategy 1 may be interpreted as follows: The bonds can be purchased at 93.87, and calls can be sold at the bid prices, so the net portfolio costs the owner 79.37 now (t_0). Later (t_1), the owner cashes the bonds and pays the call owners. It is always adequate to cover the liability for selling the put portfolio. Therefore, the owner can ask a price of 79.37 for a portfolio of two puts (39.68 per put) and still have a non-negative payoff without paying anything now. This is a weak arbitrage and getting more (45 per put contract) provides the owner with an upfront arbitrage profit.

We can also check the value of m and l. Since the price of the purchased assets (bonds) is 93.87 and the price of the sold assets is (assuming that both puts are sold at 45 per put) 104.5, then

$$m = \left(104.5 - 93.87\right) / 104.5 = 0.101722 \text{ and}$$

$$l = \left(104.5 - 93.87\right) / \left(104.5 + 93.87\right) = 0.05358.$$

The strategy shows that an ask price could have been reduced from 45 to 39.68 index points at least. If the ask price is lower than 39.68, the strategy does not work, but a new strategy might appear. Thus, if we are interested in the lowest possible ask price (actually, it is 39.68 is this case), we have to apply the general procedure provided by Theorem 2 instead of analyzing the previous strategy.

Figure 1 also shows that the final global payoff of the strategy may be easily determined by means of the final payoffs obtained when the index final value W belongs to the set of strike prices {0, 20, 40, 50, 60, 150, 200} (connecting the corresponding points with line segments). Moreover, the final payoff is never lower than zero if and only if the property holds when W belongs to this set. This is the reason why we can simplify the set of states of the world.

Let us leave Strategy 1 and go back to the general case. Because the offering of better quotes is sometimes feasible, it is interesting to measure the highest adjustments that could have been implemented in the bid (an increase) and ask (a decrease) premiums. To this aim we followed the next algorithm. Focusing on one of the quotes which gave rise to the above arbitrage opportunities, we appropriately moved up or down the implied quote only for a tick and then searched again for arbitrage opportunities. This process was iterated until reaching a total removal of the riskless arbitrage hedging. We carried over this algorithm for each price independently. The corresponding price and spread final adjustments are given in Table 4. For the National Annual derivatives, our results show price changes ranging from 2.5 percent to 100 percent along with spread reductions ranging from 5 percent to 56.52 percent. Significant adjustments were also possible for the other indices.

It should be mentioned that some refinements of this procedure point out that a more adjusted set of prices may still be reached. If the above algorithm is not carried out *ceteris paribus*, that is, if we keep the final adjusted premium before moving to the next one, we find that new arbitrage hedging strategies could appear, thereby leading to possible further reductions in the spread.[25]

In short, although the market quotes studied here do not permit a gain of arbitrage

[25] As the ordering might be relevant in this case, we do not report our results.

profits by anyone obliged to incur the cost of the bid-ask spread, better relative pricing by (strong) arbitrage methods is possible in the PCS options market for the studied periods. This is important for two reasons. First, this information is useful to traders since the whole arbitrage portfolio may be shown. Second, frictionless pricing theory suggests that competition among traders should lead to a situation with correct relative prices, i.e., a set of quotes that should exhaust any exploitable possibility of making money without any sort of risk.

Table 4
Bid-Ask Spread Reduction

This table shows the bid-ask spread reduction that can be implemented for each derivative in order to remove the second type arbitrage strategies previously detected. Those assets involved are reported on the left side; additionally, in parentheses we indicate whether changes correspond to the ask quote (a), bid quote (b) or both bid and ask quotes (b/a). CA 40 60 stands for a call spread with exercise prices 40 and 60. PS 40 60 stands for a put spread with relevant exercise prices as indicated. The price change and the spread reduction are both given in ticks (i.e., $20 or one-tenth of a point) and in percentage terms. For each asset some descriptive statistics have been computed over those days and quotes for which changes were possible.

	Price Change						Spread Reduction					
Index &	Ticks			Percent			Ticks			Percent		
Derivative	Mean	Min.	Max.	Mean	Min.	Max.	Mean	Min.	Max.	Mean	Min.	Max.
Panel A: First Period												
NAT ANN 98				10.09	5.17	22.41				16.35	5.41	56.52
CA 80 100 (a)	5	3	13	8.62	5.17	22.41	5	3	13	19.80	10.34	56.52
CA 100 120 (b/a)	3.5	2	5	12.69	10.00	15.38	3.5	2	5	9.66	5.41	13.89
CA 120 140 (a)	5.71	5	10	10.39	9.09	18.18	5	5	5	12.50	12.50	12.50
EST SEP 98												
PS 40 60 (a)	14	14	14	8.24	8.24	8.24		n.a.			n.a.	
CAL ANN 98												
CA 100 200 (b)	5.54	4	8	18.46	13.33	26.67	4	4	4	13.33	13.33	13.33
Panel B: Second Period												
NAT ANN 98				26.33	2.50	100.00				13.45	5.00	33.33
CA 40 60 (b)	3	3	3	4.00	4.00	4.00		n.a.			n.a.	
CA 60 80 (b)	18	11	25	72.00	44.00	100.00	18	11	25	25.13	16.92	33.33
CA 100 120 (b)	6	6	6	60.00	60.00	60.00	6	6	6	15.00	15.00	15.00
CA 100 200 (a)	5	5	5	2.50	2.50	2.50	5	5	5	5.00	5.00	5.00
CA 120 140 (b/a)	5.41	4	6	37.35	6.67	60.00	7.11	6	10	15.19	13.33	20.00
EST SEP 98												
PS 0 50 (a)	53.56	52	54	11.90	11.56	12.00	53.56	52	54	35.70	34.67	36.00

Some factors related to the implementation of the detected arbitrage strategies and not considered so far might explain those relative mispricings. One of them is the existence of transaction costs. In relation to the relative mispricing, it should be kept in mind that measures m and l represent relative arbitrage profits, and the levels achieved by these measures are high enough to reflect gains after discounting transaction costs.

Another factor is due to the number of units of each asset needed to implement some arbitrage strategies. This number might not be available for trading. We do not have any piece of information about the volume associated with the quotes gathered by the CBOT that constitute our sample. Nevertheless, a comparison between the volume corresponding to the transactions made during our sample periods and the number of derivatives needed to implement the detected arbitrage strategies lead us to think that in most cases this lack of information does not seem to be a real problem.

Some final reasons might be related to margin rules or illiquidity but the empirical results seem to be significant for researchers and traders anyway.

4.2. Weak Arbitrage and Hedging Portfolios

Suppose that for a fixed $j_0 \in \{2, 3, \ldots, n\}$ it is still obtained $m = 0$ after assuming that $v_{j_0} \leq v'_{j_0} \leq C_{j_0}$, is the ask price (respectively, c_{j_0} is the bid price). Then, problem (2.4) (respectively, after the modification $x_{j_0} = -1$) has been solved in order to analyze how the real bid price v_{j_0} (ask price c_{j_0}) can be improved. This case will hold when the achieved solution guarantees a return R or very close to R. Then, investors can offer a new bid (ask) $v_{j_0} \leq v'_{j_0} \leq c_{j_0}$, and the solution of (2.4) provides a portfolio that will almost guarantee the riskless return R if a new agent accepts the new price. Furthermore, this strategy could lead to great returns in some states of the world, and thus, it could be interesting for many investors.

Following this procedure hedging strategies were obtained, and they were grouped into first type arbitrage opportunities (with a guaranteed return equal to R and payoffs greater than one in at least one state of the world) and other optimal hedging strategies. Both the guaranteed net return and the maximum possible net return were computed for each detected position available for hedging; mean values are given in Table 5. We also report the corresponding mean values after subtracting the return guaranteed by the risk-free asset. For some states of the world, extraordinarily large returns might be obtained (e.g., there were first type arbitrage opportunities that involved selling one call spread 80/100 and gave rise to a possible net return of 2,215.63). The minimum net return equals that of the risk-free asset for almost all cases.

Thus, an important part of the available positions might have been hedged by means of weak (and strong) arbitrage or other optimal strategies leading in many cases to possible returns exceeding largely that of the risk-free asset. Note that this has been feasible even in a situation in which the underlying index is not tradable, and put derivatives are seldom available. Considerations akin to the ones pointed out at the end of the previous subsection, regarding the proper interpretation of these results, also apply here.

Again, for illustration purposes, Table 6 shows the optimal hedging portfolio (weak arbitrage) on July 1, 1998 for the National Annual-98, Index (Strategy 2). Figure 2 plots the portfolio payoffs pattern for different levels of the final index value.

The interpretation of Strategy 2 may be as follows: the purchase of one butterfly and the sale of one CA 40 60 provides an income equal to 2.5 index points, the bid price for the CA 60 80. Suppose that this call is bought at 2.5. Then the payoffs associated with the sold assets are dominated (strictly in some states of the world) by the payoffs associated with those purchased. So, the whole portfolio price is zero but pro-

Table 5
Optimal Hedging Portfolios

CA 40 60 stands for a call spread with exercise prices 40 and 60, and similarly for the other possible exercise prices. CB denotes a butterfly call spread with relevant exercise prices as indicated. For each derivative, the number of days for which a hedging strategy was available is given and in parentheses it is indicated whether the hedged derivative is bought (b) or sold (s) at the optimal hedging portfolios. Guaranteed and maximum returns in average terms along with the corresponding excesses over the risk-free rate *R* are also given.

	Optimal First Type Arbitrage Opport.					Other Optimal Hedging Portfolios				
Index		Mean Guaranteed Return (%)		Mean Maximum Return (%)			Mean Guaranteed Return (%)		Mean Maximum Return (%)	
& Derivative	Days	Return	Excess	Return	Excess	Days	Return	Excess	Return	Excess
Panel A: First Period										
NAT ANN 98										
CA 40 60						16 (b)	10.35	0.00	10.35	0.00
						1 (s)	10.62	0.00	10.62	0.00
CA 120 140	29 (s)	10.27	0.00	2,215.63	2,205.36					
CAL ANN 98										
CA 80 100	1 (s)	9.74	0.00	2,204.53	2,194.79					
Panel B: Second Period										
NAT ANN 98										
CA 60 80	5 (b)	8.23	0.00	2,172.77	2,164.54					
CB 40 60 80 100	1 (s)	8.11	0.00	2,170.27	2,162.16	6 (s)	6.57	–1.46	2,137.97	2,129.93

vides (strictly in some states) positive payoffs (weak arbitrage). Besides, if we add the bond to this portfolio, the final payoffs dominate the bond payoffs (strictly in some states), but we just have to pay the bond price.

4.3. Evaluating the index real distribution and the risk-neutral probability measure

To assess catastrophe options from an actuarial point of view, an analysis of the distribution of possible future values of the underlying indices is required. There are essentially two approaches to form such probability assessments. One is to use computer simulation of scenarios based on a vast amount of meteorological, seismological, and economic information. The other relies on statistical modeling based on historical data.

This subsection is partially devoted to the analysis of the distributional properties of the National Annual Index to be used in the rest of the article, and for this matter we concentrate on the statistical analysis of historical catastrophe data. We develop a nonparametric simulation procedure in order to obtain the expected final payoffs. This method does not require any distributional assumption; instead, "it lets the data talk."Furthermore, this analysis relies on the empirical distribution of waiting times and their associated losses, thereby avoiding the traditional shortage of data that is

TABLE 6
Optimal Hedging Portfolio

(Weak Arbitrage) on July 1, 1998, or the National Annual-98 Index

This table shows the optimal hedging portfolio detected on July 1, 1998, for the National Annual-98 Index. CA 30 50 stands for a call spread with exercise prices 30 and 50, and similarly for the other possible exercise prices. CB denotes a butterfly call spread with relevant exercise prices as indicated. The reported portfolio is a weak arbitrage portfolio which permits one to hedge the purchase of the CA 60 80 derivative. All derivatives available for trading together with their bid and ask prices are reported. The price for a zero-coupon bond (riskless asset) with a maturity value of one point is also given. All prices are expressed in points, each with a value of $200. Bold face is used to indicate those assets involved in the detected hedging portfolio, and the bought and sold units are given in parentheses (a negative sign indicates a sale). The last row gives the portfolio price. The same portfolio was available for 5 consecutive days.

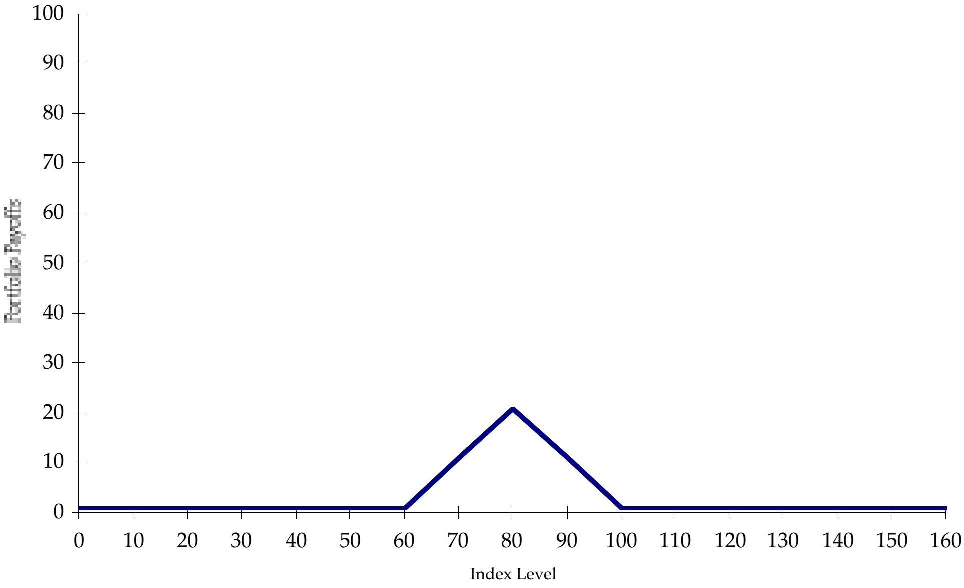
Derivative	Bid	Ask
Bond	0.92351562	**0.92351562 (1)**
CA 30 50	10	n.a.
CA 40 60	**7.5 (–1)**	n.a.
CA 60 80	**2.5 (1)**	9
CA 80 100	n.a.	7
CA 100 120	1	5
CA 100 150	n.a.	12
CA 100 200	n.a.	20
CA 120 140	n.a.	5.5
CA 150 200	4	7.5
CA 180 200	0.4	1.8
CA 200 250	n.a.	4
CA 250 300	0.5	2.5
CB 40 60 80 100	n.a.	**5 (1)**
Portfolio Price	0.92351562	

faced when using exclusively the empirical distribution of the final historical values of the index.

In addition, from a financial perspective, once we have exhausted the arbitrage and hedging pricing approaches, risk considerations come into place and therefore the use of probability assessments is also necessary. As stated by Theorems 3 and 4, the underlying index real distribution is required in order to solve minimum variance problems. In this case, variables and parameters (dates t_0 or t_1, the riskless return, securities, prices, etc.) are introduced by the procedures already mentioned, but the set of states of the world must be enlarged. Now this set must incorporate all the index-feasible final (at the expiration date t_1) values and not only the derivatives' strike prices.

Fix a day t_0 and, consequently, let us assume that all the parameters are fixed. To

FIGURE 2
Final Payoffs of Strategy 2



determine the final distribution of the underlying index W, we proceed as follows: First of all, we consider the empirical distribution of random variables T, *"time between two consecutive catastrophes,"* and L, *"losses caused by a specific catastrophe."* It is assumed that T and L can achieve several values with probabilities according to the empirical frequencies obtained from the real data described in the third section. Later, we simulate several values $T_1, T_2, \ldots, T_s$ of T till $\sum_{i=1}^{s-1} T_i \le t_1 - t_0$ and $\sum_{i=1}^{s} T_i > t_1 - t_0$, and s–1 values $L_1, L_2, \ldots, L_{s-1}$ of L. Each specific result L_i is incorporated if and only if $L_i \ge \$25$ million, and we take $L_i = 0$ otherwise. If W_0 is the index value at t_0, the simulation process provides the total value $W = W_0 + \sum_{i=1}^{s-1} L_i$ where each L_i has been previously translated into index points. The whole simulation process is repeated a high number of times in order to attain a numerical distribution of W.

The risk neutral probabilities, defined in (2.2), have also been determined. We have followed the general method proposed by Hansen and Jagannathan (1997).[26] Hence, fix a day t_0 and all the parameters of the problem. Suppose that the simulation process has already been implemented and, therefore, the (real) probability measure $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is known. Then, the (risk-neutral) measure λ is obtained by minimizing $\sum_{i=1}^{k} (\lambda_i - \mu_i)^2$ among the row-matrixes λ such that $\sum_{i=1}^{k} \lambda_i = 1, \lambda_i \ge 0, i = 1, 2, \ldots, k$, and (2.3) holds.

Once the measures μ and λ have been determined, we can give two theoretical prices per security. The first one, $(1/R)E^{\mu}\left(S_j\right)$, $j = 1, 2, \ldots, n$, is the *Pure Premium*, usual in actuarial sciences. The second (see (2.3)), $(1/R)E^{\lambda}\left(S_j\right)$, is obtained from a financial

[26] The Hansen and Jagannathan (1997) method extends the procedure provided by Rubinstein (1994) and by Jackwerth and Rubinstein (1996) to study the effect of the volatility "smile" on the risk-neutral probability measure.

point of view by considering real quotes and applying the most important topics on static asset pricing models.

The procedure above has been implemented for all the possible values of the current date t_0 (every day of our sample periods).

We restrict our observations of the value of insured losses to the sample period from 1990 through 1997 for several reasons. Figure 3 shows the annual number of catastrophes and the quantity of their associated losses since 1973. As the figure suggests, while the annual number of catastrophes remains reasonably stable along the period, there seems to be a general positive trend and a structural change in the behavior of the value of insured losses associated with each catastrophe since Hurricane Hugo hit the U.S. in 1989. However, these patterns may be only illusory. Insured losses are affected by multiple variables thus far ignored such as population growth, development, changes in insurance coverage, number of premiums, and inflation. When a sufficiently long period of time is considered, the adjustments of the loss series for these variables tend to homogenize it, approximating past losses to more recent ones.[27] Thus, a reasonable approximation to the adjusted series can be obtained by concentrating on recent (unadjusted) losses.[28] Moreover, any adjustment of the series, which becomes necessary when a long period of losses is considered, might be methodologically questionable.

With regard to the number of catastrophes, the whole sample since 1973 has been used. This is expected to surmount the difficulty of getting good estimates based on small samples for the probability of low frequency events such as catastrophes.

The simulation process described above has been implemented in order to estimate the probability distributions of the final (end of 1999) value of the National Annual-98 index for every day in both periods. A number of 100,000 replications has been used for each day. Results are given in Figures 4 and 5. As the figures show, the probability mass is mainly accumulated around the index levels lying approximately in the intervals 20 to 100 and 160 to 240. The distributions are bimodal or even trimodal because (probably) the number of catastrophes must be entire. Note also that as days go by and the final date approaches, the probability mass tends to concentrate because of the accompanying uncertainty reduction.

Once the required distributions have been estimated, the next step is to compute the discounted expected options payoffs (pure premiums). These are given in Table 7. Notice that the call spreads 100/120 and 120/140 have almost identical discounted expected payoffs. This is due to the general lack of probability mass in the interval 100 to 140. In general, pure premiums lie around options real quotes.

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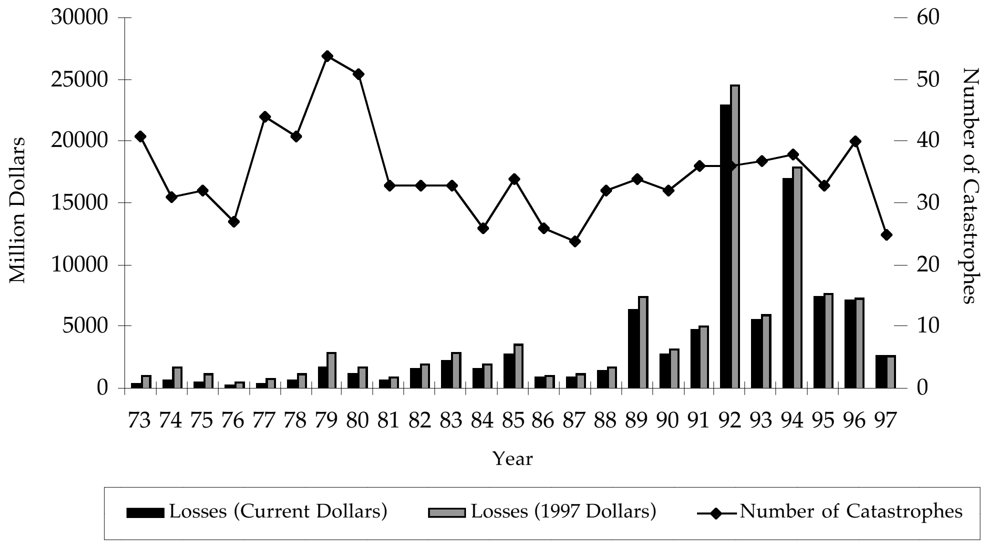
[27] An illustration of this effect may be found in Litzenberger et al. (1996). These authors adjust historical loss ratios for both the increase in population and the market penetration of catastrophe coverage.

[28] Anyway, the results of the simulation process only show slight modifications if adjusted data and long periods are considered.

FIGURE 3
Number of Catastrophes and Total Amount of Loss (1973-1997)



We now use this result to infer some conclusions on the individual prices of these options, based on our estimated future probabilities. First, mean midpoints of the bid and ask quotes being around mean pure premiums seem to indicate that, in average terms, transactions in this market could have been made at reasonable prices, close to the actuarial "fair" price during the sample periods. This is good news for those participating in the market who hedge their catastrophe insurance risks: this market seems to be an attractive alternative to traditional reinsurance, as it offers reinsurance at "fair" prices. However, those willing to participate in the (re)insurance market by selling options seeking attractive returns over the risk-free rate might find it difficult to get them, at least when trading with individual options.

Second, for our sample periods, conclusions inferred from spread midpoints may substantially change when the real bid-ask spread is taken into account. On the one hand, in general, those hedgers able to buy at bid prices will find this market more attractive than the usual reinsurance (for most options mean bid prices are lower than the mean actuarial ones), but things might turn the other way around for hedgers that buy at the ask. On the other hand, for the moment keeping aside risk considerations, investors seeking high returns should try to buy close to the bid or sell close to the ask.

Third, investors searching for new investment/financing opportunities also attend to risk considerations in their decisions. Analyzing risk and return on an option by option basis hardly makes sense as it is in a portfolio choice context where risk/return opportunities are fully understood (it is in this context where risk-pooling benefits, for instance, come into place). These are the topics considered in the next subsection.

FIGURE 4
Probability Distributions of the Index Final Value (First Period)

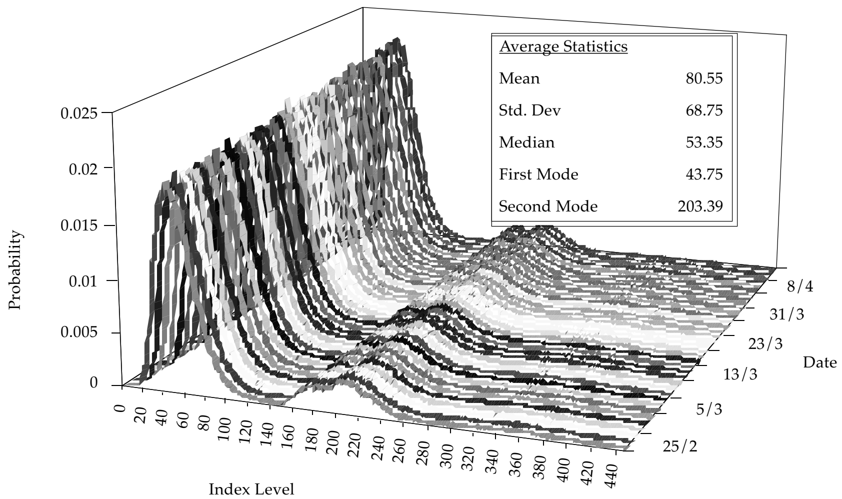


FIGURE 5
Probability Distributions of the Index Final Value (Second Period)

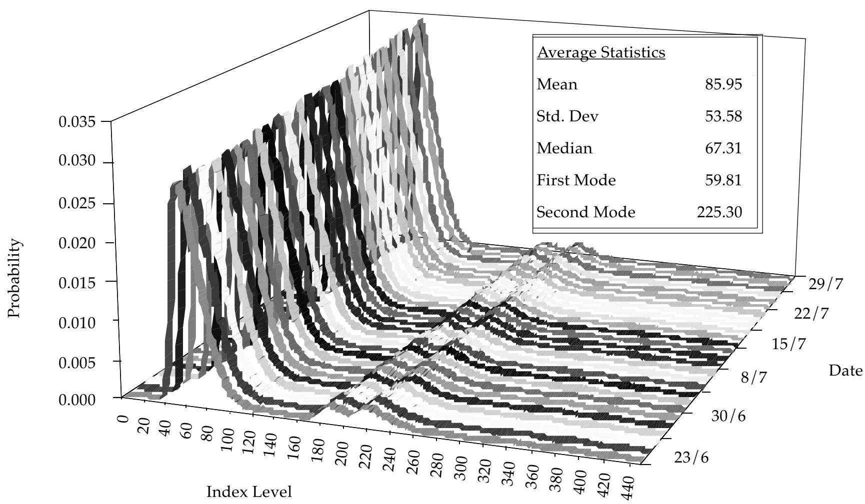


Table 7
Theoretical Prices for the National Annual-98 Index Derivatives

This table shows average theoretical prices corresponding to the National Annual-98 derivatives available for trading on any day during the sample periods. CA 40 60 stands for a call spread with exercise prices 40 and 60, and similarly for the other possible exercise prices. CB denotes a butterfly call spread with relevant exercise prices as indicated. The first two columns give the mean pure (actuarial) premiums and the mean risk-neutral (financial) price. The third column displays the absolute value of the difference between the previous two columns, and the last two columns show the mean real bid and ask quotes for comparison purposes. All figures are given in points, each point valued at $200. We also report the mean Euclidean distance between the measures μ and λ (standard deviation in parentheses) in the last row of each panel.

	Mean Theoretical Prices			Mean Real Quotes	
Derivatives	Pure Premium	Risk-Neutral Price	Discrepancy	Bid (Days)	Ask (Days)
		Panel A: First Period			
CA 40 60	10.42	10.46	0.038	8.61 (36)	9.47 (17)
CA 60 80	5.48	6.10	0.617	6.09 (36)	7.31 (34)
CA 80 100	3.70	4.35	0.644	4.12 (36)	5.96 (36)
CB 40 60 80 100	6.72	6.11	0.606	3.88 (36)	6.00 (1)
CA 100 120	3.32	3.36	0.440	1.60 (36)	5.40 (36)
CA 120 140	3.26	3.48	0.221	1.50 (1)	5.50 (36)
CA 150 200	6.79	6.08	0.712	n.a. (0)	8.00 (28)
CA 180 200	2.18	1.80	0.381	0.73 (36)	1.80 (36)
CA 250 300	0.82	0.74	0.071	0.74 (36)	2.50 (36)
Mean Square Distance (Std. Dev.):		0.000317	(0.000515)		
		Panel B: Second Period			
CA 30 50	18.17	18.19	0.024	10.00 (11)	n.a. (0)
CA 40 60	17.03	17.11	0.082	9.48 (27)	n.a. (0)
CA 60 80	8.66	8.97	0.318	4.89 (27)	9.63 (27)
CA 80 100	3.44	3.97	0.535	3.87 (3)	7.42 (26)
CB 40 60 80 100	13.92	13.40	0.525	5.00 (15)	11.25 (16)
CA 100 120	2.26	2.94	0.685	2.45 (22)	5.67 (27)
CA 100 150	5.32	7.20	1.878	n.a. (0)	12.00 (22)
CA 100 200	10.18	13.80	3.629	10.00 (7)	20.00 (21)
CA 120 140	2.11	2.77	0.658	1.00 (18)	5.59 (27)
CA 150 200	4.89	6.46	1.572	4.00 (21)	7.50 (22)
CA 180 200	1.87	1.80	0.065	0.40 (9)	1.80 (9)
CA 200 250	2.24	3.12	0.881	n.a. (0)	4.30 (15)
CA 250 300	0.37	1.46	1.088	0.50 (27)	2.95 (20)
Mean Square Distance (Std. Dev.):		0.000052	(0.000050)		

Let us now analyze the risk-neutral probabilities. We solve the minimization program that gives their possible values and we use them to obtain the corresponding theoretical risk-neutral prices. These are also given in Table 7.[29] We also report the average value of the objective function for both periods and the resulting discrepancies between pure (actuarial) premiums and prices calculated with the risk-neutral probabilities. As these figures show, it might well be concluded that real prices, as summarized by the linear pricing rule extracted from them, are reasonably close to their "fair" value.

Lane and Movchan (1999) also consider real market mid-year 1998 prices and compute a compatible risk neutral probability measure.[30] They follow a different and interesting approach because they do not use any real distribution μ. Instead, they impose the constraints (2.3) and minimize the differences between the prices of real transactions and the theoretical prices provided by the second term of (2.3). This methodology could show some advantages with respect to the one we followed because we could commit errors when evaluating μ (although the authors assume a specific type of distribution for the relevant random variables). However, bearing in mind our purposes, we preferred to obtain λ from μ for several reasons. First, one of this article's main objectives is to give well diversified portfolios and (as pointed out by Theorem 4), consequently, the measure μ is required. Second, the theoretical prices obtained by Lane and Movchan are very close to the risk-neutral prices provided in Table 7. Third, as will be pointed out in the next subsection, the main conclusions concerning well diversified portfolios seem to be more robust if we take λ as close to μ as possible.

4.4. Looking for Well Diversified Portfolios

Consider an arbitrary date t_0, the probability measures λ and μ obtained for t_0 in the previous subsection, and the linear pricing rule π such that $\pi_j = (1/R)E^{\lambda}(S_j)$, $j = 1, 2, \ldots, n$. Then, the (unique) discount factor d of Theorem 3 may be easily found by means of the following conditions:

$$\begin{cases} d^t \in span(A) \\ \sum_{i=1}^{k} a_{ij} d_i \mu_i = \pi_j, \; j = 1, 2, \ldots, n \end{cases}$$

The condition $d^t \in span(A)$ leads to a simple linear system of equations. In fact, consider the matrix $\mathbf{B} = (A, d^t)$ (i.e., add the column d^t to $\mathbf{A}$) and impose that the ranges of $\mathbf{A}$ and $\mathbf{B}$ must be identical.

According to Theorem 4, the minimum-variance strategies are obtained by selling the portfolio $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $\mathbf{A}\mathbf{x}^{\mathrm{T}} = d^t$. The portfolio $\mathbf{x}^* = (0, x_2, \ldots, x_n)$[31] will also provide a very useful information. Depending on the sign of its theoretical

[29] Risk-neutral prices certainly verify the bid and ask quote restrictions day by day, even though the reported mean risk-neutral prices do not have to lie inside the mean bid-ask spread; the bid/ask quote may not be present whenever these prices are computed (we use the term "risk-neutral prices" as has become usual in finance).

[30] We thank an anonymous referee for pointing out this reference.

[31] That is, the portfolio $\mathbf{x}$ once the bond has been excluded.

price $\sum_{j=2}^{n} \pi_j x_j$, we know when variance-averse individuals must sell or buy derivatives in their reinsurance-linked portfolios. As will be shown later, it turned out that quite often investors in the PCS options market must sell derivatives (the sign is positive).[32]

For the empirical analysis we first excluded those redundant derivatives for each day. See Table 8 for a summary of the main results. With regard to the x^* portfolio, this was mainly composed of bonds (87.98 percent and 68.36 percent in average terms for the first and the second periods, respectively), and its mean (gross) return equals .9460 and .9096. The portfolio x^* has a positive value for 77.78 percent of the days in the first period and 81.48 percent in the second. Thus, risk-averse investors should view the PCS options market as a very profitable source of capital that allows them to finance their investments in other markets.

Let us remark that the last conclusion concerning the sign of the price of x^* seems to be robust with respect to the measure λ used in the analysis. In fact, when λ is obtained by minimizing the distance to μ, we are minimizing the *Risk Premium*, the difference between the risk-neutral price and the actuarial pure premium. Thus, if we take a new measure λ' for which (2.3) holds but such that $\lambda' \neq \lambda$, the risk premium will probably increase and short positions in PCS options will probably be more interesting for traders.

In order to further illustrate the relationship between probabilities μ and λ, and the portfolio x^* payoffs (appropriately standardized), these variables are plotted in Figures 6 and 7 for a representative day of both periods. It is clear that the portfolio final payoff becomes significantly negative only for states of the world (index final values) with slight probability.

5. Concluding Remarks

Throughout, this article has shown that arbitrage arguments, in a static setting, very often allows us to price catastrophe-linked derivatives and reduce their bid-ask spread. Though the empirical literature concerning the existence of arbitrage in real financial markets usually focuses on concrete well-known arbitrage portfolios (put-call parity, relationship between spot and future prices, etc.), this procedure does not apply to catastrophe-linked derivatives. We have followed more complex methodologies that are based on the main principles of asset valuation and provide arbitrage portfolios without previously specifying the exact nature of the arbitrage strategy to be used. This seems to be a significant difference with respect to other financial papers. Furthermore, the procedure to detect arbitrage portfolios has been given, and some illustrative examples have been presented.

Hedging arguments have also been applied and, again, they have shown many possibilities to price these derivatives. The hedging portfolios have been computed by using general procedures, too, rather than usual particular methods that only apply in special situations. Concrete examples of hedging portfolios have been given.

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[32] Notice that we are considering investors willing to participate in the catastrophe insurance market through PCS options in order to benefit from their new risk/return opportunities when included in diversified stock and bond portfolios. Of course, insurers, who must hedge their proper liabilities when the final index value becomes large, follow different portfolio criteria.

TABLE 8
Well Diversified Portfolios

The first two rows show the risk/return characteristics of the entire portfolio by providing the expected return and its standard deviation, respectively (returns defined as payoffs divided by the risk-neutral price). The third row gives the weight of $\tilde{\mathbf{x}}^*$ (the derivatives portfolio) over the entire diversified portfolio, while the fourth row shows the price of $\tilde{\mathbf{x}}^*$ in index points. Finally, the number of days in which $\tilde{\mathbf{x}}^*$ has positive price is indicated in the last row.

Portfolio Characteristics	Descriptive Statistics Mean	Std. Dev.	Min.	Median	Max.
Panel A: First Period					
Expected Return	0.9460	0.0809	0.8262	0.9330	1.0589
Std.Dev. of the Return	5.1902	10.8703	1.4038	1.7278	64.1093
Cat derivatives weight	0.1202	0.1875	–0.2386	0.1319	0.3374
Price of $\tilde{x}^*$ Portfolio	0.1249	0.1930	–0.2345	0.1185	0.3664
Days with positive price 28 (77.78%)					
Panel B: Second Period					
Expected Return	0.9096	0.1753	0.6622	0.9911	1.0826
Std.Dev. of the Return	1.4279	1.4056	0.3566	0.9453	5.5590
Cat derivatives weight	0.3164	0.2907	–0.0980	0.5053	0.8079
Price of $\tilde{x}^*$ Portfolio	0.3808	0.3519	–0.0845	0.5727	0.8648
Days with positive price 22 (81.48%)					

TABLE 9
An Example of Two Well Diversified Portfolios

This table shows the asset weights of the derivatives portfolio $\tilde{x}^*$ for two selected days. Weights were calculated as value invested in the asset divided by the portfolio value (a negative sign indicates a sale). The corresponding risk-neutral prices are also reported. The weights for these derivative portfolios over the entire (bond included) portfolios are 0.29 and 0.63 for panels A and B, respectively.

Panel A: Date February 2, 1998 (First Period)			Panel B: Date July 30, 1998 (Second Period)		
Derivative	Weight	Risk-Neutral Price	Derivative	Weight	Risk-Neutral Price
CA 40 60	0.06	11.66	CA 40 60	0.36	17.88
CA 60 80	0.05	7.00	CA 60 80	–0.02	9.84
CA 80 100	0.32	5.00	CA 80 100	–0.08	4.43
CA 100 120	3.92	4.25	CA 100 120	1.74	3.50
CA 120 140	–1.52	3.83	CA 100 150	–4.16	8.43
CA 150 200	–2.58	6.23	CA 120 140	4.24	3.32
CA 180 200	0.72	1.80	CA 150 200	–1.59	7.5
CA 250 300	0.00	0.80	CA 250 300	0.52	2.70
Portfolio	1	0.30	Portfolio	1	0.86

FIGURE 6
Standardized Payoffs of Portfolio x^* on February 25, 1998, Along With the Corresponding Risk-Neutral and Real Probabilities

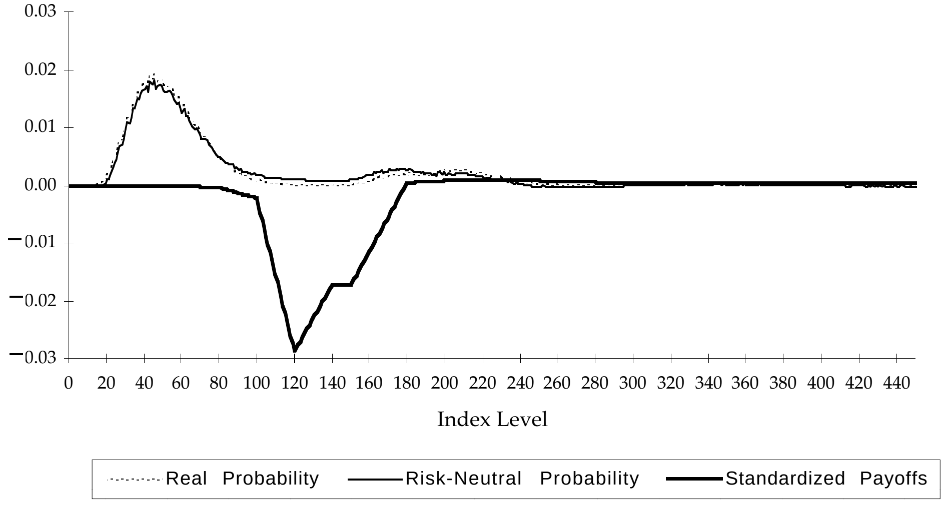
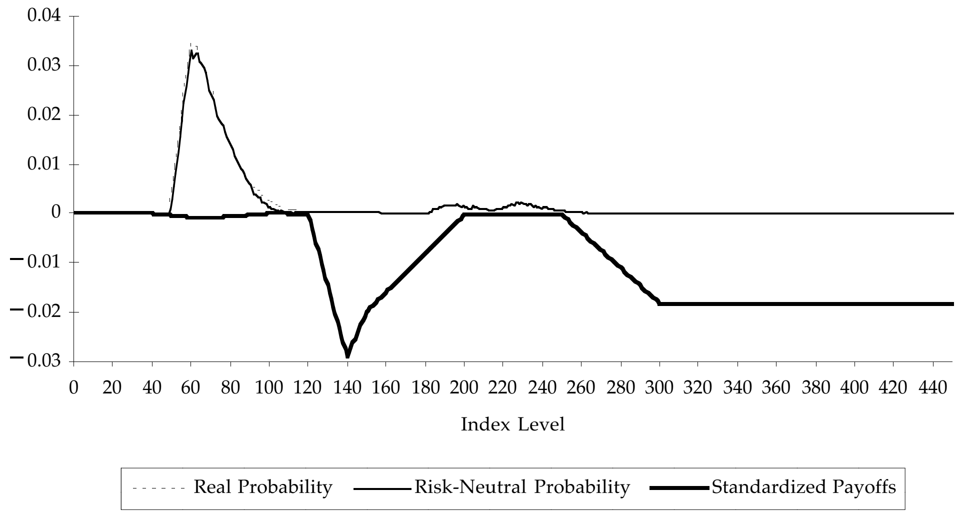


FIGURE 7
Standardized Payoffs of Portfolio x^* on date July 30, 1998, Along With the Corresponding Risk-Neutral and Real Probabilities



The Theory of Portfolio Selection also yields suitable strategies to invest. Moreover, if the bid-ask spread is reduced by arbitrage, the real quotes available in the market show very significant particularities. Linear pricing rules compatible with the quotes usually imply theoretical prices quite close to the actuarial pure premiums. However, even though a well diversified portfolio (in a variance-aversion context) is composed of different catastrophe-linked derivatives in short and long positions, its price is usually negative (i.e., the total price of the sold derivatives is greater than the total price of those purchased), and this capital must be invested in shares and bonds.

Catastrophe-linked derivatives can usually be added to a portfolio of other assets to the mean-variance advantage of the portfolio holder. This result emanates from the fact that catastrophe derivative outcomes are uncorrelated with bonds or stock outcomes. But the conclusion presented in the previous paragraph provides important additional information. In a well diversified portfolio, the price of the catastrophe derivatives is usually negative.

The last comment leads to significant implications. Insurers can consider the market in order to buy reinsurances and hedge their liabilities. The price paid may be adequate with respect to the pure premiums. On the contrary, variance-averse investors whose risk does not depend on the indices can use catastrophe-linked derivatives to compose portfolios with negative price that must be invested in other type of assets. This makes the market very attractive because it allows us to appropriately diversify among many investors the risk due to catastrophic events. However, we must notice that these properties hold for linear pricing rules and, therefore, it is important to reduce the real bid-ask spreads observed in the market. As mentioned above, this is possible by arbitrage and hedging arguments.

Well diversified portfolios have also been determined by applying a very general method. In fact, the optimal mean-variance strategies are given by densities between risk-neutral measures and initial probability measures, rather than by correlations associated with the returns of the available catastrophe-linked derivatives.

The applied methodology seems to reveal two interesting advantages. It is useful to traders because practical criteria and strategies to invest are provided. Moreover, it seems to be general enough to apply in many other contexts. Only two properties are needed. The underlying uncertainty must be easily identified (arbitrage and hedging), and the probability space that explains its behavior, along with the risk-neutral probability, needs to be determined (variance aversion).

REFERENCES

Balbás, A., and M. J. Muñoz, 1998, Measuring the Degree of Fulfillment of the Law of One Price. Applications to Financial Market Integration. *Investigaciones Económicas*, 22(2): 153-177.

Balbás, A., P. J. Guerra, and M. J. Muñoz, 1998, Measuring the Arbitrage Opportunities in an Intertemporal Dynamic Asset Pricing Model, in: F. J. Girón, ed., *Applied Decision Analysis* (Boston, MA: Kluwer), 159-172.

Canter, M. S., J. B. Cole, and R. L. Sandor, 1996, Insurance Derivatives: A New Asset Class for the Capital Markets and a New Hedging Tool for the Insurance Industry, *Journal of Derivatives*, 4(2): 89-105.

Chamberlain, G., and M. Rothschild, 1983, Arbitrage, Factor Structure and Mean-Variance Analysis of Large Assets, *Econometrica*, 51(5): 1281-1304.

Cummins, J. D., and H. Geman, 1995, Pricing Catastrophe Insurance Futures and Call Spreads: An Arbitrage Approach, *Journal of Fixed Income*, 4(4): 46-57.

D'Arcy, S. P., and V. G. France, 1992, Catastrophe Futures: A Better Hedge for Insurers, *Journal of Risk and Insurance*, 59(4): 575-601.

Duffie, D., 1988, *Security Markets: Stochastic Models* (Boston, MA: Academic Press).

Garman, M. B., 1976, An Algebra for Evaluating Hedge Portfolios, *Journal of Financial Economics*, 3(4): 403-427.

Geman, H., and M. Yor, 1997, Stochastic Time Changes in Catastrophe Option Pricing, *Insurance: Mathematics and Economics*, 21(3): 185-194.

Hansen, L. P., and R. Jagannathan, 1997, Assessing Specification Errors in Stochastic Discount Factor Models, *Journal of Finance*, 52(2): 567-590.

Hansen, L. P., and S. F. Richard, 1987, The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models, *Econometrica*, 55(3): 587-613.

Ingersoll, J. E., 1987, *Theory of Financial Decision Making* (Totowa, NJ: Rowman & Littlefield Publishers, Inc.).

Jackwerth, J. C., and M. Rubinstein, 1996, Recovering Probability Distributions from Option Prices, *Journal of Finance*, 50(6): 1611-1631.

Jaffee, D. M., and T. Russell, 1997, Catastrophe Insurance, Capital Markets and Insurable Risk, *Journal of Risk and Insurance*, 64(2): 205-230.

Jouini, E., and H. Kallal, 1995, Martingales and Arbitrage in Securities Markets with Transaction Costs, *Journal of Economic Theory*, 66(1): 178-197.

Lane, M., and O. Movchan, 1999, The Perfume of the Premium II, *Derivatives Quarterly*, 5(3): 27-40.

Litzenberger, R. H., R. D. Beaglehole, and C. E. Reynolds, 1996, Assessing Catastrophe Reinsurance-Linked Securities as a New Asset Class, *Journal of Portfolio Management, Special Issue*: 76-86.

Prisman, E. J., 1986, Valuation of Risky Assets in Arbitrage Free Economies with Frictions, *Journal of Finance*, 41(3): 545-556.

O'Brien, T., 1997, Hedging Strategies Using Catastrophe Insurance Options, *Insurance: Mathematics and Economics*, 21(2): 153-162.

Rubinstein, M., 1994, Implied Binomial Trees, *Journal of Finance*, 69(3): 771-818.

Tomas, M. T., 1998, A Note on Pricing PCS Single-Event Options, *Derivatives Quarterly*, 4(3): 23-28.