# Generalized flows and singular ODEs on differentiable manifolds

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**Abstract**

Based on the concept of manifold valued generalized functions we initiate a study of nonlinear ordinary differential equations with singular (in particular: distributional) right hand sides in a global setting. After establishing several existence and uniqueness results for solutions of such equations and flows of singular vector fields we compare the solution concept employed here with the purely distributional setting. Finally, we derive criteria securing that a sequence of smooth flows corresponding to a regularization of a given singular vector field converges to a measurable limiting flow.

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## 1 Introduction

The need for considering ordinary differential equations involving generalized functions on differentiable manifolds occurs naturally in a number of applications. Examples include singular Hamiltonian mechanics ([22], [23]), symmetry group analysis of differential equations involving singularities ([3], [6], [15]) and geodesic equations in singular space-times in general relativity ([2, 16, 28, 29]).
An appropriate setting for developing a theory capable of handling this question is provided by Colombeau's theory of nonlinear generalized functions. Introduced in [4, 5] primarily as a tool for the treatment of nonlinear partial differential equations in the presence of singularities (cf.

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[26]), the theory has undergone a quite substantial "geometrization" in recent years owing to an increasing number of applications in a predominantly geometric context. For the so-called full version of the construction, distinguished by the existence of a canonical embedding of the space of distributions, this restructuring was carried out in [8, 10, 12]. For the special version of the theory, which will form the basic setting of the present article, a global construction was developed in [7, 17].
Finally, an extension of Colombeau's special construction to a "nonlinear distributional geometry" capable of modeling generalized functions taking values in differentiable manifolds was given in [14, 18, 19]. A comprehensive presentation of these developments can be found in [9]. The aim of the present paper is to extend this setting to a theory of singular ordinary differential equations on differentiable manifolds.

For the convenience of the reader we review the geometric theory of generalized functions in the following section where we also introduce our notational conventions. The basic existence theory for singular ODEs on manifolds and generalized flows is the subject of Section 3. We compare our setting with the purely distributional framework put forward in [22] in Section 4. We introduce a number of notions of association relations for manifold valued generalized functions in Section 5 which in turn are used in the final Section 6 to give necessary criteria for the limit of a generalized flow to obey the flow property.

## 2 Linear and nonlinear distributional geometry

In this section we collect some basic definitions from linear and nonlinear distributional geometry needed in the sequel. Our notational conventions will be based on [9] throughout.

In what follows, $C$ will always denote a generic constant. $X$ will be a smooth paracompact Hausdorff manifold of dimension $n$. $K \subset\subset A$ (with $A \subseteq X$) means that $K$ is a compact subset of $A^\circ$. For any vector bundle $\pi_X : E \to X$ over $X$ we denote by $\Gamma(X, E)$ resp. $\Gamma_c(X, E)$ the space of smooth (resp. smooth compactly supported) sections of $E$. The space of differential operators $\Gamma(X, E) \to \Gamma(X, E)$ is denoted by $\mathcal{P}(X, E)$ resp. $\mathcal{P}(X)$ in case $E = X \times \mathbb{R}$ (cf. [13]). The space of smooth sections of the tangent bundle $TX$, i.e., the space of smooth vector fields on $X$ is denoted by $\mathfrak{X}(X)$. The volume bundle over $X$ will be written as $\mathrm{Vol}\,(X)$, its smooth sections are called one-densities. The space $\mathcal{D}'(X, E)$ of $E$-valued distributions on $X$ is defined as the dual of the space of compactly supported sections of the bundle $E^* \otimes \mathrm{Vol}\,(X)$:

$$\mathcal{D}'\,(X, E) := [\Gamma_c(X, E^* \otimes \mathrm{Vol}\,(X))]'\,.$$

For $E = X \times \mathbb{R}$ we obtain $\mathcal{D}'(X) := \mathcal{D}'(X, E)$, the space of distributions on $X$. We have the following isomorphism of $\mathcal{C}^\infty(X)$-modules:

$$\mathcal{D}'(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X, E) \;\cong \mathcal{D}'(X, E)\,,$$

i.e., distributional sections may be viewed as sections with distributional coefficients.

Setting $I = (0, 1]$ and $\mathcal{E}(X) = \mathcal{C}^\infty(X)^I$, we define the spaces of moderate and negligible nets in $\mathcal{E}(X)$ by

$$\begin{aligned}
\mathcal{E}_M(X) \;&:=\; \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(X) :\ \forall K \subset\subset X,\ \forall P \in \mathcal{P}(X)\ \exists N \in \mathbb{N} : \\
&\qquad\qquad \sup_{p\in K} |Pu_\varepsilon(p)| = O(\varepsilon^{-N})\} \\
\mathcal{N}(X) \;&:=\; \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X) :\ \forall K \subset\subset X,\ \forall q \in \mathbb{N}_0 :\ \sup_{p\in K} |u_\varepsilon(p)| = O(\varepsilon^q)\}\,.
\end{aligned}$$

(Note that in the definition of $\mathcal{N}(X)$ no conditions on the derivatives of $(u_\varepsilon)_\varepsilon$ are necessary, cf. [17].) $\mathcal{G}(X) := \mathcal{E}_M(X)/\mathcal{N}(X)$ is called the (special) Colombeau algebra on $X$, its elements are written as $u = [(u_\varepsilon)_\varepsilon)]$. $\mathcal{G}(\_)$ is a fine sheaf of differential algebras with respect to the Lie derivative along smooth vector fields. $\mathcal{C}^\infty(X)$ is a subalgebra of $\mathcal{G}(X)$ and there exist injective sheaf morphisms embedding $\mathcal{D}'(\_)$ into $\mathcal{G}(\_)$.

Elements of $\mathcal{G}(X)$ are uniquely determined by their values on generalized points in the following way: $(p_\varepsilon)_\varepsilon \in X^I$ is called compactly supported if $\exists\varepsilon_0$, $K \subset\subset X$ such that $p_\varepsilon \in K$ for $\varepsilon < \varepsilon_0$; the set of compactly supported points is denoted by $X_c$. Two nets $(p_\varepsilon)$, $(q_\varepsilon)_\varepsilon \in X_c$ are called equivalent, $(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon$, if $d_h(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)$ for each $m > 0$, where $d_h$ denotes the distance function induced on $X$ by one (hence every) Riemannian metric $h$. The quotient space $\tilde{X}_c$ of the set of compactly supported points modulo $\sim$ is called the space of compactly supported generalized points on $X$ and we write $\tilde{p} = [(p_\varepsilon)_\varepsilon]$. For $X = \mathbb{R}$ we use the notation $\mathcal{R}_c$ instead of $\tilde{X}_c$. For $\tilde{p} \in \tilde{X}_c$, $u \in \mathcal{G}(X)$, $[(u_\varepsilon(p_\varepsilon))_\varepsilon]$ gives a well-defined element of $\mathcal{K} = \mathcal{R}$ resp. $\mathcal{C}$ (the space of generalized numbers (corresponding to $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{C}$ and defined as the set of moderate nets of numbers $(r_\varepsilon)_\varepsilon \in \mathbb{K}^I$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some $N$ modulo negligible nets $|r_\varepsilon| = O(\varepsilon^m)$ for each $m$). $u \in \mathcal{G}(X)$ is uniquely determined by its point values on $\tilde{X}_c$, i.e., $u = v \Leftrightarrow u(\tilde{p}) = v(\tilde{p})\ \forall \tilde{p} \in \tilde{X}_c$ ([27], [17]).

Colombeau generalized sections of $E \to X$ are defined analogously to $\mathcal{G}(X)$ using asymptotic estimates with respect to the norm on the fibers of $E$ induced by any Riemannian metric $h$ on $X$, which we will denote by $\|\ \|_h$ throughout. Setting $\Gamma_{\mathcal{E}}(X,E) = \Gamma(X,E)^I$ we define

$$\begin{aligned}
\Gamma_{\mathcal{E}_M}(X,E) &:= \{(s_\varepsilon)_\varepsilon \in \Gamma_{\mathcal{E}}(X,E) :\ \forall P \in \mathcal{P}(X,E)\, \forall K \subset\subset X\, \exists N \in \mathbb{N} : \\
&\qquad \sup_{p \in K} \|P u_\varepsilon(p)\|_h = O(\varepsilon^{-N})\} \\
\Gamma_{\mathcal{N}}(X,E) &:= \{(s_\varepsilon)_\varepsilon \in \Gamma_{\mathcal{E}_M}(X,E) :\ \forall K \subset\subset X\, \forall m \in \mathbb{N} : \\
&\qquad \sup_{p \in K} \|u_\varepsilon(p)\|_h = O(\varepsilon^{m})\}\,.
\end{aligned}$$

Then $\Gamma_{\mathcal{G}}(X,E) := \Gamma_{\mathcal{E}_M}(X,E)/\Gamma_{\mathcal{N}}(X,E)$. $\Gamma_{\mathcal{G}}(\_,E)$ is a fine sheaf of projective and finitely generated $\mathcal{G}(X)$-modules, moreover

$$\Gamma_{\mathcal{G}}(X,E) = \mathcal{G}(X) \otimes_{\mathcal{C}^\infty(X)} \Gamma(X,E)\,.$$

In case $E$ is some tensor bundle $T^r_s X$ we use the notation $\mathcal{G}^r_s(X)$ for $\Gamma_{\mathcal{G}}(X, T^r_s X)$.

Next we turn to the definition of the space of manifold valued generalized functions ([14, 19]). Both solutions of generalized ODEs and flows of generalized vector fields will be modeled as elements of this space. A net $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X,Y)^I$ ($Y$ another manifold) is called compactly bounded (c-bounded) if for each $K \subset\subset X$ $\exists\varepsilon_0$, $K' \subset\subset Y$ such that $u_\varepsilon(K) \subseteq K'$ for all $\varepsilon < \varepsilon_0$. The space $\mathcal{G}[X,Y]$ of c-bounded generalized Colombeau functions from $X$ to $Y$ is defined as the quotient of the set of $\mathcal{E}_M[X,Y]$ of moderate, c-bounded maps from $X$ to $Y$ modulo the equivalence relation $\sim$, where $\mathcal{E}_M[X,Y]$ is the set of all $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(X,Y)^I$ satisfying

(i) $(u_\varepsilon)_\varepsilon$ is c-bounded.

(ii) $\forall k \in \mathbb{N}$, for each chart $(V,\varphi)$ in $X$, each chart $(W,\psi)$ in $Y$, each $L \subset\subset V$ and each $L' \subset\subset W$ there exists $N \in \mathbb{N}$ with

$$\sup_{x \in L \cap u_\varepsilon^{-1}(L')} \|D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^{-N})\,,$$

and $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon \in \mathcal{E}_M[X,Y]$ are called equivalent, $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$, if

(i) $\forall K \subset\subset X$, $\sup_{p \in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0$ ($\varepsilon \to 0$) for some (hence every) Riemannian metric $h$ on $Y$.

(ii) $\forall k \in \mathbb{N}_0$ $\forall m \in \mathbb{N}$, for each chart $(V,\varphi)$ in $X$, each chart $(W,\psi)$ in $Y$, each $L \subset\subset V$ and each $L' \subset\subset W$:

$$\sup_{x \in L \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')} \|D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1} - \psi \circ v_\varepsilon \circ \varphi^{-1})(\varphi(p))\| = O(\varepsilon^m).$$

Moderateness and equivalence of nets in $\mathcal{C}^\infty(X,Y)^I$ can be tested equivalently by composition with smooth functions, i.e., $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M[X,Y]$ iff $(f\circ u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ $\forall f\in\mathcal{C}^\infty(Y)$ ([19], 3.2) and two nets $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ in $\mathcal{E}_M[X,Y]$ are equivalent iff $(f\circ u_\varepsilon - f\circ v_\varepsilon)_\varepsilon \in \mathcal{N}(X)$ $\forall f\in\mathcal{C}^\infty(Y)$ ([19], 3.3).
Inserting a compactly supported point $\tilde p \in \tilde X_c$ into $u\in\mathcal{G}[X,Y]$ yields a well-defined element $[u_\varepsilon(p_\varepsilon)_\varepsilon]\in \tilde Y_c$ and again these generalized point values characterize elements of $\mathcal{G}[X,Y]$. Typically, elements of $\mathcal{G}[X,Y]$ are capable of modeling jump-discontinuities.
In order to be able to form tangent maps of manifold valued generalized functions, the concept of generalized vector bundle homomorphisms in the following sense is needed ([14]). Let $\pi_Y: F\to Y$ be a vector bundle over $Y$ and ${\mathcal{E}_M}^{\mathrm{VB}}[E,F]$ be the set of all $(u_\varepsilon)_\varepsilon \in \mathrm{Hom}(E,F)^I$ satisfying

(i) $(\underline{u_\varepsilon})_\varepsilon \in \mathcal{E}_M[X,Y]$.

(ii) $\forall k\in\mathbb{N}_0$ $\forall(V,\Phi)$ vector bundle chart in $E$, $\forall(W,\Psi)$ vector bundle chart in $F$, $\forall L\subset\subset V$ $\forall L'\subset\subset W$ $\exists N\in\mathbb{N}$ $\exists\varepsilon_1>0$ $\exists C>0$ with
$$\|D^{(k)}(u^{(2)}_{\varepsilon\Psi\Phi}(\varphi(p)))\| \le C\varepsilon^{-N}$$
for all $\varepsilon<\varepsilon_1$ and all $p\in L\cap \underline{u_\varepsilon}^{-1}(L')$, where $\|\,.\,\|$ denotes any matrix norm.

Here, $\underline{u}_\varepsilon$ is the unique element of $\mathcal{C}^\infty(X,Y)$ such that $\pi_Y\circ u_\varepsilon = \underline{u}_\varepsilon\circ\pi_X$ and $u_{\varepsilon\Psi\Phi} := \Psi\circ u_\varepsilon\circ\Phi^{-1} = (x,\xi)\mapsto (u^{(1)}_{\varepsilon\Psi\Phi}(x), u^{(2)}_{\varepsilon\Psi\Phi}(x)\cdot\xi)$.
$(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon \in {\mathcal{E}_M}^{\mathrm{VB}}[E,F]$ are called $vb$-equivalent, $(u_\varepsilon)_\varepsilon \sim_{vb} (v_\varepsilon)_\varepsilon$, if

(i) $(\underline{u_\varepsilon})_\varepsilon \sim (\underline{v_\varepsilon})_\varepsilon$ in $\mathcal{E}_M[X,Y]$.

(ii) $\forall k\in\mathbb{N}_0$ $\forall m\in\mathbb{N}$ $\forall(V,\Phi)$ vector bundle chart in $E$, $\forall(W,\Psi)$ vector bundle chart in $F$, $\forall L\subset\subset V$ $\forall L'\subset\subset W$ $\exists\varepsilon_1>0$ $\exists C>0$ such that:
$$\|D^{(k)}(u^{(2)}_{\varepsilon\Psi\Phi} - v^{(2)}_{\varepsilon\Psi\Phi})(\varphi(p))\| \le C\varepsilon^{m}$$
for all $\varepsilon<\varepsilon_1$ and all $p\in L\cap\underline{u_\varepsilon}^{-1}(L')\cap\underline{v_\varepsilon}^{-1}(L')$.

Then $\mathrm{Hom}_{\mathcal{G}}[E,F] := {\mathcal{E}_M}^{\mathrm{VB}}[E,F]/\sim_{vb}$. For $u\in\mathrm{Hom}_{\mathcal{G}}[E,F]$, $\underline{u} := [(\underline{u_\varepsilon})_\varepsilon]$ is a well-defined element of $\mathcal{G}[X,Y]$ uniquely characterized by $\underline{u}\circ\pi_X = \pi_Y\circ u$. The tangent map $Tu := [(Tu_\varepsilon)_\varepsilon]$ of any $u\in\mathcal{G}[X,Y]$ is a well-defined element of $\mathrm{Hom}_{\mathcal{G}}[TX,TY]$.
Finally, we need the space $\mathcal{G}^h[X,F]$ of hybrid generalized functions defined on $X$ and taking values in the vector bundle $F$ ([18]); this space will be used to define the notion of a vector field on a curve. It is defined as follows: Let $\mathcal{E}^h_M[X,F]$ the set of all nets $(u_\varepsilon)_\varepsilon\in\mathcal{C}^\infty(X,F)^{(0,1]}$ satisfying (with $\underline{u_\varepsilon} := \pi_Y\circ u_\varepsilon$)

(i) $(\underline{u_\varepsilon})_\varepsilon$ is c-bounded.

(ii) $\forall k\in\mathbb{N}_0$ $\forall(V,\varphi)$ chart in $X$ $\forall(W,\Psi)$ vector bundle chart in $F$ $\forall L\subset\subset V$ $\forall L'\subset\subset W$ $\exists N\in\mathbb{N}$ $\exists\varepsilon_1>0$ $\exists C>0$ such that
$$\|D^{(k)}(\Psi\circ u_\varepsilon\circ\varphi^{-1})(\varphi(p))\| \le C\varepsilon^{-N}$$
for each $\varepsilon<\varepsilon_1$ and each $p\in L\cap\underline{u_\varepsilon}^{-1}(L')$.

In particular, $(u_\varepsilon)_\varepsilon\in\mathcal{E}^h_M[X,F]$ implies $(\underline{u_\varepsilon})_\varepsilon\in\mathcal{E}_M[X,Y]$. $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon\in\mathcal{E}^h_M[X,F]$ are called equivalent, $(u_\varepsilon)_\varepsilon\sim_h(v_\varepsilon)_\varepsilon$, if the following conditions are satisfied:

(i) For each $K\subset\subset X$, $\sup_{p\in K} d_h(\underline{u_\varepsilon}(p),\underline{v_\varepsilon}(p))\to 0$ for some (hence every) Riemannian metric $h$ on $Y$.

(ii) $\forall k\in\mathbb{N}_0$ $\forall m\in\mathbb{N}$ $\forall(V,\varphi)$ chart in $X$, $\forall(W,\Psi)$ vector bundle chart in $F$, $\forall L\subset\subset V$ $\forall L'\subset\subset W$ $\exists\varepsilon_1>0$ $\exists C>0$ such that
$$\|D^{(k)}(\Psi\circ u_\varepsilon\circ\varphi^{-1} - \Psi\circ v_\varepsilon\circ\varphi^{-1})(\varphi(p))\| \le C\varepsilon^{m}$$
for each $\varepsilon<\varepsilon_1$ and each $p\in L\cap\underline{u_\varepsilon}^{-1}(L')\cap\underline{v_\varepsilon}^{-1}(L')$.

If $u\in\mathcal{G}[X,Y]$, $v\in\Gamma_{\mathcal{G}}[Y,F]$ then $v\circ u := [(v_\varepsilon\circ u_\varepsilon)_\varepsilon]$ is a well-defined element of $\mathcal{G}^h[X,F]$. We will make use of this fact in analyzing the flow property of generalized flows (cf. Theorem 3.6 below).

# 3 Basic existence and uniqueness theorems

To begin with we consider the system of autonomous nonlinear ODEs on $\mathbb{R}^n$

$$\dot{x}(t) = F(x(t)) \tag{1}$$

subject to the initial conditions

$$x(t_0) = x_0\,. \tag{2}$$

In contrast to previous treatments in the literature (cf. [9], sec. 1.5, [11, 20, 21, 25]) we seek solutions to (1) in the space $\mathcal{G}[\mathbb{R},\mathbb{R}^n]$ of c-bounded generalized functions (cf. [14], [18], [19]) rather than in $\mathcal{G}(\mathbb{R})^n$. We will therefore suppose $F$ to be c-bounded rather than a tempered Colombeau generalized function to give sense to the composition of generalized functions on the right hand side of equation (1). It is precisely this shift in the overall setting which will allow for the treatment of the flow of a generalized vector field on a differentiable manifold as a generalized function valued in a smooth manifold.
To begin with we present basic existence and uniqueness results for the above initial value problem.

**3.1 Theorem.** *Let $\tilde{x}_0 \in \mathcal{R}_c^n$ and let $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)^n$ satisfy*

*(i) $\exists C, \varepsilon_0 > 0$ such that $|F_\varepsilon(x)| \le C(1+|x|)$ $(x \in \mathbb{R}^n,\ \varepsilon < \varepsilon_0)$, and*

*(ii) $|\nabla F|$ is locally of $L^\infty$-log-type (cf. [11], Def. 2.3(c)), i.e.,*

$$\forall K \subset\subset \mathbb{R}^n \ \sup_{x\in K} |\nabla F_\varepsilon(x)| = O(|\log\varepsilon|)\,.$$

*Then the initial value problem (1), (2) has a unique solution in $\mathcal{G}[\mathbb{R},\mathbb{R}^n]$. Moreover, $\dot{x}$ is c-bounded.*

**Proof.** We start by establishing *existence*. By (i) classical ODE theory provides us with globally defined solutions on the level of representatives, i.e., for any (fixed) $\varepsilon$ there exists $x_\varepsilon \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n)$ such that

$$\begin{aligned} \dot{x}_\varepsilon(t) &= F_\varepsilon(x_\varepsilon(t)) \\ x_\varepsilon(t_0) &= x_{0\varepsilon}, \end{aligned} \tag{3}$$

where $[(x_{0\varepsilon})_\varepsilon] = \tilde{x}_0 \in \mathcal{R}_c^n$. From (i) we obtain using Gronwall's lemma $|x_\varepsilon(t)| \le Ce^{Ct}$; hence $x_\varepsilon$ as well as $\dot{x}_\varepsilon$ is c-bounded. To show moderateness of $x_\varepsilon$ we write

$$|\ddot{x}_\varepsilon(t)| = |\nabla F_\varepsilon(x_\varepsilon(t))|\,|\dot{x}_\varepsilon(t)| \le C\varepsilon^{-N} \tag{4}$$

by the c-boundedness of $x_\varepsilon$ and its derivative and the moderateness of $F$. The higher order derivatives of $x_\varepsilon$ are now estimated inductively by differentiating equation (4).
To prove *uniqueness* suppose $[(y_\varepsilon)_\varepsilon]$ is another (c-bounded) solution subject to the same initial conditions. Then since $[(y_{0\varepsilon})_\varepsilon] = [(x_{0\varepsilon})_\varepsilon]$ there exist $\tilde{n}_\varepsilon, n_\varepsilon \in \mathcal{N}(\mathbb{R})^n$ such that

$$\begin{aligned} (x_\varepsilon - y_\varepsilon)(t) &= x_{0\varepsilon} - y_{0\varepsilon} + \int_0^t \left(F_\varepsilon(x_\varepsilon(s)) - F_\varepsilon(y_\varepsilon(s)) + n_\varepsilon(s)\right) ds \\ &= \int_0^t \tilde{n}_\varepsilon(s)\,ds + \int_0^t\int_0^1 \nabla F_\varepsilon((1-\sigma)y_\varepsilon(s) + \sigma x_\varepsilon(s))\,d\sigma\,(x_\varepsilon - y_\varepsilon)(s)\,ds\,. \end{aligned}$$

Hence on $|t - t_0| \le T$ by assumption (ii) for any $m > 0$ and $\varepsilon$ sufficiently small we obtain

$$|(x_\varepsilon - y_\varepsilon)(t)| \le C\varepsilon^m\, e^{-TC\log\varepsilon} \le C\varepsilon^{m-1}\,.$$

□

For later use (cf. Theorem 3.3) we note the following stronger set of conditions that also gives an existence and uniqueness result.

**3.2 Corollary.** *Let* $\tilde{x}_0 \in \mathcal{R}_c^n$, $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)^n$ *and suppose that there exist* $C, \varepsilon_0 > 0$ *such that*

*(i)* $|F_\varepsilon(0)| \leq C \quad (\varepsilon < \varepsilon_0)$, *and*

*(ii)* $|\nabla F_\varepsilon(x)| \leq C \quad (x \in \mathbb{R}^n,\, \varepsilon < \varepsilon_0)$.

*Then the initial value problem (1), (2) has a unique solution in* $\mathcal{G}[\mathbb{R}, \mathbb{R}^n]$. *Moreover,* $\dot{x}$ *is c-bounded.*

**Proof.** From (i) and (ii) it follows that $F$ in fact satisfies the hypotheses of Theorem 3.1. □

Next we give the basic theorem on the flow of system (1).

**3.3 Theorem.** *Let* $F \in \mathcal{G}(\mathbb{R}^n)^n$ *satisfy the assumptions (i) and (ii) of Theorem 3.1. Then there exists a unique generalized function* $\Phi \in \mathcal{G}[\mathbb{R}^{n+1}, \mathbb{R}^n]$, *the* generalized flow *of system (1) such that*

$$\frac{d}{dt}\Phi(t,x) = F(\Phi(t,x)) \quad \text{in } \mathcal{G}[\mathbb{R}^{1+n}, \mathbb{R}^n] \tag{5}$$

$$\Phi(0,.) = \mathrm{id}_{\mathbb{R}^n} \quad \text{in } \mathcal{G}[\mathbb{R}^n, \mathbb{R}^n] \tag{6}$$

$$\Phi(t+s,.) = \Phi(t, \Phi(s,.)) \quad \text{in } \mathcal{G}[\mathbb{R}^{2+n}, \mathbb{R}^n]\,. \tag{7}$$

*Moreover,* $\frac{d}{dt}\Phi$ *is c-bounded and under the assumptions of Corollary 3.2* $\nabla_x \Phi$ *is c-bounded as well.*

As usual we shall often write $\Phi_t$ instead of $\Phi(t,.)$ and use the notation $\Phi_t = [(\Phi_t^\varepsilon)_\varepsilon]$.

**Proof.** Classical theory provides us with a unique and globally defined flow $\Phi^\varepsilon$ for fixed $\varepsilon$.
To prove *existence*, we conclude from the integral equation corresponding to (5) that $\Phi^\varepsilon$ and $\frac{d}{dt}\Phi^\varepsilon$ are c-bounded as functions in $(t,x)$. The higher order $t$-derivatives are estimated as in the proof of Theorem 3.1. To estimate the $x$-derivatives we write

$$\nabla_x \Phi^\varepsilon(t,x) = x + \int_0^t \nabla F_\varepsilon(\Phi^\varepsilon(s,x)) \nabla_x \Phi^\varepsilon(s,x)\, ds\,. \tag{8}$$

Since $\Phi^\varepsilon$ is c-bounded, (ii) and Gronwall's inequality imply on any $\tilde{K} = [0,T] \times K \subset\subset \mathbb{R}^{1+n}$

$$|\nabla_x \Phi(t,x)| \leq C e^{-CT \log \varepsilon} = O(1/\varepsilon^{CT})\,.$$

For $F$ satisfying the assumptions of Corollary 3.2, an analogous estimate establishes c-boundedness of $\nabla_x \Phi$.
The higher order $x$-derivatives are now estimated by successively differentiating equation (8) and using the estimates already obtained. Similarly, the mixed $x,t$-derivatives may be estimated by differentiating the equations for the $x$-derivatives with respect to $t$.
To prove *uniqueness* assume that $\Psi$ is another solution in $\mathcal{G}[\mathbb{R}^{1+n}, \mathbb{R}^n]$. Then fixing any $\tilde{x}_0 = [(x_0)_\varepsilon] \in \mathcal{R}_c^n$, both $t \mapsto \Phi(t, \tilde{x}_0)$ and $t \mapsto \Psi(t, \tilde{x}_0))$ solve the initial value problem

$$\dot{x}(t) = F(x(t))$$

$$x(0) = \tilde{x}_0\,.$$

By the uniqueness part of Theorem 3.1 we have for all $\tilde{x} \in \mathcal{R}_c^n$ : $\Phi(.,\tilde{x}) = \Psi(.,\tilde{x})$ in $\mathcal{G}[\mathbb{R}, \mathbb{R}^n]$. Hence by [19], Th. 3.5, $\Phi(\tilde{t}, \tilde{x}) = \Psi(\tilde{t}, \tilde{x})$ for all $(\tilde{t}, \tilde{x}) \in \mathcal{R}_c^{1+n}$. Therefore, another appeal to [19], Th. 3.5 establishes $\Phi = \Psi$ in $\mathcal{G}[\mathbb{R}^{1+n}, \mathbb{R}^n]$.
Finally, the *flow properties* (6), (7) hold on the level of representatives by the classical theory. Hence again by the point value characterization [19], Th. 3.5, the claim follows. □

In order to prove analogous theorems on a manifold we introduce the following notions of boundedness in terms of Riemannian metrics on $X$.

**3.4 Definition.** *Let $\xi \in \mathcal{G}^1_0(X)$.*

*(i) We say that $\xi$ is* locally bounded *resp. locally of $L^\infty$*-log-type *if for all $K \subset\subset X$ and one (hence every) Riemannian metric $h$ on $X$ we have for one (hence every) representative $\xi_\varepsilon$*

$$\sup_{p\in K} \| \xi_\varepsilon|_p \|_h \leq C \quad \textit{resp.} \quad \sup_{p\in K} \| \xi_\varepsilon|_p \|_h \leq C|\log\varepsilon|\,,$$

*where $\| \quad \|_h$ denotes the norm induced on $T_pX$ by $h$.*

*(ii) $\xi$ is called* globally bounded *with respect to $h$ if for some (hence every) representative $(\xi_\varepsilon)_\varepsilon$ of $\xi$ there exists $C > 0$ with*

$$\sup_{p\in X} \| \xi_\varepsilon|_p \|_h \leq C\,.$$

Contrary to the local notions in (i) above, global boundedness obviously depends on the Riemannian metric $h$.

**3.5 Theorem.** *Let $(X, h)$ be a complete Riemannian manifold, $\tilde{x}_0 \in \tilde{X}_c$ and $\xi \in \mathcal{G}^1_0(X)$ such that*

*(i) $\xi$ is globally bounded with respect to $h$.*

*(ii) For each differential operator $P \in \mathcal{P}(X, TX)$ of first order $P\xi$ is locally of $L^\infty$-log-type.*

*Then the initial value problem*

$$\begin{aligned} \dot{x}(t) &= \xi(x(t)) \\ x(t_0) &= \tilde{x}_0 \end{aligned} \tag{9}$$

*has a unique solution $x$ in $\mathcal{G}[\mathbb{R}, X]$.*

Note that equality (9) holds in the space $\mathfrak{X}_{\mathcal{G}}(x)$ of generalized sections along the generalized mapping $x \in \mathcal{G}[\mathbb{R}, X]$, defined by (cf. [18] Def. 4.6) $\mathfrak{X}_{\mathcal{G}}(x) := \{v \in \mathcal{G}^h[\mathbb{R}, TX] \,|\, \pi_X \circ v = x\}$.

**Proof.** Choose a representative $(\xi_\varepsilon)_\varepsilon$ of $\xi$. By (i) each $\xi_\varepsilon$ is globally bounded with respect to $h$. Then due to the completeness of $(X, h)$, for each $\varepsilon \in I$ there exists a globally defined solution $x_\varepsilon$ of

$$\begin{aligned} \dot{x}_\varepsilon(t) &= \xi_\varepsilon(x_\varepsilon(t)) \\ x_\varepsilon(t_0) &= \tilde{x}_{0\varepsilon} \end{aligned} \tag{10}$$

(cf. [24], Ch. 5, R20).
Let $t_1 < t_2 \in \mathbb{R}$. Then denoting by $L$ the length of a curve we have from (i)

$$L(x_\varepsilon|_{[t_1,t_2]}) = \int_{t_1}^{t_2} \|\dot{x}_\varepsilon(s)\|_h \, ds = \int_{t_1}^{t_2} \|\xi_\varepsilon(x_\varepsilon(s))\|_h \, ds \leq C|t_2 - t_1| \tag{11}$$

for all $\varepsilon$. Let $K \subset\subset X$, $\varepsilon_0 > 0$ such that $x_\varepsilon(t_1) \in K$ for all $\varepsilon < \varepsilon_0$. Then by the above $\bigcup_{\varepsilon<\varepsilon_0} x_\varepsilon[t_1, t_2] \subseteq \{p \in X \mid d_h(p, K) \leq C|t_2 - t_1|\}$. Since the latter set is compact by the Hopf-Rinow theorem, it follows that $(x_\varepsilon)_\varepsilon$ is c-bounded. Due to this fact, moderateness of $(x_\varepsilon)_\varepsilon$ follows as in the local case taking into account [14], Def. 2.2, using the moderateness of $(\xi_\varepsilon)_\varepsilon$ and applying the differential equation for $x_\varepsilon$ inductively.
To establish *uniqueness*, let $a > 0$ and choose $\varepsilon_0 \in I$, $K \subset\subset X$ such that $x_\varepsilon([-a-1, a+1]) \cup y_\varepsilon([-a-1, a+1]) \subseteq K$ for all $\varepsilon < \varepsilon_0$. Let $t_0 \in (-a, a)$ and suppose that $(x_\varepsilon)_\varepsilon$ satisfies (10), and $(y_\varepsilon)_\varepsilon \in \mathcal{E}_M[\mathbb{R}, X]$ solves

$$\begin{aligned} \dot{y}_\varepsilon(t) &= \xi_\varepsilon(y_\varepsilon(t)) + n_\varepsilon(t) \\ y_\varepsilon(t_0) &= \tilde{y}_{0\varepsilon} \end{aligned} \tag{12}$$

Here, $\pi_X \circ n_\varepsilon = y_\varepsilon$ for each $\varepsilon$ and $(n_\varepsilon)_\varepsilon \sim_h (0 \circ y_\varepsilon)_\varepsilon$ in $\mathcal{E}^h_M[\mathbb{R}, TX]$, where 0 denotes the zero element in $\Gamma_{\mathcal{G}}(X, TX)$ (cf. [19], Prop. 5.7). Also $(\tilde{x}_{0\varepsilon})_\varepsilon$, $(\tilde{y}_{0\varepsilon})_\varepsilon \in X_c$ satisfy $[(\tilde{x}_{0\varepsilon})] = [(\tilde{y}_{0\varepsilon})]$ in

$\tilde{X}_c$. By [1], Th. 1.36 there exists some $r > 0$ such that $K$ can be covered by finitely many metric balls $B_r(p_i)$ ($p_i \in K, 1 \le i \le k$) with each $B_{4r}(p_i)$ a geodesically convex domain for the chart $\psi_i := \exp_{p_i}^{-1}$. Choose $\varepsilon_1 < \varepsilon_0$ such that

$$d_h(x_\varepsilon(t_0), y_\varepsilon(t_0)) = d_h(x_{0\varepsilon}, y_{0\varepsilon}) < r$$

for all $\varepsilon < \varepsilon_1$. With $C$ as in (11) we choose $0 < d < \min(r/C, 1)$. For fixed $\varepsilon < \varepsilon_1$ there exists some $i \in \{1, \dots, k\}$ (depending on $\varepsilon$) with $x_{0\varepsilon} \in B_r(p_i)$. Then by convexity, for each $t$ with $|t - t_0| < d$ the entire line connecting $\psi_i(x_\varepsilon(t))$ and $\psi_i(y_\varepsilon(t))$ is contained in $\psi_i(B_{3r}(p_i))$. Given any $m > 0$, we may therefore employ the Gronwall argument from the proof of Theorem 3.1 to conclude that there exists $\varepsilon_2 < \varepsilon_1$ such that for $\varepsilon < \varepsilon_2$

$$|\psi_i \circ x_\varepsilon(t) - \psi_i \circ y_\varepsilon(t)| \le C' \varepsilon^m .$$

Here, $\varepsilon_2$, $C'$ only depend on $n = [(n_\varepsilon)_\varepsilon]$, $K$, $(\tilde{x}_{0\varepsilon})_\varepsilon$, $(\tilde{y}_{0\varepsilon})_\varepsilon$ and $\psi_i$ (on a compact subset of its domain), hence can be chosen uniformly in $i \in \{1, \dots, k\}$ and $t \in [t_0 - d, t_0 + d]$. Therefore,

$$\sup_{t \in [t_0 - d, t_0 + d]} d_h(x_\varepsilon(t), y_\varepsilon(t)) \le C'' \varepsilon^m ,$$

for $\varepsilon < \varepsilon_2$, so $(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon$ on $(t_0 - d, t_0 + d)$ by [19], Th. 3.3. Since $d$ depends exclusively on $K$ and $C$ it follows that if $x$ and $y$ coincide in any $t_0 \in (-a, a)$ then in fact they agree on an interval of fixed minimal length around $t_0$, hence they are identical on all of $(-a, a)$. Since $a$ was arbitrary it follows that $x$ and $y$ agree globally as elements of $\mathcal{G}[\mathbb{R}, X]$. □

Based on this result we are now able to establish the following flow theorem in the global context.

**3.6 Theorem.** *Let $(X, h)$ be a complete Riemannian manifold and suppose that $\xi \in \mathcal{G}^1_0(X)$ satisfies conditions (i) and (ii) of Theorem 3.5. Then there exists a unique generalized function $\Phi \in \mathcal{G}[\mathbb{R} \times X, X]$, the* generalized flow of $\xi$, *such that*

$$\begin{aligned}
\frac{d}{dt}\Phi(t,x) &= \xi(\Phi(t,x)) \quad \textit{in } \mathcal{G}^h[\mathbb{R} \times X, TX] &(13)\\
\Phi(0,.) &= \mathrm{id}_X \quad \textit{in } \mathcal{G}[X,X] &(14)\\
\Phi(t+s,.) &= \Phi(t, \Phi(s,.)) \quad \textit{in } \mathcal{G}[\mathbb{R}^2 \times X, X] . &(15)
\end{aligned}$$

**Proof.** *Existence:* Choosing a representative $(\xi_\varepsilon)_\varepsilon$ such that each $\xi_\varepsilon$ is globally bounded with respect to $h$ we obtain a global smooth flow $\Phi^\varepsilon$ for each $\varepsilon$. (13)–(15) then clearly hold componentwise for $(\Phi^\varepsilon)_\varepsilon$. Since $h$ is complete, an argument as in (11) shows that any compact subset of $\mathbb{R} \times X$ remains bounded (hence relatively compact) upon application of $\Phi^\varepsilon$, uniformly in $\varepsilon$. Thus $(\Phi^\varepsilon)_\varepsilon$ is c-bounded.
To show moderateness of $(\Phi^\varepsilon)_\varepsilon$, we first note that $t$-derivatives of $\Phi^\varepsilon$ may be estimated according to [14], Def. 2.2, precisely as in the proof of Theorem 3.5. Next, let $[0, t'] \subset\subset \mathbb{R}$, $K \subset\subset X$ be given and fix $p \in K$. Then there exist $t_0 = 0, t_1, \dots, t_k = t'$ such that each $\{\Phi^\varepsilon(t,p) \mid t_i \le t \le t_{i+1}\}$ lies entirely within a chart domain. We may therefore iterate an integral argument as in (8) to obtain a moderateness estimate on the first (local) $x$-derivative of $\Phi^\varepsilon$ on $[0, t'] \times \{p\}$. Since only finitely many charts are needed to cover $[0, t'] \times K$ and the constants in the resulting estimates can be chosen uniformly in $p \in K$, we obtain the moderateness estimate for first order $x$-derivatives of $\Phi^\varepsilon$ (the case $t' < 0$ is treated analogously). Higher order $x$-derivatives as well as mixed $x$, $t$ derivatives are estimated in the same manner, so $(\Phi^\varepsilon)_\varepsilon$ is indeed moderate. Moreover, we conclude from the above that $(\frac{d}{dt}\Phi^\varepsilon)_\varepsilon \in \mathcal{E}^h_M[\mathbb{R} \times X, TX]$. Also, the composition on the right hand side of (13) yields a well-defined element of $\mathcal{G}^h[\mathbb{R} \times X, TX]$ by [18], Th. 4.2. (13) therefore holds since it was already established on the level of representatives. Similarly, (14), (15) hold for $\Phi = [(\Phi^\varepsilon)_\varepsilon]$.
Finally, *uniqueness* of the flow follows from the point value characterization of manifold valued generalized functions and Theorem 3.5, precisely as in the proof of 3.3. □

**3.7 Definition.** *We call a generalized vector field $\xi \in \mathcal{G}_0^1(X)$* $\mathcal{G}$-complete *if there exists a unique global generalized flow $\Phi \in \mathcal{G}[\mathbb{R} \times X, X]$ satisfying (13), (14), (15).*

# 4 The distributional setting

Our next aim is an analysis of the interrelation between the theory introduced in the previous section and a purely distributional approach, as provided by Marsden in [22].
Any distributional theory of ordinary differential equations on manifolds faces a number of principal obstacles resulting from the basic structure of the theory of distributions itself. In fact, consider the initial value problem

$$\begin{aligned} \dot{x}(t) &= \zeta(x(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{16}$$

with $\zeta \in \mathcal{D}'(X, TX)$ a distributional vector field. The first question to be answered in treating this problem is in which setting the solution $x$ is to be sought (there is no concept of distributions taking values in a differentiable manifold). A similar problem occurs upon trying to introduce a notion of distributional flow for (16). Marsden in [22] employs a regularization approach to cope with these problems, introducing a sequence of smooth vector fields $\xi_\varepsilon$ approximating $\zeta$. Each $\xi_\varepsilon$ has a classical flow $\Phi^\varepsilon$ and under certain assumptions the assignment $\Psi = \lim_{\varepsilon \to 0} \Phi^\varepsilon$ allows one to associate a measurable function $\Psi$ to the distributional vector field $\zeta$. However, the question arises under which conditions on $\zeta$, resp. the regularizing sequence, the limiting map $\Psi$ is indeed a flow, i.e., satisfies $\Psi_{t+s} = \Psi_t \circ \Psi_s$. The answer provided by Th. 6.2 in [22] turns out to be wrong as we shall see below by an explicit counter-example. This fact is particularly unfortunate as the main flow theorems in [22] both in the general (Th. 6.3) and in the Hamiltonian case (Th. 8.4) rest upon Th. 6.2.
With a view to a smooth presentation of these considerations we first recall the following definition ([22], Def. 6.1, with the index set of the regularizing sequence changed from $\mathbb{N}$ to $I$ to ease comparison with the present setting):
Let $\zeta \in \mathcal{D}'(X, TX)$ be a distributional vector field on the manifold $X$ and let $(\xi_\varepsilon)_\varepsilon$ be a net of smooth vector fields with complete flows $\Phi^\varepsilon(t, .)$ and $\xi_\varepsilon \to \zeta \in \mathcal{D}'(X, TX)$. $\zeta$ is called a *vector field with measurable flow* $\Psi_t$ if

(i) $\Phi^\varepsilon(t, .) \to \Psi(t, .)$ almost everywhere on $X$ for all $t$ (in particular, $\Psi_t$ is measurable), and

(ii) For each $t \in \mathbb{R}$ and each $C \subset\subset X$ there exists $\varepsilon_0 \in I$ and $K \subset\subset X$ with $C \subseteq K$ such that $\Phi^\varepsilon(t, C) \subseteq K$ for all $\varepsilon$.

Note that in our terminology, (ii) says that $\Phi^\varepsilon(t, \,.\,)$ is c-bounded. Moreover, if $(\xi_\varepsilon)_\varepsilon$ is additionally supposed to be moderate and $\xi = [(\xi_\varepsilon)_\varepsilon] \in \mathcal{G}_0^1(X)$ then $\xi_\varepsilon \to \zeta \in \mathcal{D}'(X, TX)$ is the same as requiring that $\xi$ is associated with $\zeta$ (see Section 5 below). As remarked in [22], $\Psi$ in general depends on the chosen regularizing net $(\xi_\varepsilon)_\varepsilon$.
The basic theorem on flows of distributional vector fields then takes the following form
**Theorem 6.2 of [22]**
*Let $\zeta \in \mathcal{D}'(X, TX)$ be a vector field with measurable flow $\Psi_t$; then the flow property holds in the following sense*

$$\Psi_{t+s} = \Psi_t \circ \Psi_s \text{ almost everywhere on } X, \forall s, t \in \mathbb{R}.$$

In order to analyze the validity of this claim we consider the following initial value problem on $X = S^1$

$$\dot{x}(t) = \zeta(x(t)) \tag{17}$$
$$x(0) = e^{i\alpha_0}, \tag{18}$$

with the vector field $\zeta$ given by

$$\zeta(e^{i\alpha}) = \left(e^{i\alpha}, H(\alpha + \frac{\pi}{2}) - H(\alpha - \frac{\pi}{2})\right), \tag{19}$$

where $H$ denotes the Heaviside function.
We proceed by replacing $H$ by a suitable regularization. We choose a *scaling function* $\sigma : (0, \infty) \to (0, \infty)$ satisfying $\sigma(\varepsilon) \to 0$ $(\varepsilon \to 0)$ and a mollifier $\rho \in \mathcal{D}(\mathbb{R})$ with $\rho \geq 0$, $\mathrm{supp}(\rho) \subseteq [-1, 1]$ and $\int \rho = 1$. Then we set

$$\rho_{\sigma(\varepsilon)} := \frac{1}{\sigma(\varepsilon)} \rho\left(\frac{x}{\sigma(\varepsilon)}\right) \quad \text{and finally} \tag{20}$$

$$H_\varepsilon(x) := \int_{-\infty}^{x} \rho_{\sigma(\varepsilon)}(s)\, ds. \tag{21}$$

Equipping $S^1$ with the standard metric we have the following

**4.1 Proposition.** *Let $\alpha_0 \in [-\pi, \pi]$. The initial value problem*

$$\dot{x}(t) = \xi(x(t)) \tag{22}$$

$$x(0) = e^{i\alpha_0}, \tag{23}$$

*with the vector field $\xi = [(\xi_\varepsilon)_\varepsilon]$ given by*

$$\xi_\varepsilon(e^{i\alpha}) = \left(e^{i\alpha}, H_\varepsilon(\alpha + \frac{\pi}{2}) - H_\varepsilon(\alpha - \frac{\pi}{2})\right), \tag{24}$$

*and $\sigma(\varepsilon) := |\log(\varepsilon)|^{-1}$ has a unique solution $x = [(x_\varepsilon)_\varepsilon]$ in $\mathcal{G}[\mathbb{R}, S^1]$.*
*Moreover if $\alpha_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x_\varepsilon$ has the following (continuous) pointwise limit*

$$x_\varepsilon(t) \to x_{\alpha_0}(t) := \begin{cases} e^{-i\frac{\pi}{2}} & -\infty < t \leq -\alpha_0 - \frac{\pi}{2} \\ e^{i(\alpha_0 + t)} & -\alpha_0 - \frac{\pi}{2} \leq t \leq -\alpha_0 + \frac{\pi}{2} \\ e^{i\frac{\pi}{2}} & -\alpha_0 + \frac{\pi}{2} \leq t < \infty. \end{cases} \tag{25}$$

**Proof.** Existence and uniqueness follows by Theorem 3.5 due to our assumptions on $\sigma$.
To prove the statement on the limit we use the following notation: $x_\varepsilon(t) = e^{i\gamma_\varepsilon(t)}$. First note that for all $\alpha_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have $\xi_\varepsilon(e^{i\alpha_0}) = (e^{i\alpha_0}, 1)$ for $\varepsilon$ small enough. So $e^{i(\alpha_0 + t)}$ is a solution as long as $-\frac{\pi}{2} + \sigma(\varepsilon) < \alpha_0 + t = \gamma_\varepsilon(t) < \frac{\pi}{2} - \sigma(\varepsilon)$, that is $-\frac{\pi}{2} + \sigma(\varepsilon) - \alpha_0 < t < \frac{\pi}{2} - \sigma(\varepsilon) - \alpha_0$. Hence $\gamma_\varepsilon(t) \to \alpha_0 + t$ for $-\frac{\pi}{2} - \alpha_0 < t < \frac{\pi}{2} - \alpha_0$.
On the other hand if $t \leq -\frac{\pi}{2} + \sigma(\varepsilon) - \alpha_0$ resp. $t \geq \frac{\pi}{2} - \sigma(\varepsilon) - \alpha_0$ then $-\frac{\pi}{2} - \sigma(\varepsilon) \leq \gamma_\varepsilon(t) \leq -\frac{\pi}{2} + \sigma(\varepsilon)$ resp. $\frac{\pi}{2} - \sigma(\varepsilon) \leq \gamma_\varepsilon(t) \leq \frac{\pi}{2} + \sigma(\varepsilon)$ by the fact that $e^{-i(\frac{\pi}{2} + \sigma(\varepsilon))}$ resp. $e^{i(\frac{\pi}{2} + \sigma(\varepsilon))}$ are equilibrium points and the monotonicity of $\gamma_\varepsilon$. Hence the claim follows. □

Note that if $\alpha_0 \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$ the solution equals $e^{i\alpha_0}$ for all times $t$. If $\alpha_0 = \pm\frac{\pi}{2}$ the limit of the solution will in general depend on the choice of $\rho$. The most "generic" choice is (a) to suppose that $0 < \gamma_- \leq H_\varepsilon(0) \leq \gamma_+ < 1$ for all $\varepsilon$. In this case we obtain

$$x_\varepsilon(t) \to x_{\pm\frac{\pi}{2}}(t) = \begin{cases} e^{-i\frac{\pi}{2}} & -\infty < t \leq \mp\frac{\pi}{2} - \frac{\pi}{2} \\ e^{i(\pm\frac{\pi}{2} + t)} & \mp\frac{\pi}{2} - \frac{\pi}{2} \leq t \leq \mp\frac{\pi}{2} + \frac{\pi}{2} \\ e^{i\frac{\pi}{2}} & \mp\frac{\pi}{2} + \frac{\pi}{2} \leq t < \infty\,. \end{cases} \tag{26}$$

Indeed for $\alpha_0 = -\frac{\pi}{2}$ (for $\alpha_0 = \frac{\pi}{2}$ just adapt the argument accordingly) and $t \leq 0$ we use the same arguments as in the last part of the above proof to conclude that $-\frac{\pi}{2} - \sigma(\varepsilon) \leq \gamma_\varepsilon(t) \leq -\frac{\pi}{2}$. Hence $\gamma_\varepsilon(t) \to -\frac{\pi}{2}$ for $0 \leq t$.
To deal with nonnegative $t$ we first observe that $\mathrm{pr}^2(\xi_\varepsilon(e^{-i\frac{\pi}{2}})) = H_\varepsilon(0) \geq \gamma_- > 0$ hence $\dot{\gamma}_\varepsilon(t) \geq \gamma_-$ for all $t \geq 0$ small enough, i.e., such that $\gamma_\varepsilon(t) \leq \frac{\pi}{2} - \sigma(\varepsilon)$. So for all such $t$ we obtain $\gamma_\varepsilon(t) \geq$

$\gamma_- \cdot t - \frac{\pi}{2}$. In particular for $t > \sigma(\varepsilon)/\gamma_-$ we have $\gamma_\varepsilon(t) \geq -\frac{\pi}{2} + \sigma(\varepsilon)$. So there exists $t_\varepsilon \leq \sigma(\varepsilon)/\gamma_-$ such that $\gamma_\varepsilon(t_\varepsilon) = -\frac{\pi}{2} + \sigma(\varepsilon)$. This in turn implies that for $t_\varepsilon \leq t \leq \pi - 2\sigma(\varepsilon) + t_\varepsilon$ the solution takes the form $\gamma_\varepsilon(t) = -\frac{\pi}{2} + \sigma(\varepsilon) + (t - t_\varepsilon)$. So $\gamma_\varepsilon(t) \to -\frac{\pi}{2} + t$ for $0 \leq t \leq \pi$.
Finally for $t \geq \pi$ we again use the monotonicity of $\gamma_\varepsilon$ and the fact that $e^{i(\frac{\pi}{2}+\sigma(\varepsilon))}$ is an equilibrium point to establish the claim.

If we choose (b) $H_\varepsilon(0) = 0$ (resp. (c) $H_\varepsilon(0) = 1$) one sees by adapting the above line of arguments that the limiting solution with initial value $\alpha_0 = -\frac{\pi}{2}$ ($\alpha_0 = \frac{\pi}{2}$) will be trapped at $e^{i\alpha_0}$ and equal $x_{\alpha_0}$ for $\alpha_0 = \frac{\pi}{2}$ ($\alpha_0 = -\frac{\pi}{2}$). However, we still could use cases (b) and (c) in the construction to follow. In case we drop the assumption $\rho \geq 0$ the limiting behavior can be more complicated since the solution then may be trapped between different equilibria.

Now we are going to show that the above proposition provides a counter-example to Marsden's theorem. We consider the flow $\Phi^\varepsilon(t, e^{i\alpha}) = x_\varepsilon(t)$, where $x_\varepsilon$ is the solution with $x_\varepsilon(0) = e^{i\alpha}$ provided by the Proposition. By Theorem 3.6 $\Phi = [(\Phi^\varepsilon)_\varepsilon]$ is in $\mathcal{G}[R \times S^1, S^1]$ and has the flow properties (14), (15). Defining $\Psi = \lim_{\varepsilon\to 0} \Phi^\varepsilon$ conditions (i) (even with convergence everywhere) and (ii) of Marsden's definition are satisfied, hence $\zeta$ (given by eq. (19)) is a vector field with measurable flow $\Psi$. So $\Psi$ ought to have the flow property in the sense of [22], Th. 6.2. However, we have by the second part of Proposition 4.1 and the remark following its proof (using case (a))

$$\Phi^\varepsilon(t, e^{i\alpha}) \;\to\; \Psi(t, e^{i\alpha}) := \begin{cases} x_\alpha(t) & \text{if } \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ e^{i\alpha} & \text{if } \alpha \notin [-\frac{\pi}{2}, \frac{\pi}{2}], \end{cases} \tag{27}$$

where $x_\alpha(t)$ denotes the limiting function in (25) with $\alpha = \alpha_0$. Hence

$$\Psi(-\pi, e^{i\alpha}) = \begin{cases} e^{-i\frac{\pi}{2}} & \text{if } \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ e^{i\alpha} & \text{otherwise} \end{cases} \tag{28}$$

and

$$\Psi(\pi, e^{i\alpha}) = \begin{cases} e^{i\frac{\pi}{2}} & \text{if } \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ e^{i\alpha} & \text{otherwise.} \end{cases} \tag{29}$$

This in turn implies

$$\Psi(\pi, \Psi(-\pi, e^{i\alpha})) = \begin{cases} e^{i\frac{\pi}{2}} & \text{if } \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ e^{i\alpha} & \text{otherwise.} \end{cases} \tag{30}$$

So $\Psi_\pi \circ \Psi_{-\pi} \neq id$ for all $e^{i\alpha}$ with $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ contradicting the assertion of Theorem 6.2 in [22]. Note that if we choose cases (b) or (c) above we obtain similar results; only the range of $\alpha$ changes form the closed interval to the half open interval $(-\frac{\pi}{2}, \frac{\pi}{2}]$ resp. $[-\frac{\pi}{2}, \frac{\pi}{2})$ and again the flow property fails to hold for $e^{i\alpha}$ in a set of positive measure.

Since the approach in [22] is built upon pointwise convergence almost everywhere of the regularizing flows its failure motivates the study of different notions of convergence for generalized functions taking values in a manifold to allow for a corrected version of the Theorem. We do so in the following section.

# 5 Notions of Association

In all variants of spaces of Colombeau generalized functions taking values in a linear space compatibility with respect to the distributional setting is affected through the notion of association. We call $u \in \mathcal{G}(X)$ associated with zero, $u \approx 0$, if one (hence every) representative $u_\varepsilon$ converges to zero weakly (cf. also Definition 5.1 (v) below). The assignment $u \approx v :\Leftrightarrow u_\varepsilon - v_\varepsilon \approx 0$ gives rise to an equivalence relation on $\mathcal{G}(X)$ and a linear quotient space $\mathcal{G}(X)/\approx$, generalizing distributional equality to the level of $\mathcal{G}(X)$. Moreover if $\lim_{\varepsilon\to 0} \int_X u_\varepsilon \nu = \langle \omega, \nu \rangle$ for some distribution $\omega$ and every compactly supported one-density $\nu$ we write $u \approx \omega$ and call $\omega$ the distributional shadow of $u \in \mathcal{G}(X)$.
In this section we are going to introduce a number of notions of association in the space $\mathcal{G}[X, Y]$ (cf. also [19], Sec. 6) and clarify their respective interrelations.

**5.1 Definition.** *Let $u = [(u_\varepsilon)_\varepsilon], v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[X, Y]$, and let $h$ be a Riemannian metric on $X$ with distance function $d_h$.*

*(i) $u$ is called* zero-associated *(cf. [19], Def 6.1) with $v$,*

$$u \approx_0 v \ :\Leftrightarrow \sup_{p\in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0 \quad \forall K \subset\subset X. \tag{31}$$

*(ii) $u$ is called* pointwise-associated (pw-associated) *with $v$,*

$$u \approx_{\mathrm{pw}} v \ :\Leftrightarrow d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0 \quad \forall p \in X. \tag{32}$$

*(iii) $u$ is called* pointwise-associated almost everywhere (pwae-associated) *with $v$,*

$$u \approx_{\mathrm{pwae}} v \ :\Leftrightarrow d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0 \ \textit{ for almost all } \ p \in X. \tag{33}$$

*(iv) $u$ is called* model-associated *with $v$,*

$$u \approx_{\mathcal{M}} v \ :\Leftrightarrow f \circ u_\varepsilon - f \circ v_\varepsilon \to 0 \ \textit{ in } \ \mathcal{D}'(X) \quad \forall f \in \mathcal{C}^\infty(Y). \tag{34}$$

*That is, $f \circ u \approx f \circ v$ in $\mathcal{G}(X)$ for all $f \in \mathcal{C}^\infty(Y)$ (see (v) below).*

*(v) If $Y = \mathbb{R}^n$ then $u$ is called* associated *with $v$,*

$$u \approx v :\Leftrightarrow u_\varepsilon - v_\varepsilon \to 0 \ \textit{ in } \ \mathcal{D}'(X)^n.$$

It is straightforward to check that notions (i)–(iii) are independent of the Riemannian metric $h$ employed (cf. [9], Lemma 3.2.4) and that all of the above definitions are independent of the representatives chosen for $u$ and $v$.

**5.2 Theorem.** *We have the following chain of implications:*

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v),$$

*where the last implication holds in case $Y = \mathbb{R}^n$. None of the above implications can be reversed.*

**Proof.** (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear as well as (iv) $\Rightarrow$ (v).
(iii) $\Rightarrow$ (iv): Let $f \in \mathcal{C}^\infty(Y)$ and let $(u_\varepsilon)_\varepsilon$ resp. $(v_\varepsilon)_\varepsilon$ be representatives of $u$ resp. $v$. Then $|f \circ u_\varepsilon(p) - f \circ v_\varepsilon(p)| \to 0$ a.e. and is bounded uniformly in $\varepsilon$ on compact sets by the c-boundedness of $u_\varepsilon$ and $v_\varepsilon$. Hence by dominated convergence $\int (f \circ u_\varepsilon(p) - f \circ v_\varepsilon(p))\, \mu(p) \to 0$ for all compactly supported one-densities $\mu$ on $X$.
(v) $\not\Rightarrow$ (iv): Set $u_\varepsilon = \sin\left(\frac{x}{\varepsilon}\right)$ and $v_\varepsilon = 0$. Then $u \approx v$ in $\mathcal{G}(\mathbb{R})$ but for $f(x) = x^2$ we have $f \circ u_\varepsilon \to 1/2$ pointwise hence $f \circ u_\varepsilon - f \circ v_\varepsilon \not\to 0$ in $\mathcal{D}'(\mathbb{R})$.
(iv) $\not\Rightarrow$ (iii): Let $\rho_0 \in \mathcal{D}(\mathbb{R})$, $\rho_0(\mathbb{R}) \subseteq [0, 1]$, $\mathrm{supp}(\rho_0) \subseteq [-1, 1]$, $\int \rho_0 = 1$ and $\rho_0(0) = 1$. Furthermore, set

$$\rho(x) := \sum_{n=-\infty}^{n=\infty} \rho_0(2^{|n|}(x - n))$$

and $\rho_\varepsilon(x) := \rho(x/\varepsilon)$. Then there is no $x \in \mathbb{R}$ such that $\rho_\varepsilon(x) \to 0$. Indeed, $\forall x\ \forall n\ \exists \varepsilon_n$, $\varepsilon_n \to 0$ such that $\rho_{\varepsilon_n}(x) = 1$. On the other hand $\rho_\varepsilon \approx_{\mathcal{M}} 0$ since for any $f \in \mathcal{C}^\infty(\mathbb{R})$ and any test function $\varphi$ we have

$$\begin{aligned} |\int (f \circ \rho_\varepsilon(x))\varphi(x)\, dx| &\leq \|\nabla f\|_{\infty,[0,1]} \int |\rho_\varepsilon(x)|\, |\varphi(x)|\, dx \\ &= \|\nabla f\|_{\infty,[0,1]} \int \rho\left(\frac{x}{\varepsilon}\right) |\varphi(x)|\, dx \\ &\leq \varepsilon \|\nabla f\|_{\infty,[0,1]}\, \|\varphi\|_\infty\, \|\rho\|_1 \\ &\to 0\,. \end{aligned}$$

(iii) $\not\Rightarrow$ (ii) $\not\Rightarrow$ (i) is clear. □

**5.3 Definition.** *Let $u = [(u_\varepsilon)_\varepsilon]$ in $\mathcal{G}[X,Y]$ and $v : X \to Y$ a map. $v$ is called a* shadow *of $u$ in the sense of zero-, pw-, pwae-, resp. model-association (or, for short, zero-, pw-, pwae-, resp. model-associated with $u$) if (31), (32), (33), resp. (34) holds with $v$ replacing $v_\varepsilon$.*

Of course the hierarchy of Theorem 5.2 carries over to shadows of the types introduced above. In the following we shall also need a notion which encodes information on the order of convergence with respect to $\varepsilon$.

**5.4 Definition.** *Let $u \in \mathcal{G}[X,Y]$.*

*(i) $u$ is called* fast-associated *with $v \in \mathcal{G}[X,Y]$,*

$$u \approx_{\mathrm{f}} v \;:\Leftrightarrow d_h(u_\varepsilon(p), v_\varepsilon(p)) = O(\varepsilon^m) \;\forall p \in X \;\forall \varepsilon \in \mathbb{N}$$

*for one (hence every) representative $(u_\varepsilon)_\varepsilon$ of $u$ and one (hence every) representative $(v_\varepsilon)_\varepsilon$ of $v$ where again $d_h$ denotes the Riemannian distance with respect to any Riemannian metric $h$ on $Y$.*

*(ii) $u$ is called* fast-associated *with $v : X \to Y$,*

$$u \approx_{\mathrm{f}} v \;:\Leftrightarrow d_h(u_\varepsilon(p), v(p)) = O(\varepsilon^m) \;\forall p \in X \;\forall \varepsilon \in \mathbb{N}$$

*for one (hence every) representative $(u_\varepsilon)_\varepsilon$ of $u$.*

Note that for $u\,, v \in \mathcal{G}[X,Y]$, $u \approx_{\mathrm{f}} v$ if and only if the generalized number $d_h(u(p), v(p))$ is 0 for each $p \in X$ and one (each) Riemannian metric $h$, i.e., if and only if $u(p) = v(p)$ for all $p \in X$. This notion is *strictly* weaker than equality in $\mathcal{G}[X,Y]$ (cf. [27]). $\approx_{\mathrm{f}}$ implies $\approx_{\mathrm{p}w}$ and the converse implication is clearly wrong. However, there is no relation between $\approx_{\mathrm{f}}$ and $\approx_0$.

# 6 Limiting flows

Having introduced a number of notions of association in the previous section we now have the tools at hand to analyze the following question: Let $\xi \in \mathcal{G}^1_0(X)$ be $\mathcal{G}$-complete with $\Phi = [(\Phi^\varepsilon)_\varepsilon]$ its (unique) flow in $\mathcal{G}[\mathbb{R} \times X, X]$. If $\Phi$ admits a shadow $\Psi$ in the sense of one of the notions introduced above, does this imply that $\Psi$ has the flow property? We are going to answer this question in the following but first turn to some preliminaries.

**6.1 Definition.** *We say a function $u \in \mathcal{G}[X,Y]$ is of* locally bounded derivative *if for all $K \subset\subset X$ and for one (hence every) pair of Riemannian metrics $g$ on $X$ resp. $h$ on $Y$ there exists $C > 0$ and $\varepsilon_0 > 0$ such that*

$$\sup_{p \in K} \|T_p u_\varepsilon\|_{g,h} \le C \qquad \forall \varepsilon \le \varepsilon_0$$

*for one (hence every) representative $(u_\varepsilon)_\varepsilon$ of $u$. Here $\|T_p f\|_{g,h}$ denotes the norm of the linear map $T_p f : (T_p X, \|\ \ \|_g) \to (T_{f(p)} Y, \|\ \ \|_h)$ (cf. [9], 3.2.54).*

**6.2 Lemma.** *Let $(X,h)$ be a complete Riemann manifold, let $\xi \in \mathcal{G}^1_0(X)$ satisfy the assumptions of Theorem 3.5 and denote by $\Phi = [(\Phi^\varepsilon)_\varepsilon] \in \mathcal{G}[\mathbb{R} \times X, X]$ the generalized flow of $\xi$. If $P\xi$ is locally bounded for all differential operators $P \in \mathcal{P}(X, TX)$ of first order then $\Phi(t, .)$ is of locally bounded derivative.*

**Proof.** For each $\varepsilon$ let $\Phi^\varepsilon$ be the complete flow corresponding to some globally bounded (w.r.t. $h$) representative $\xi_\varepsilon$ of $\xi$. Then each $T\Phi^\varepsilon(t, .)$ satisfies the following ODE

$$\begin{aligned} \frac{d}{dt} T\Phi^\varepsilon(t,\,.\,) &= T\xi(\Phi^\varepsilon(t,\,.\,))\; T\Phi^\varepsilon(t,\,.\,) \\ T\Phi^\varepsilon(0,\,.\,) &= \mathrm{id}. \end{aligned}$$

As we only need to consider $(t,p)$ varying in a compact subset of $\mathbb{R}\times X$ we may employ the same argument as in the uniqueness part of the proof of Theorem 3.6 to successively estimate over pieces of the integral curves $s\mapsto\Phi^\varepsilon(s,p)$ of $\xi_\varepsilon$, each contained in a single chart. Therefore we may work locally to obtain

$$D_x\Phi^\varepsilon(t,x)=x+\int_0^t D_x\xi_\varepsilon(\Phi^\varepsilon(s,x))\ D_x\Phi^\varepsilon(s,x)\,ds$$

and by Gronwall's inequality

$$\|D\Phi^\varepsilon(t,x)\|\le C\ \exp\left(\int_0^t\|D_x\xi_\varepsilon(\Phi^\varepsilon(s,x))\|\,ds\right).$$

From this and the c-boundedness of the flow the assertion follows. □

**6.3 Theorem.**

*(i) Let $\xi\in\mathcal{G}^1_0(X)$ satisfy the assumptions of Lemma 6.2. If for all $t\in\mathbb{R}$, $\Phi(t,\,.\,)\approx_{\mathrm{f}}\Psi(t,\,.\,)$ then $\Psi$ has the flow property.*

*(ii) Let $\xi\in\mathcal{G}^1_0(X)$ be a $\mathcal{G}$-complete generalized vector field with flow $\Phi$. If for each $t\in\mathbb{R}$, $\Phi(t,\,.\,)\approx_0\Psi(t,\,.\,)$ then $\Psi$ has the flow property.*

**Proof.** (i) On the complete Riemannian manifold $(X,h)$ we have to show that

$$d_h(\Psi(s+t,p),\Psi(s,\Psi(t,p)))=0\,.$$

for all $s,t\in\mathbb{R}$ and for all $p\in X$. Since for all $s,t,p$

$$\Psi(s+t,p)=\lim_{\varepsilon\to0}\Phi^\varepsilon(s+t,p)=\lim_{\varepsilon\to0}\Phi^\varepsilon(s,\Phi^\varepsilon(t,p))$$

it suffices to show that

$$d_h(\Phi^\varepsilon(s,\Phi^\varepsilon(t,p)),\Psi(s,\Psi(t,p)))\to0$$

as $\varepsilon\to0$. We introduce the following splitting

$$d_h(\Phi^\varepsilon_s(\Phi^\varepsilon_t(p)),\Psi_s(\Psi_t(p)))\ \le\ d_h(\Phi^\varepsilon_s(\Phi^\varepsilon_t(p)),\Phi^\varepsilon_s(\Psi_t(p)))+d_h(\Phi^\varepsilon_s(\Psi_t(p)),\Psi_s(\Psi_t(p))).\tag{35}$$

Here the second term converges to zero since $\Phi^\varepsilon(s,q)\to\Psi(s,q)$ pointwise and we are left with the first term. On a complete Riemannian manifold any two points can be joined by a minimizing geodesic segment. So we choose such segments $\gamma_\varepsilon:[0,b_\varepsilon]\to X$ with $\gamma_\varepsilon(0)=\Psi(t,p)$ and $\gamma_\varepsilon(b_\varepsilon)=\Phi^\varepsilon(t,p)$. Then by the $c$-boundedness of $\Phi^\varepsilon$ and the Hopf-Rinow theorem we may choose $K\subset\subset X$ and $\varepsilon_0>0$ such that each $\gamma_\varepsilon$ stays entirely within $K$ for $\varepsilon<\varepsilon_0$. Thus

$$\begin{aligned}
d_h(\Phi^\varepsilon(s,\Phi^\varepsilon(t,p)),\Phi^\varepsilon(s,\Psi(t,p)))\ &\le\ \int_0^{b_\varepsilon}\|(\Phi^\varepsilon(s,.)\circ\gamma_\varepsilon)'(\lambda)\|_h\,d\lambda\\
&\le\ \int_0^{b_\varepsilon}\|T_{\gamma_\varepsilon(\lambda)}\Phi^\varepsilon(s,.)\|_{h,h}\,\|\gamma_\varepsilon'(\lambda)\|_h\,d\lambda\\
&\le\ \sup_{q\in K}\|T_q\Phi^\varepsilon(s,.)\|_{h,h}\int_0^{b_\varepsilon}\|\gamma_\varepsilon'(\lambda)\|_h\,d\lambda \qquad (36)\\
&\le\ \frac{C}{\varepsilon^N}\ d_h(\Phi^\varepsilon(s,p)),\Psi(s,p)),
\end{aligned}$$

for some $C,\,N > 0$. Since $\Phi(t,p) \approx_{\mathrm{f}} \Psi(t,p)$, $d_h(\Phi^\varepsilon(s,\Phi^\varepsilon(t,p)),\Phi^\varepsilon(t,\Psi(s,p)))$ converges to zero for $\varepsilon \to 0$, as desired.
(ii) In this case we use the splitting

$$d_h(\Phi^\varepsilon_s(\Phi^\varepsilon_t(p)),\Psi_s(\Psi_t(p))) \;\le\; d_h(\Phi^\varepsilon_s(\Phi^\varepsilon_t(p)),\Psi_s(\Phi^\varepsilon_t(p))) + d_h(\Psi_s(\Phi^\varepsilon_t(p)),\Psi_s(\Psi_t(p))) \tag{37}$$

Here the first term converges to 0 since $\Phi^\varepsilon(s,\,.\,) \to \Psi(s,\,.\,)$ locally uniformly and $(\Phi^\varepsilon)_\varepsilon$ is c-bounded and the second one since $\Psi_s$ is necessarily continuous. □

As was already mentioned in the proof, in case (ii) the limiting flow $\Psi$ necessarily is continuous in $p$. On the other hand, we will give an explicit example of a discontinuous limiting flow below. First, however, we turn to another set of assumptions guaranteeing a continuous limiting flow.

**6.4 Corollary.** *Let $\xi \in \mathcal{G}^1_0(X)$ satisfy the assumptions of 6.2 (ii). For each $t \in \mathbb{R}$, let $\Phi(t,\,.\,) \approx_{\mathrm{pw}} \Psi(t,\,.\,)$. Then $\Psi$ is continuous in $p$ and has the flow property.*

**Proof.** Let $\Phi = [(\Phi^\varepsilon)_\varepsilon]$. By Lemma 6.2 (ii) it follows that $(\Phi^\varepsilon)_\varepsilon$ is locally uniformly equicontinuous. But then by the theorem of Arzela-Ascoli $\Phi^\varepsilon$ in fact converges locally uniformly to $\Psi$, i.e., $\Phi(t,\,.\,) \approx_0 \Psi(t,\,.\,)$ for all $t$. Hence the claim follows from Theorem 6.3 (ii). □

**6.5 Example.** Let $X = T^2 = S^1 \times S^1$ and $\xi = [(\xi)_\varepsilon] = \mathcal{G}^1_0(X)$ be given by

$$\xi_\varepsilon(e^{i\alpha},e^{i\beta}) = (e^{i\alpha},e^{i\beta};1,1-\rho_{\sigma(\varepsilon)}(\alpha)).$$

Although $\xi$ does not satisfy the boundedness assumption of Theorem 3.6, we may nevertheless establish its $\mathcal{G}$-completeness as follows. First, since $X$ is compact, each $\xi_\varepsilon$ possesses a global flow $\Phi^\varepsilon$. $(\Phi^\varepsilon)_\varepsilon$ is moderate by the proof of Theorem 3.6 and that $\Phi := [(\Phi^\varepsilon)_\varepsilon]$ is indeed the unique flow of $\xi$ follows readily by choosing appropriate charts on $X$ and applying the local case (Theorem 3.3).
Moreover, $\Phi$ has a discontinuous pointwise limit $\Psi$, namely

$$\Phi^\varepsilon(t;e^{i\alpha},e^{i\beta}) = \begin{pmatrix} e^{i(\alpha+t)} \\ e^{i(\beta+t-\int_\alpha^{\alpha+t}\rho_{\sigma(\varepsilon)}(\gamma)\,d\gamma)} \end{pmatrix} \to \begin{pmatrix} e^{i(\alpha+t)} \\ e^{i(\beta+t-H(\alpha+t)+H(\alpha))} \end{pmatrix}, \tag{38}$$

(in fact we even have $\Phi^\varepsilon(t,\,.\,) \approx_{\mathrm{f}} \Psi(t,\,.\,)$ for all $t$) which by direct verification satisfies the flow property $\Psi_{s+t} = \Psi_s \circ \Psi_t$ for all $s.\,t \in \mathbb{R}$.

To conclude this section let us summarize the above results in the following way. From the counterexample in Section 4 we know that $\Phi \approx_{\mathrm{pw}} \Psi$ does not imply the flow property of $\Psi$. Neither does convergence of $\Phi^\varepsilon(t,.)$ to $\Psi(t,.)$ locally in $L^p$ for any $1 \le p < \infty$ secure the flow property of $\Psi$, as can also be seen from the example given in Section 4. On the other hand, convergence locally in $L^\infty$ (i.e., (ii) of Theorem 6.3) implies the flow property of $\Psi$ while at the same time forcing the limiting flow to be continuous. Hence the above example lies precisely in the gap which allows for a discontinuous limiting *flow*.

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