RANDOM COINCIDENCE POINTS FOR MULTI-VALED NON-LINEAR CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES

N. Shafqat *, N. Yasmin b, Z. Akhter *

a,b Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, 60800, Pakistan.

c Department of Mathematics, The Islamia University Bhawalpur

naeembzu77@gmail.com, nusyasmin@yahoo.com, zahid9896@yahoo.com

ABSTRACT: The aim of this work is to derive the results for random coincidence points for multi-valued nonlinear contractions in partially ordered metric spaces. We do it from two different approaches, the first is Δ -symmetric property and the second is by using g - mixed monotone property. These results are the random versions of Hussain and Alotabi [Fixed Point Theory and Appl., 2011, 2011:82]. The present theorems extend certain results due to Ciric, Samet and Vetro.

Key Words: Partially ordered set, Δ -symmetric property, mixed g -monotone property, Compatible maps, Couple random coincidence point.

AMS 2010 Subject Classification: Primary 47H10, Secondary 54H25

1. INTRODUCTION

Random fixed point theorems are stochastic generalization of classical fixed point theorems. Random fixed point theorems for contraction mappings on separable complete metric spaces were first proved by Spacek [23] and Hans [5,6]. The survey article by Bharucha-Ried [1] attrenched the attention of several mathematicians (see Zhang and Huang [26], Hans [5,6], Huang [7], Itoh [10], Lin [14], Papageorgiou [16,17], Shahzad and Hussain [21], Shahzad and Latif [22], Tan and Yuan [24] and give wings to this theory. Itoh [10] extended Spacek and Hans’s theorem to multi-valued contraction mappings. The stochastic version of the well-known Schauder’s fixed point theorem was proved by Sehgal and Singh [20], Ciric and Lakshmikantham [4], Zhu and Xiao [27], Hussain et all [9] and Khan et all [11] proved some coupled random fixed point and coupled random coincidence results in partially ordered complete metric spaces. Ciric et all [3] proved fixed point theorems for single-valued mappings, extended to a coincidence theorems for a pair of a random operator f : Ω × X → X and a multi-valued random operator T : Ω × X → CB(X).

The aim of this article is to prove a stochastic analog of the Hussain and Alotabi [8] coupled coincidences for multi-valued contractions in partially ordered metric spaces for a pair of random operators g : Ω × X → X and a multi-valued random operator T : Ω × X → CL(X).

2. Preliminaries

Let (X, d) be a metric space. We denote by CB(X) the collection of non-empty closed bounded subsets of X. For A, B ∈ CB(X) and x ∈ X, suppose that D(x, A) = inf d(x, a) and

H(A, B) = max {sup a∈A d(a, B), sup b∈B d(b, A)},

such a mapping H is called a Hausdorff metric on CB(X) induced by d.

Definition 2.1. [8] An element x ∈ X is said to be a fixed point of a multi-valued mapping T : X → CB(X) if and only if x ∈ Tx.

Definition 2.2. [2]. Let X be a nonempty set and F : X × X → X be a given mapping. An element (x, y) ∈ X × X is said to be a coupled fixed point of the mapping F if F(x, y) = x and F(y, x) = y.

Definition 2.3. [13]. Let (x, y) ∈ X × X, F : X × X → X and g : X → X. We say that the pair (x, y) is a coupled coincidence point of F and g if F(x, y) = gx and F(y, x) = gy for all x, y ∈ X.

Definition 2.4 [8]. A function f : X → R is called lower semi-continuous if and only if for any sequence \{x_n\}, \{y_n\} ⊂ X and (x, y) ∈ X × X, we have,

lim(x_n, y_n) = (x, y)

implies

f(x, y) ≤ lim inf \(n→∞\) f(x_n, y_n).

Let CL(X) := {A ⊂ X | A ≠ Ø, A = A}, where A denotes the closure of A in the metric space (X, d). Let (X, d) be a metric space endowed with a partial order and G : X → X be a given mapping. We define the set Δ ⊆ X × X by Δ := \{(x, y) ∈ X × X | G(x) ≤ G(y)\}.

In [18], Samet and Vetro introduced the binary relation R on CL(X) defined by ARB ⇔ A × B ⊆ Δ, where A, B ∈ CL(X).

Definition 2.5 [8]. Let F : X × X → CL(X) be a given mapping. We say that F is a Δ -symmetric mapping if and only if (x, y) ∈ Δ ⊆ F(x, y) R F(y, x).

Example 2.6. [8]. Suppose that X = [0,1], endowed with the usual order ≤. Let G : [0,1] → [0,1] be the mapping defined by G(x) = M for all x ∈ [0,1], where M is a constant in [0,1]. Then Δ = [0,1] × [0,1] and F : X × X → CL(X) is a Δ -symmetric mapping.

Definition 2.7. [19]. Let F : X × X → CL(X) be a given mapping. We say that (x, y) ∈ X × X is a coupled fixed point of F if and only if (x, y, F(x, y), F(y, x)) ∈ Δ.
point of \( F \) if and only if \( x \in F(x, y) \) and \( y \in F(y, x) \).

**Definition 2.8.** [8]. Let \( F : X \times X \to CL(X) \) be a given mapping and let \( g : X \to X \). We say that \( (x, y) \in X \times X \) is a coupled coincidence point of \( F \) and \( g \) if and only if \( g(x) \in F(x, y) \) and \( g(y) \in F(y, x) \).

Hussain and Alotaibi [8] proved the following theorems for the existence of coupled coincidence for multi-valued nonlinear contractions using two different approaches, first is based on \( \Delta \)-symmetric property recently studied in [19] and second one is based on mixed \( g \)-monotone property studied by Lakshmikantham and Cirić [13].

**Theorem 2.9.** [8]. Let \((X, d)\) be a metric space endowed with a partial order \( \preceq \) and \( \Delta \neq \emptyset \). Suppose that \( F : X \times X \to CL(X) \) is \( \Delta \)-symmetric mapping

\( g : X \to X \) is continuous, \( gX \) is complete, the function \( f : g(X) \times g(X) \to [0, +\infty) \) defined by

\[ f(g(x), g(y)) = D(gx, F(x, y)) + D(gy, F(y, x)) \]

for all \( x, y \in X \) and \( \varphi : [0, \infty) \to [a, 1), 0 < a < 1 \) satisfying

\[ \lim_{r \to r^+} \varphi(r) < 1 \] for each \( t \in [0, +\infty) \). Assume that for any \( (x, y) \in \Delta \) there exist \( gu \in F(x, y) \) and \( gv \in F(y, x) \) satisfying

\[ \sqrt{\varphi(f(gx, gy))[d(gx, gu) + d(gy, gv)]} \leq f(gx, gy) \]

Such that

\[ f(gu, gv) \leq \varphi(f(gx, gy))[d(gx, gu) + d(gy, gv)] \]

Then, \( F \) and \( g \) have a coupled coincidence point, that is, there exists \( gz = (gz_1, gz_2) \in X \times X \) such that

\[ gz_1 \in F(z_1, z_2) \quad \text{and} \quad gz_2 \in F(z_2, z_1) \].

**Theorem 2.10.** [8]. Let \((X, d)\) be a metric space endowed with a partial order \( \preceq \) and \( \Delta \neq \emptyset \). Suppose that \( F : X \times X \to CL(X) \) is \( \Delta \)-symmetric mapping

\( g : X \to X \) is continuous, \( gX \) is complete. Suppose that the function \( gX \times gX \to [0, +\infty) \) defined in \( \text{Theorem 2.9} \) is a lower semi-continuous and that there exists a function \( \varphi : [0, \infty) \to [a, 1), 0 < a < 1 \) satisfying

\[ \lim_{r \to r^+} \varphi(r) < 1 \] for each \( t \in [0, +\infty) \). Assume that for any \( (x, y) \in \Delta \), there exist \( gu \in F(x, y) \) and \( gv \in F(y, x) \) satisfying

\[ \sqrt{\varphi(d(gx, gu) + d(gy, gv))[d(gx, gu) + d(gy, gv)]} \]

\[ \leq [D(gx, F(x, y)) + D(gy, F(y, x))] \]

such that

\[ D(gu, F(u, v)) + D(gv, F(v, u)) \leq \varphi(d(gx, gu) + d(gy, gv)) \]

Then, \( F \) and \( g \) have a coupled coincidence point, that is, there exists \( gz = (gz_1, gz_2) \in X \times X \) such that

\[ gz_1 \in F(z_1, z_2) \quad \text{and} \quad gz_2 \in F(z_2, z_1) \].

Using the concept of commuting maps and mixed \( g \)-monotone property, Lakshmikantham and Cirić [13] established the existence of coupled coincidence point results to generalize the results of Bhaskar and Lakshmikantham [2]. Hussain and Alotaibi [8] proved the following results by using the mixed \( g \)-monotone property for compatible maps \( F \) and \( g \) in partially ordered metric space, where \( F \) is the multi-valued mapping.

**Theorem 2.11.** [8]. Let \( F : X \times X \to CL(X) \), \( g : X \to X \) be such that

there exists \( k \in (0, 1) \) with

\[ H(F(x, y), F(u, v)) \leq \frac{k}{2} [d((gx, gy), (gu, gv)) \]

for all \((gx, gy),(gu, gv)\),

\[ g(x_i) \leq g(x_{i+1}) , g(y_i), g(y_{i+1}) \quad x_i, y_i \in X(i = 1, 2) \quad \text{and for all} \]

\( g(u_i) \in F(x_i, y_i) \), there exists

\( g(u_2) \in F(x_2, y_2) \) with \( g(u_1) \leq g(u_2) \) and for all \( g(v_i) \in F(y_i, x_i) \), there exists \( g(v_2) \in F(y_2, x_2) \) with \( g(v_2) \leq g(v_1) \) provided,

\[ d((gu_1, gv_1),(gu_2, gv_2)) < 1 \];

\( F \) has the mixed \( g \)-monotone property, provided

\[ d((gu_1, gv_1),(gu_2, gv_2)) < 1 \],

there exists \( x_0, y_0 \in X \) and some

\( gx_0 \in F(x_0, y_0), gy_0 \in F(y_0, x_0) \) with

\[ g(x_0) \leq g(x_1), g(y_0) \leq g(y_1) \quad \text{such that} \]

\[ d((gx_0, gy_0),(gx_1, gy_1)) < 1 - k \quad \text{with} \quad k \in (0, 1) \]

if a non-decreasing sequence \( \{x_n\} \to x \), then \( x_n \leq x \) for all \( n \) and if non-increasing sequence \( \{y_n\} \to y \), then \( y \leq y_n \) for all \( n \) and if \( gX \) is complete, then \( F \) and \( g \) have a coupled coincidence point.

**Theorem 2.12.** [8]. Let \( F : X \times X \to CB(X) \), \( g : X \to X \) be such that condition (i)-(iii) of Theorem 2.11 hold. Let \( X \) be complete, \( F \) and \( g \) be continuous and compatible. Then \( F \) and \( g \) have a coupled coincidence point.

### 3. MAIN RESULTS

Let \((\Omega, \Sigma)\) be a measurable space with a \( \Sigma \)-sigma algebra of subsets of \( \Omega \) and let \((X, d)\) be metric space. We denote by \( 2^X \) the family of all subsets of \( X \). A mapping \( T : \Omega \to 2^X \) is called \( \Sigma \)-measurable if for any open subset
Let $X$, $T^{-1}(U) = \{w : T(w) \cap U \neq \emptyset \} \in \Sigma$. In what follows, when we speak of measurability, we will mean $\Sigma$-measurability. A mapping $f : \Omega \times X \to X$ is called a random operator if for any $x \in X$, $f(., x)$ is measurable. A mapping $T : \Omega \times X \to CL(X)$ is called a multivalued random operator if for every $x \in X$, $T(., x)$ is measurable. A mapping $s : \Omega \to X$ is called a measurable selector of a measurable multifunction $T : \Omega \to 2^X$ if $s$ is measurable and $s(\omega) \in T(\omega)$ for all $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to X$ is called a random fixed point of a random multifunction $T : \Omega \times X \to CL(X)$ if $\xi(\omega) \in T(\omega, \xi(\omega))$ for every $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to X$ is called a random coincidence of $T : \Omega \times X \to CL(X)$ and $f : \Omega \times X \to X$ if $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for every $\omega \in \Omega$.

Theorem 3.1. Let $(X, \leq_d, d)$ be a complete separable partially ordered metric space, $(\Omega, \Sigma)$ be a measurable space and let $F : \Omega \times (X \times X) \to CL(X)$ be a $\Delta$-symmetric mapping, $g : \Omega \times X \to X$ be continuous, such that

(i) $F(., v)$ and $g(., x)$ are measurable for all $v \in X \times X$ and $x \in X$ respectively,

(ii) $F(\omega, .)$ is continuous for all $\omega \in \Omega$.

The function $f : \Omega \times (gX \times gX) \to [0, \infty)$ defined by $f(\omega, (g(\omega, x), g(\omega, y))) = D(g(\omega, x), F(\omega, (x, y))) + D(g(\omega, y), F(\omega, (y, x)))$ for all $x, y \in X$, $\omega \in \Omega$ is lower semi-continuous and there exists a function $\phi : [0, \infty) \to [a, 1)$, $0 < a < 1$ satisfying,

$$\lim_{r \to -r^+} \phi(r) = 1$$

Assume that for any $(\omega, (x, y)) \in \Omega \times \Delta$, there exist $g(\omega, u) \in F(\omega, (x, y))$ and $g(\omega, v) \in F(\omega, (y, x))$ satisfying

$$\sqrt{\phi \left[ f(\omega, (g(\omega, x), g(\omega, u))) \right]} \left[ d(g(\omega, x), g(\omega, u)) \right] + d(g(\omega, y), g(\omega, v)) \leq f(\omega, (g(\omega, x), g(\omega, u)))$$

such that

$$f(\omega, (g(\omega, x), g(\omega, u))) \leq \phi \left[ f(\omega, (g(\omega, x), g(\omega, u))) \right] d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v))$$

If $g(\omega \times X) = X$, then there exist measurable mappings $\zeta, \theta : \Omega \to X$ such that $g(\omega, \zeta(\omega)) \in F(\omega, (\zeta(\omega), \theta(\omega)))$ and $g(\omega, \theta(\omega)) \in F(\omega, (\theta(\omega), \zeta(\omega)))$ for all $\omega \in \Omega$, that is $F$ and $g$ have a coupled random coincidence point.

Proof. Let $\Psi = \{\zeta : \Omega \to X\}$ be a family of measurable mappings. Define a function $h : \Omega \times X \to R^+$ as follows

$$h(\omega, x) = d(\omega, F(\omega, x))$$

Since $x \to F(\omega, x)$ is continuous for all $\omega \in \Omega$, we conclude that $h(\omega, .)$ is continuous for all $\omega \in \Omega$. Also, since $x \to F(\omega, x)$ is measurable for all $x \in X$, we conclude that $h(\omega, .)$ is measurable (see Wanger [25, page 868]) for all $\omega \in \Omega$. Thus $h(\omega, x)$ is the Caratheodory function. Therefore, if $\zeta : \Omega \to X$ is a measurable mapping, then $\omega \to h(\omega, \zeta(\omega))$ is also measurable (see [18]). Now, we shall construct two sequences of measurable mappings $\{\zeta_n\}$ and $\{\eta_n\}$ in $\Psi$, and two sequences $\{g(\omega, \zeta_n(\omega))\}$ and $\{g(\omega, \eta_n(\omega))\}$ in $X$ as follows.

Let $(\omega, (\zeta_0, \eta_0)) \in \Omega \times \Delta$ be arbitrary, then the multifunction $G : \Omega \to CB(X)$ defined by $G(\omega) = F(\omega, (\zeta_0, \eta_0))$ is measurable. From the Kuratowski and Ryll-Nardzewski [11] selector theorem, there are measurable selectors $\lambda_1, \lambda_1 : \Omega \to X$ such that

$$\lambda_1(\omega) = F(\omega, (\zeta_0, \eta_0)) \quad \text{and} \quad \lambda_1(\omega) = F(\omega, (\eta_0, \zeta_0))$$

for all $\omega \in \Omega$. Since $\lambda_1(\omega) \in F(\omega, (\eta_0, \zeta_0)) \subseteq X = g(\omega \times X)$ and $\lambda_1(\omega) \in F(\omega, (\eta_0, \zeta_0))$, then there are $(\omega, (\lambda_1(\omega), \mu_1(\omega))) \in \Omega \times \Delta$ such that

$$g(\omega, \zeta_1(\omega)) = \lambda_1(\omega), \quad g(\omega, \eta_1(\omega)) = \mu_1(\omega).$$

Thus $g(\omega, \zeta_1(\omega)) \in F(\omega, (\zeta_0, \eta_0))$ and $g(\omega, \eta_1(\omega)) \in F(\omega, (\eta_0, \zeta_0))$.

Now by (2) and (3), we have

$$\phi \left[ f(\omega, (g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega)))) \right] \left[ d(g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega))) \right] + d(g(\omega, \eta_1(\omega)), g(\omega, \eta_1(\omega))) \leq f(\omega, (g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega))))$$

and

$$\leq \phi \left[ f(\omega, (g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega)))) \right] d(g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega))) \leq f(\omega, (g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega))))$$

and

$$\leq \phi \left[ f(\omega, (g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega)))) \right] d(g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega))) \leq f(\omega, (g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega))))$$

Since by definition of $\phi$, we have $\phi(f(\omega, (x, y))) < 1$ for each $(\omega, (x, y)) \in \Omega \times (X \times X)$, it follows from (5) and (6).
9f(ω, (g(ω, ζ(ω)), g(ω, η(ω))))
≤ φ \left( f(\omega, (g(\omega, \zeta(\omega)), \frac{d(g(\omega, \zeta(\omega)), g(\omega, \zeta(\omega)))}{g(\omega, \eta(\omega))}) \right)
+ d(\omega, \eta(\omega)) \right).

= \sqrt{\phi \left( f(\omega, (g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right)}
\left( f(\omega, (g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right)
\left( d(g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right)
\left( d(g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right)
\left( d(g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right).

\leq \sqrt{\phi \left( f(\omega, (g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right)}
\left( f(\omega, (g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))) \right).

\text{Since } F \text{ is a } \Delta \text{-symmetric mapping and}
\text{(ω, (ζ0, η0)) ∈ } Ω \times Δ, \text{ we have,}
\text{F(ω, (ζ0(ω), η0(ω)))RF(ω, (η0(ω), ζ0(ω))) implies that}
\text{(ω, (ζ(ω), η(ω))) ∈ } Ω \times Δ. \text{ Similarly, as}
g(ω, ζ1(ω)) ∈ F(ω, ζ0(ω), η0(ω))) \text{ and}
g(ω, η1(ω)) ∈ F(ω, η0(ω), ζ0(ω))) \text{ there are measurable selectors}
g(ω, ζ2(ω)), g(ω, η1(ω)) \text{ of}
\text{F(ω, (ζ(ω), η(ω))) and F(ω, (η1(ω), ζ1(ω))) respectively, we have again from (2) and (3),}
\sqrt{\phi \left( f(ω, (g(ω, ζ(ω)), g(ω, η1(ω))) \right)}
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right).

\text{Hence, we get}
f(ω, (g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( f(ω, (g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right).

\text{with } (ω, (ζ2(ω), η1(ω))) \in Ω \times Δ. \text{ Continuing this process, such that for all } n \in \mathbb{N}, \text{ we have}
\text{(ω, (ζn(ω), ηn(ω))) ∈ Ω \times Δ,}
g(ω, ζn+1(ω)) ∈ F(ω, (ζn(ω), ηn(ω))) \text{ and}
g(ω, ηn+1(ω)) ∈ F(ω, (ηn(ω), ζn(ω))). \text{ Again from (2) and (3), we have}
\sqrt{\phi \left( f(ω, (g(ω, ζ(ω)), g(ω, ηn(ω))) \right)}
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right)
\left( d(g(ω, ζ(ω)), g(ω, η1(ω))) \right).

\text{Now, we shall show that}
f(ω, g(ω, ζn(ω)), g(ω, ηn(ω))) → 0 \text{ as } n \to \infty. \text{ We shall assume that}
f(ω, g(ω, ζn(ω)), g(ω, ηn(ω))) > 0 \text{ for all } n \in \mathbb{N}, \text{ since if}
f(ω, g(ω, ζn(ω)), g(ω, ηn(ω))) = 0, \text{ then we get}
D(g(ω, ζn(ω)), F(ω, ζn(ω), ηn(ω))) = 0, \text{ which implies that}
g(ω, ζn(ω)) ∈ F(ω, (ζn(ω), ηn(ω))) = F(ω, (ζn(ω), ηn(ω))) \text{ and}
D(g(ω, ηn(ω)), F(ω, ηn(ω), ζn(ω))) = 0, \text{ implies that}
g(ω, ηn(ω)) ∈ F(ω, (ηn(ω), ζn(ω))) = F(ω, (ηn(ω), ζn(ω))). \text{ In this case,}
(ζn(ω), ηn(ω)) \text{ is a random couple-d coincidence point of } F \text{ and } g \text{ and the assertion of the theorem is proved. From (9) and } φ(0) < 1, \text{ we deduced that}
\{f(ω, g(ω, ζn(ω)), g(ω, ηn(ω)))\} \text{ is a strictly decreasing sequence of positive real numbers. Therefore, there is some } δ > 0 \text{ such that}
lm f(ω, g(ω, ζn(ω)), g(ω, ηn(ω))) = δ. \text{ (10)}

Now, we will show that } \delta = 0. \text{ Suppose that this is not the case, by (1) and taking the limit on both sides of (9), we have}
\delta \leq \lim_{n \to \infty} \sup_{\frac{f(ω, g(ω, ζn(ω)), g(ω, ηn(ω)))}{g(ω, ηn(ω))}} \delta < \delta,
\text{ a contradiction. Thus, } \delta = 0, \text{ that is}
lm f(ω, g(ω, ζn(ω)), g(ω, ηn(ω))) = 0. \text{ (11)}

Now, we have to show that \{g(ω, ζn(ω))\} \text{ and}
\{g(ω, ηn(ω))\} \text{ are Cauchy sequences in } g(ω \times X) = X. \text{ Suppose that}
\alpha = \lim_{n \to \infty} \sup_{\frac{f(ω, g(ω, ζn(ω)), g(ω, ηn(ω)))}{g(ω, ηn(ω))}} \delta < \delta.
\text{ (12)}
\text{ Then, by (1), we have } \alpha < \frac{1}{2}. \text{ Let } \alpha < q < 1, \text{ then there is some } n_0 \in \mathbb{N}\text{ such that}
\sqrt{\phi(f(ω, g(ω, ζn(ω)), g(ω, ηn(ω))))} < q. \text{ (13)}
\text{ for each } n \geq n_0.
\text{ Thus, from (9) and (13), we have}
\sqrt{\phi(f(ω, g(ω, ζn+1(ω)), g(ω, ηn+1(ω))))} \leq q \sqrt{f(ω, g(ω, ζn+1(ω)), g(ω, ηn+1(ω)))} \text{, (14)}
for each \( n \geq n_0 \). Hence, by induction

\[
\begin{align*}
  f(\omega, g(\omega, \xi_{n+1}(\omega)), g(\omega, \eta_{n+1}(\omega))) \\
  \leq q^{n-1-n_0} \left( f(\omega, g(\omega, \xi_{n}(\omega)), g(\omega, \eta_{n}(\omega))) \right)
\end{align*}
\]

(15)

for each \( n \geq n_0 \). Since \( \phi(t) \geq \alpha > 0 \) for all \( t \geq 0 \), from (8) and (15), we have

\[
\begin{align*}
  \frac{1}{\sqrt{\alpha}} q^{-n} \left( f(\omega, g(\omega, \xi_{n}(\omega)), g(\omega, \eta_{n}(\omega))) \right)
\end{align*}
\]

(16)

From (16) and since \( q < 1 \), we conclude that \( \{g(\omega, \xi_{n}(\omega))\} \) and \( \{g(\omega, \eta_{n}(\omega))\} \) are Cauchy sequences in \( g(\omega \times X) = X \). Now, since \( gX \) is complete, there exist \( \xi'_{0}(\omega), \eta'_{0}(\omega) \in X \) such that

\[
\lim_{n \to \infty} g(\omega, \xi_{n}(\omega)) = g(\omega, \xi'_{0}(\omega))
\]

and

\[
\lim_{n \to \infty} g(\omega, \eta_{n}(\omega)) = g(\omega, \eta'_{0}(\omega)).
\]

As \( g(\omega, \xi'_{0}(\omega)) \) and \( g(\omega, \eta'_{0}(\omega)) \) are measurable, then the functions \( \zeta(\omega), \theta(\omega) \) defined by,

\[
\zeta(\omega) = g(\omega, \xi'_{0}(\omega)) \text{ and } \theta(\omega) = g(\omega, \eta'_{0}(\omega))
\]

Since by assumption \( f \) is lower semi-continuous, so from (11), we get

\[
0 \leq f(\omega, (g(\omega, \zeta(\omega)), g(\omega, \eta(\omega))))
\]

\[
= D(g(\omega, \xi(\omega), F(\omega, (\zeta(\omega), \theta(\omega))))
\]

\[
+ D(g(\omega, \theta(\omega), F(\omega, (\theta(\omega), \zeta(\omega))))
\]

\[
\leq \liminf_{n \to \infty} f(\omega, g(\omega, \zeta_{n}(\omega)), g(\omega, \eta_{n}(\omega))
\]

\[
= 0.
\]

Hence,

\[
D(g(\omega, \xi(\omega), F(\omega, (\zeta(\omega), \theta(\omega))))
\]

\[
= D(g(\omega, \theta(\omega), F(\omega, (\theta(\omega), \zeta(\omega))))
\]

\[
= 0,
\]

which implies that

\[
g(\omega, \zeta(\omega)) \in F(\omega, (\zeta(\omega), \theta(\omega)))
\]

and

\[
g(\omega, \theta(\omega)) \in F(\omega, (\theta(\omega), \zeta(\omega))).
\]

This proves that \( F \) and \( g \) have a coupled random coincidence points. □

**Theorem 3.2:** Let \((X, \preceq, d)\) be a complete separable partially ordered metric space, \((\Omega, \Sigma)\) be a measurable space and let \( F : \Omega \times (X \times X) \to \mathcal{C}(\mathcal{L}(X)) \) be a \( \Delta \)-symmetric mapping, \( g : \Omega \times X \to X \) be continuous, such that

\[ F(\cdot, v) \text{ and } g(\cdot, x) \text{ are measurable for all } v \in X \times X \]

and \( x \in X \) respectively.

\( F(\omega, \cdot) \) is continuous for all \( \omega \in \Omega \).

The function \( f : \Omega \times (gX \times gX) \to [0, +\infty) \) defined by

\[
f(\omega, (g(\omega, x), g(\omega, y))) = D(g(\omega, x), F(\omega, (x, y))) + D(g(\omega, y), F(\omega, (y, x)))
\]

(17)

for all \( x, y \in X, \omega \in \Omega \) is lower semi-continuous and there exists a function \( \phi : [0, \infty) \to [a, 1], 0 < a < 1 \) satisfying

\[
\limsup_{r \to \infty} \phi(r) < 1
\]

(18)

for each \( r \in [0, \infty) \).

Assume that for any \((\omega, (x, y)) \in \Omega \times \Delta\), there exist \( g(\omega, u) \in F(\omega, (x, y)) \) and \( g(\omega, v) \in F(\omega, (y, x)) \) satisfying

\[
\sqrt{\phi(d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v))}
\]

\[
[ d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v))]
\]

\[
\leq D(g(\omega, x), F(\omega, (x, y))) + D(g(\omega, y), F(\omega, (y, x)))
\]

(19)

such that

\[
D(g(\omega, u), F(\omega, (u, v))) + D(g(\omega, v), F(\omega, (v, u)))
\]

\[
\leq \phi(d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v)))
\]

\[
[ d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v))]
\]

(20)

If \( g(\omega \times X) = X \), then there exist measurable mappings \( \zeta, \theta : \Omega \to X \) such that,

\[
g(\omega, \zeta(\omega)) \in F(\omega, (\zeta(\omega), \theta(\omega)))
\]

and

\[
g(\omega, \theta(\omega)) \in F(\omega, (\theta(\omega), \zeta(\omega)))
\]

for all \( \omega \in \Omega \), that is, \( F \) and \( g \) have a coupled random coincidence point.

**Proof:** Replacing \( \phi(f(\omega, (g(\omega, x), g(\omega, y)))) \) with \( \phi(d(g(\omega, x), g(\omega, u)) + d(g(\omega, y), g(\omega, v))) \) and as in the proof of Theorem 3.1, we can construct sequences \( \{g(\omega, \xi_{n}(\omega))\} \) and \( \{g(\omega, \eta_{n}(\omega))\} \) in \( g(\omega \times X) = X \) such that for all \( n \in \mathbb{N} \), we have \( (\omega, (\zeta_{n}(\omega), \eta_{n}(\omega))) \in \Omega \times \Delta \),

\[
g(\omega, \xi_{n+1}(\omega)) \in F(\omega, (\zeta_{n}(\omega), \eta_{n}(\omega)))
\]

and

\[
g(\omega, \eta_{n+1}(\omega)) \in F(\omega, (\eta_{n}(\omega), \zeta_{n}(\omega))).
\]

Now by (19)

\[
\sqrt{\phi(d(g(\omega, \zeta_{n}(\omega)), g(\omega, \xi_{n}(\omega)))) + d(g(\omega, \eta_{n}(\omega)), g(\omega, \eta_{n+1}(\omega)))}
\]

\[
[d(g(\omega, \xi_{n}(\omega)), g(\omega, \zeta_{n+1}(\omega)))]
\]

\[
+ [d(g(\omega, \eta_{n}(\omega)), g(\omega, \eta_{n+1}(\omega)))]
\]

\[
\leq D(g(\omega, \zeta_{n}(\omega)), F(\omega, (\zeta_{n}(\omega), \eta_{n}(\omega)))) + D(g(\omega, \eta_{n}(\omega)), F(\omega, (\eta_{n}(\omega), \zeta_{n}(\omega))))
\]

and

\[
\frac{1}{\sqrt{\alpha}} q^{-n} \left( f(\omega, g(\omega, \xi_{n}(\omega)), g(\omega, \eta_{n}(\omega))) \right)
\]

(16)
Assume to contrary that $\beta > \delta$. Then, $\beta - \delta > 0$ and so from (23) and (25) there is a positive integer $n_0$ such that

$$
\begin{align*}
&\left\{ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \\
&+ D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega))))
\right\} < \delta + \frac{\beta - \delta}{4}
\end{align*}
$$

and

$$
\beta - \frac{\beta - \delta}{4} < \left\{ d(g(\omega, \zeta_{n+1}(\omega)), g(\omega, \zeta_n(\omega))) \right\}
$$

for all $n \geq n_0$. Then, combining (21), (26) and (27), we get

$$
\begin{align*}
\sqrt{\phi} & \left\{ d(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega))) \\
&+ d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))) \right\} \\
&< \delta + \frac{\beta - \delta}{4}
\end{align*}
$$

for all $n \geq n_0$. Hence, we get

$$
\begin{align*}
\sqrt{\phi} & \left\{ d(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega))) \\
&+ d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))) \right\} \\
&< \delta + \frac{\beta - \delta}{4}
\end{align*}
$$

for all $n \geq n_0$. Put $h = \frac{\beta + 3\delta}{3\beta + \delta} < 1$. Now, form (22) and (28), it follows that

$$
\begin{align*}
&\left\{ D(g(\omega, \zeta_{n+1}(\omega)), F(\omega, (\zeta_{n+1}(\omega), \eta_{n+1}(\omega)))) \\
&+ D(g(\omega, \eta_{n+1}(\omega)), F(\omega, (\eta_{n+1}(\omega), \zeta_{n+1}(\omega))))
\right\} \\
&< h \left\{ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \\
&+ D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega)))) \right\}
\end{align*}
$$

for all $n \geq n_0$. Since, we assume that $\delta > 0$ and $h < 1$, proceeding by induction and combining the above inequalities, it follows that

$$
\begin{align*}
&\left\{ D(g(\omega, \zeta_{n+k_0}(\omega)), F(\omega, (\zeta_{n+k_0}(\omega), \eta_{n+k_0}(\omega)))) \\
&+ D(g(\omega, \eta_{n+k_0}(\omega)), F(\omega, (\eta_{n+k_0}(\omega), \zeta_{n+k_0}(\omega))))
\right\} \\
&< \delta
\end{align*}
$$

for a positive integer $k_0$, which is a contradiction to the assumption that $\beta > \delta$ and so we must have $\beta = \delta$. Now, we shall show that $\beta = 0$. Since
\[ \beta = \delta \]
\[ \leq \left[ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \right] \]
\[ + D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega)))) \]
\[ \leq \left[ d(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega))) \right] \]
\[ + d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega))) \].

S.0, we can read (23) as
\[ \lim \inf \left\{ \frac{d(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega)))}{+d(g(\omega, \eta_n(\omega)), g(\omega, \eta_{n+1}(\omega)))} \right\} = \beta^*. \]

Thus, there exists a subsequence
\[ \left\{ d(g(\omega, \zeta_n(\omega)), g(\omega, \zeta_{n+1}(\omega))) \right\} = \beta_n^* \]

such that
\[ \lim_{n \to \infty} \beta_n^* = \beta^*. \]

Now, by (18), we have
\[ \lim_{\beta_n^* \to \beta^*} \sup \sqrt{\phi(\beta_n^*)} < 1. \quad (29) \]

From (22),
\[ \left[ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \right] \]
\[ + D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega)))) \]
\[ \leq \sqrt{\phi(\beta_n^*)} \left[ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \right] \]
\[ + D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega)))) \].

Taking the limit as \( k \to +\infty \) and using (23), we get
\[ \delta = \lim_{k \to +\infty} \left[ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \right] \]
\[ + D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega)))) \]
\[ \leq (\lim_{k \to +\infty} \sup \sqrt{\phi(\beta_n^*)} \left[ D(g(\omega, \zeta_n(\omega)), F(\omega, (\zeta_n(\omega), \eta_n(\omega)))) \right] \]
\[ + D(g(\omega, \eta_n(\omega)), F(\omega, (\eta_n(\omega), \zeta_n(\omega)))) \].

From the last inequality, if we suppose that \( \delta > 0 \), we get
\[ 1 \leq (\lim_{\beta_n^* \to \beta^*} \sup \sqrt{\phi(\beta_n^*)}) \]

a contradiction with (29). Thus, \( \delta = 0 \). Then from (23) and (24) we have
\[ \alpha = \lim_{\beta_n^* \to \beta^*} \sup \sqrt{\phi(\beta_n^*)} \]

\[ < 1. \]

Once again, proceeding as in the proof of Theorem 3.1, one can prove that \[ \{ g(\omega(\xi_n(\omega)) \} \text{ and } \{ g(\omega(\eta_n(\omega)) \} \in g(\omega \times X) = X \]
and that \( F \) and \( g \) have a coupled random coincidence point. 

4. Random coupled coincidence point by mixed \( g \)-monotone property

Definition 4.1: [13] Let \((X, \leq)\) be a partially ordered set and \( F : X \times X \to X \) and \( g : X \to X \). We say that \( F \)
has the mixed \( g \)-monotone property if \( F \) is monotone \( g \)-non-decreasing in its first argument and is monotone \( g \)-non-increasing in its second argument, that is, for any \( x, y \in X \), \( x_1, x_2 \in X \), \( g(x_1) \leq g(x_2) \)
implies \( F(x_1, y) \leq F(x_2, y) \) and \( y_1, y_2 \in X \), \( g(y_1) \leq g(y_2) \) implies \( F(x, y_1) \geq F(x, y_2) \).

Definition 4.2: [13] Let \((X, \leq)\) be a partially ordered set, \( F : X \times X \to CL(X) \) and \( g : X \to X \) be mappings. We say that the mapping \( F \) has the mixed \( g \)-monotone property, if for all \( x_1, x_2, y_1, y_2 \in X \) with \( g(x_1) \leq g(x_2) \) and \( g(y_1) \geq g(y_2) \), we get for all \( g(u_1) \in F(x_1, y_1) \), there exists \( g(u_2) \in F(x_2, y_2) \) such that \( g(u_1) \leq g(u_2) \)
and for all \( g(v_1) \in F(y_1, x_1) \), there exists \( g(v_2) \in F(y_2, x_2) \) such that \( g(v_1) \geq g(v_2) \).

Lemma 4.3: [15] If \( A, B \in CB(X) \) with \( H(A, B) < \epsilon \), then for each \( a \in A \) there exists an element \( b \in B \) such that \( d(a, b) < \epsilon \).

Lemma 4.4: [15] Let \( \{A_n\} \) be a sequence in \( CB(X) \) and
\[ \lim_{n \to \infty} H(A_n, A) = 0 \] for \( A \in CB(X) \). If \( x_n \in A_n \) and
\[ \lim_{n \to \infty} d(x_n, x) = 0, \] then \( x \in A \).

Let \((X, \leq)\) be a partially ordered set and \( d \) be a metric on \( X \) such that \((X, d)\) is a complete metric space. We define the partial order on the product space \( X \times X \) as:
for \( (u, v), (x, y) \in X \times X \), \((u, v) \leq (x, y)\) if and only if \( u \leq x, v \geq y \). The product metric on \( X \times X \) is defined as:
\[ d((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + d(y_1, y_2) \]
for all \( x_1, y_1 \in X \) \((i = 1, 2)\).

For notational convenience, we use the symbol \( d \) for the product metric as well as for the metric on \( X \).

Now, we prove the following result that provide the existence of a coupled random coincidence point for compatible maps \( F \) and \( g \) in partially ordered metric spaces, where \( F \) is the multi-valued mapping.

Theorem 4.5: Let \((X, \leq, d)\) be a complete separable partially ordered metric space, \((\Omega, \Sigma)\) be a measurable space and \( F : \Omega \times (X \times X) \to CB(X) \) and \( g : \Omega \times X \to X \) be measurable mappings. If \( g : \Omega \times X \to X \) be continuous and there exists \( k \in (0, 1) \) such that
\[ H(F(\omega, (x, y)), F(\omega, (u, v))) \]
\[ \leq \frac{k}{2} d((g(\omega, x), g(\omega, y)), (g(\omega, u), g(\omega, v))) \]
for all \( x, y, u, v \in X \), \( \omega \in \Omega \) for each March-April
Suppose that if 
\( g(\omega, x_i) \leq g(\omega, x_j) \), \( g(\omega, y_i) \leq g(\omega, y_j) \),
for all \( x_i, y_i \in X \) (i = 1, 2), then for all 
\( g(\omega, u_i) \in F(\omega, (x_i, y_i)) \), there exists
\( g(\omega, u_2) \in F(\omega, (x_2, y_2)) \) with \( g(\omega, u_1) \leq g(\omega, u_2) \)
and for all \( g(\omega, v_i) \in F(\omega, (y_i, x_i)) \), there exists
\( g(\omega, v_2) \in F(\omega, (y_2, x_2)) \) with \( g(\omega, v_1) \leq g(\omega, v_2) \)
provided
\[ d((g(\omega, u_1), g(\omega, v_1), (g(\omega, u_2), g(\omega, v_2))) < 1, \]
(i) There exist \( x_0, y_0 \in X \), \( \omega \in \Omega \) and some
\( g(\omega, x_i) \in F(\omega, (x_0, y_0)) \), \( g(\omega, y_i) \in F(\omega, (y_0, x_0)) \)
with \( g(\omega, x_0) \leq g(\omega, x_i) \), \( g(\omega, y_0) \leq g(\omega, y_i) \) such that
\[ d((g(\omega, x_0), g(\omega, y_0), (g(\omega, x_i), g(\omega, y_i))) < 1 \]
for all \( \omega \in \Omega \), that is \( F \) and \( g \) have a coupled random coincidence point.

**Proof:** Let \( \Psi = \{ \zeta : \Omega \rightarrow X \} \) be a family of measurable mappings. Define a function \( g : \Omega \times X \rightarrow R^+ \) as follows
\[ h(\omega, x) = d(\omega, F(\omega, x)) \]
Since \( x \rightarrow F(\omega, x) \) is continuous for all \( \omega \in \Omega \), we conclude that \( h(\omega, \cdot) \) is continuous for all \( \omega \in \Omega \). Also, since \( x \rightarrow F(\omega, x) \) is measurable for all \( x \in X \), we conclude that \( h(\omega, \cdot) \) is measurable (see Wanger [25, page 868]) for all \( \omega \in \Omega \). Thus \( h(\omega, x) \) is the Caratheodory function. Therefore, if \( \zeta : \Omega \rightarrow X \) is a measurable mapping, then \( \omega \rightarrow h(\omega, \zeta(\omega)) \) is also measurable (see [18]). Now, we shall construct two sequences of measurable mappings \( \{ \zeta_n \} \) and \( \{ \eta_n \} \) in \( \Psi \), and two sequences
\[ \{ g(\omega, \zeta_n(\omega)) \} \) and \( \{ g(\omega, \eta_n(\omega)) \} \) in \( X \) as follows.
Let \( \zeta_0, \eta_0 \in \Psi \) such that by (ii), there exist
\( g(\omega, \zeta_1(\omega)) \in F(\omega, (\zeta_0(\omega), \eta_0(\omega))) \),
\( g(\omega, \eta_1(\omega)) \in F(\omega, (\eta_0(\omega), \zeta_0(\omega))) \) with 
\[ g(\omega, \zeta_0(\omega)) \leq g(\omega, \zeta_1(\omega)) \]
and 
\[ g(\omega, \eta_0(\omega)) \geq g(\omega, \eta_1(\omega)) \) such that
\[ d\left( \begin{array}{c} g(\omega, \zeta_0(\omega)), g(\omega, \eta_1(\omega)) \\ g(\omega, \zeta_1(\omega)), g(\omega, \eta_0(\omega)) \end{array} \right) < 1 - k \]. (31)
Since
\[ g(\omega, \zeta_0(\omega)), g(\omega, \eta_0(\omega)) \leq g(\omega, \zeta_1(\omega)), g(\omega, \eta_1(\omega)) \]
using (30) and (31), we have
\[ H\left( F(\omega, (\zeta_0(\omega), \eta_0(\omega))), h(\omega, \zeta_1(\omega)), g(\omega, \eta_1(\omega)) \right) \]
\[ \leq k^2 \frac{d}{2} \left( \begin{array}{c} (g(\omega, \zeta_1(\omega)), g(\omega, \eta_0(\omega)) \\ (g(\omega, \zeta_0(\omega)), g(\omega, \eta_1(\omega))) \end{array} \right) \] (32)
and similarly,
\[ H\left( F(\omega, (\eta_0(\omega), \zeta_1(\omega))), h(\omega, \eta_1(\omega)), g(\omega, \zeta_1(\omega)) \right) \]
\[ \leq k^2 \frac{d}{2} \left( \begin{array}{c} (g(\omega, \eta_0(\omega)), g(\omega, \zeta_1(\omega)) \\ (g(\omega, \eta_1(\omega)), g(\omega, \zeta_1(\omega))) \end{array} \right) \] (33)
Using (i) and Lemma (4.3), we have the existence of 
\( g(\omega, \zeta_2(\omega)) \in F(\omega, (\zeta_1(\omega), \eta_1(\omega))) \) and 
\( g(\omega, \eta_2(\omega)) \in F(\omega, (\eta_1(\omega), \zeta_1(\omega))) \) with
\( g(\omega, \zeta_1(\omega)) \leq g(\omega, \zeta_2(\omega)) \) and 
\( g(\omega, \eta_1(\omega)) \geq g(\omega, \eta_2(\omega)) \)
\[ d((g(\omega, \zeta_1(\omega)), g(\omega, \zeta_2(\omega))) \leq \frac{k^2}{2} (1 - k) \] (34)
and
\[ d((g(\omega, \eta_1(\omega)), g(\omega, \eta_2(\omega))) \leq \frac{k^2}{2} (1 - k) \] (35)
From (34) and (35), we have 
\[ H\left( F(\omega, (\zeta_1(\omega), \eta_1(\omega))), F(\omega, (\zeta_2(\omega), \eta_2(\omega))) \right) \]
\[ \leq \frac{k^2}{2} (1 - k) \] (36)
and
\[ H\left( F(\omega, (\eta_1(\omega), \zeta_1(\omega))), F(\omega, (\eta_2(\omega), \zeta_2(\omega))) \right) \]
\[ \leq \frac{k^2}{2} (1 - k) \] (33)
From Lemma (4.3) and (i), we have the existence of 
\( g(\omega, \zeta_3(\omega)) \in F(\omega, (\zeta_2(\omega), \eta_2(\omega))) \) and 
\( g(\omega, \eta_3(\omega)) \in F(\omega, (\eta_2(\omega), \zeta_2(\omega))) \) with 
\( g(\omega, \zeta_2(\omega)) \leq g(\omega, \zeta_3(\omega)) \) and
Let $(X,d)$ be a separable metric space, $(\Omega,\Sigma)$ be a measurable space and $F: \Omega \times (X \times X) \to CB(X)$ and $g: \Omega \times X \to X$ be mappings. We say that $F$ and $g$ are compatible if

\[
\lim_{n\to\infty} \left( g(\omega, F(\omega, (x_n, y_n))), F(\omega, (g(\omega, x_n), g(\omega, y_n))) \right) = 0
\]

and

\[
\lim_{n\to\infty} \left( g(\omega, F(\omega, (y_n, x_n))), F(\omega, (g(\omega, y_n), g(\omega, x_n))) \right) = 0,
\]

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$, such that

\[
\lim_{n\to\infty} F(\omega, (x_n, y_n)) = \lim_{n\to\infty} g(\omega, x_n) = x
\]

and

\[
\lim_{n\to\infty} F(\omega, (y_n, x_n)) = \lim_{n\to\infty} g(\omega, y_n) = y
\]
\[
\lim_{n \to \infty} F(\omega, (y_n, x_n)) = \lim_{n \to \infty} g(\omega, y_n) = y \text{ for all } \omega \in \Omega \text{ and } \ x, \ y \in X.
\]

**Theorem 4.7:** Let \((X, \preceq, d)\) be a completely separable partially ordered metric space, \((\Omega, \Sigma, \mathbb{P})\) be a measurable space and \(F : \Omega \times (X \times X) \to CB(X)\) and \(g : \Omega \times X \to X\) be measurable mappings. If \(F\) and \(g\) be continuous and compatible mappings, there exist \(k \in (0, 1)\) such that
\[
H\left(F(\omega, (x, y)), \frac{k}{2} \left( (g(\omega, x), g(\omega, y)) \right) \right) \leq d\left( (g(\omega, u), g(\omega, v)) \right)
\]
for all \((x, y, u, v) \in X\), \(\omega \in \Omega\) for which
\[
(g(\omega, x), g(\omega, y)) \preceq (g(\omega, u), g(\omega, v)).
\]
Suppose that
(i) \(g(\omega, x_i) \preceq g(\omega, x_j)\), \(g(\omega, y_i) \preceq g(\omega, y_j)\), for all \(x_i, y_i \in X\) (\(i = 1, 2\)), then for all \(g(\omega, u_i) \in F(\omega, (x, y))\), there exists \(g(\omega, u_i) \in F(\omega, (x, y))\) with \(g(\omega, x_i) \preceq g(\omega, x_j)\) and for all \(g(\omega, v_i) \in F(\omega, (y, x))\), there exists \(g(\omega, v_i) \in F(\omega, (y, x))\) with \(g(\omega, v_i) \preceq g(\omega, v_j)\) provided
\[
d((g(\omega, u), g(\omega, v)), (g(\omega, u), g(\omega, v))) < 1,
\]
(ii) There exist \(x_0, y_0 \in X\), \(\omega \in \Omega\) and some
\(g(\omega, x_i) \in F(\omega, (x_0, y_0))\), \(g(\omega, y_i) \in F(\omega, (y_0, x_0))\) with \(g(\omega, x_i) \preceq g(\omega, x_0)\) and \(g(\omega, y_i) \preceq g(\omega, y_0)\) such that
\[
d((g(\omega, x_0), g(\omega, y_0)), (g(\omega, x_i), g(\omega, y_i))) < 1 - k,
\]
where \(k \in (0, 1)\); (iii) If a non-decreasing sequence \(\{x_n\} \to x\), then \(x_n \leq x\) for all \(n\) and if a non-increasing sequence \(\{y_n\} \to y\), then \(y \leq y_n\) for all \(n\).

If \(g(\omega \times X) = X\) is complete, then there exist measurable mappings \(\zeta, \theta : \Omega \to X\) such that
\[
g(\omega, \zeta(\omega)) \in F(\omega, (\zeta(\omega), \theta(\omega))) \text{ and } g(\omega, \theta(\omega)) \in F(\omega, (\theta(\omega), \zeta(\omega))) \text{ for all } \omega \in \Omega, \text{ that is } F \text{ and } g \text{ have a coupled random coincidence point.}
\]

**Proof:** As in the proof of Theorem 4.5, we obtain the Cauchy sequences \(\{g(\omega, \zeta_n(\omega))\}\) and \(\{g(\omega, \eta_n(\omega))\}\). Since \(X\) is complete and \(g(\omega \times X) = X\), therefore \(\zeta, \theta \in \Psi\) such that \(g(\omega, \zeta_n(\omega)) \to g(\omega, \zeta(\omega))\) and \(g(\omega, \eta_n(\omega)) \to g(\omega, \theta(\omega))\). Since \(F : \Omega \times (X \times X) \to CB(X)\) and \(g : \Omega \times X \to X\) are compatible maps, we have
\[
\lim_{n \to \infty} \left( g(\omega, F(\omega, (\xi_n(\omega), \eta_n(\omega)))) \right) = 0
\]
and
\[
\lim_{n \to \infty} \left( g(\omega, (g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega)))) \right) = 0.
\]
Since \(\{g(\omega, \zeta_n(\omega))\}\) and \(\{g(\omega, \eta_n(\omega))\}\) are sequences in \(X\), such that
\[
\zeta(\omega) = \lim_{n \to \infty} g(\omega, \zeta_n(\omega)) \in \lim_{n \to \infty} F(\omega, (g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega))))
\]
and
\[
\theta(\omega) = \lim_{n \to \infty} g(\omega, \eta_n(\omega)) \in \lim_{n \to \infty} F(\omega, (g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega))))
\]
are satisfied. For all \(n \geq 0\), we have
\[
D(g(\omega, \zeta(\omega)), F(\omega, (g(\omega, \xi_n(\omega)), g(\omega, \eta_n(\omega)))) \leq D(g(\omega, \xi_n(\omega)), g(\omega, F(\omega, (\xi_n(\omega), \eta_n(\omega))))
\]
\[
+ \left( g(\omega, F(\omega, (\xi_n(\omega), \eta_n(\omega)))) \right)
\]
Taking the limit as \(n \to \infty\) and using the fact that \(F\) and \(g\) are continuous. We get from (39) that \(D(g(\omega, \zeta(\omega)), F(\omega, (g(\omega, \xi_n(\omega)), g(\omega, \theta(\omega)))) \to 0\), which implies that, \(g(\omega, \zeta(\omega)) \in F(\omega, (g(\omega, \zeta(\omega)), g(\omega, \theta(\omega))))\).

Similarly from (40), we can prove that \(D(g(\omega, \theta(\omega)), F(\omega, (g(\omega, \theta(\omega)), g(\omega, \zeta(\omega)))) \to 0\), which implies that \(g(\omega, \theta(\omega)) \in F(\omega, (g(\omega, \theta(\omega)), g(\omega, \zeta(\omega))))\). Thus \(F\) and \(g\) have a coupled random coincidence points.

**REFERENCES**


