CONTINUOUS-DILATION DISCRETE-TIME SELF-SIMILAR SIGNALS AND LINEAR SCALE-IN Variant SYSTEMS

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ABSTRACT
In this paper we present a novel model for purely discrete-time self-similar processes and scale-invariant systems. The results developed are based on a new interpretation of the discrete-time scaling (equivalently dilation or contraction) operation which is defined through a mapping between discrete and continuous time. It is shown that it is possible to have continuous scaling factors through this operation even though the signal itself is discrete-time. We study both deterministic and stochastic discrete-time self-similar signals. We then derive the existence conditions of discrete-time deterministically self-similar signals with respect to some specific mappings. Finally, we discuss the construction of discrete-time linear scale-invariant (LSI) system and present results related to white noise driven system models of stochastic self-similar signals. Unlike continuous-time self-similar signals, it is possible to construct a wide class of non-trivial discrete-time self-similar signals.

1. INTRODUCTION
This paper addresses the problem of defining and representing discrete-time self-similar signals and systems.

The study of the discrete-time self-similar processes in this paper is motivated in part by the previous work of Wornell and colleagues [3, 4, 5] in continuous time. They provide formulations involving continuous-time, scale-invariant signals and systems. They also provide a detailed study of such systems for dyadic scale factors. Our paper here provides answers to questions such as: Is it possible to define purely discrete-time, self-similar signals? Are there formulations of discrete-time, scale-invariant systems? How do we provide a definition of dilation or scaling of discrete-time signal that is general enough to provide non-trivial self-similar signals and scale-invariant systems? The answer to the third question holds the key to answering the first two questions. A key result of this paper is the demonstration of the fact that it is possible to define scaling or dilation in such a way that is continuous even though the signal itself is discrete-time. Hereafter, we will use the term scaling exclusively to mean dilation. Using this definition of scaling, we develop definitions and constructions of deterministic and stochastic, discrete-time, self-similar signals and discrete-time scale-invariant systems.

2. SCALING IN DISCRETE-TIME

2.1. Discrete-Time Scaling Operation
Generally the scaling or dilation operation of a discrete-time signal \( x(n) \) by an arbitrary factor is not well defined. It is difficult to obtain an interpretation of scaling in the discrete-time domain that is as unambiguous as that in the continuous-time domain. Operations such as upsampling, interpolation, downsampling and fractional sampling rate alteration [2] can have a scaling interpretation. However, such operations cannot handle scaling factors over a continuum. We present here a different approach to discrete-time scaling that can handle continuous scaling factors. We define the discrete-time scaling operation in a way that effectively amounts to converting \( x(n) \) into a continuous-time signal through an invertible mapping, applying the scaling operation to the continuous-time signal and finally inverse mapping the signal back to the discrete-time domain. The actual definition is based on operations in the frequency domain.

Let \( f(t) \) be a continuous-time signal and \( F(\Omega) \) its Fourier transform:

\[
F(\Omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t) e^{-j\Omega t} dt, \quad (1)
\]

where \( -\infty < \Omega < +\infty \). If \( f(t) \) is scaled by \( a \) (\( a > 0 \)), its
Fourier transform becomes
\[ \mathcal{F}\{f(t/a)\} = aF(a\Omega), \quad -\infty < \Omega < +\infty. \] (2)
Thus, for a continuous-time signal, a scaling in time can be accomplished in principle by a frequency-scaling of its Fourier transform in the opposite direction along with an amplitude scaling. Now, consider a discrete-time sequence \( x(n) \) whose Fourier transform is
\[ X(\omega) \equiv \mathcal{G}\{x(n)\} = \sum_{n} x(n)e^{-j\omega n}. \] (3)
The function \( X(\omega) \) is 2\( \pi \)-periodic. If we try to define a discrete-time scaling operation by adapting (2) to (3), it will only work for integer values of \( a \) because of the 2\( \pi \)-periodicity requirement on the Fourier transform of a discrete-time signal. This corresponds to upsampling the discrete-time signal by an integer factor of \( a \). The implementation of our discrete-time continuous scaling operation is as follows (see Figure 1).

1. Given is a discrete-time signal \( x(n) \) with the (2\( \pi \)-periodic) Fourier transform \( X(\omega) \).
2. Map the principal interval \( \omega \in [-\pi, \pi] \) to continuous frequency \( \Omega \) (the real line) through an invertible transformation \( \Omega = f(\omega) \).
3. Dilate \( Y(\Omega) \equiv X(f^{-1}(\Omega)) \) by the required dilation factor \( a \) to form \( Y_a(\Omega) \equiv aY(\alpha \Omega) \).
4. Form \( X_a(\omega) = Y_a(f[\omega]) \)
5. The sequence \( x_a(n) \) resulting from the inverse Fourier transformation of \( X_a(\omega) \) is the continuous dilatation of \( x(n) \) by \( a \)
\[ x_a(n) = aG^{-1}\{X[f^{-1}(a f(\omega))]\}. \] (4)
where \( G^{-1} \) denotes inverse discrete-time Fourier transform.
Some examples of \( f(\omega) \) (\( \omega \in [-\pi, \pi] \)) are:
- Bilinear transform. \( \Omega = f(\omega) = 2\tan(\omega/2) \).
- \( 1/\omega \)-based transform. \( \Omega = f(\omega) = \frac{\pi}{\pi-|\omega|} \).
- log-based transform. \( \Omega = f(\omega) = \text{sgn}(\omega) \ln \left( \frac{\pi}{|\omega|} \right) \).

### 2.2. System Properties
Let \( S_a \) denote the discrete-time scaling operator defined above. It is straightforward to verify that \( S_a \) has the following properties:
1. \( S_a \) is a linear operator.
2. \( S_a \) \((a \neq 1)\) is a time-varying operator.
3. \( S_1\{x(n)\} = x(n) \) as expected. This corresponds to the non-scaling case.
4. The inverse operator \( S_a^{-1}\{x(n)\} = S_{1/a}\{x(n)\} \) is discrete-time scaling operation with parameter \( 1/a \).
5. Commutativity
\[ S_a\{S_b\{x(n)\}\} = S_b\{S_a\{x(n)\}\} = S_{ab}\{x(n)\} \] (5)
6. If the discrete-time Fourier spectrum in the principal interval \([−\pi, \pi]\) of an input discrete-time signal is a function of \( f(\omega) \), i.e.,
\[ X(\omega) = T[f(\omega)], \] (6)
and the function \( T(\omega') \) satisfies
\[ T(a\omega') = C(a)T(\omega'), \] (7)
where \( C(a) \) is a function of \( a \), then the output of the discrete-time scaling operator is
\[ S_a\{x(n)\} = aC(a)x(n). \] (8)

Property 6 provides some interesting insights into the discrete-time scaling operation. It implies that if the inverse Fourier transform of the function \( T[f(\omega)] \) exists, the corresponding time sequence represents an eigen-function of the system. Also, when the input spectrum satisfies (6) and (7), for example,
\[ T(\omega') = \omega'^r \] and hence \( X(\omega) = T[f(\omega)] = [f(\omega)]^r, \) (9)
the output spectrum is identical to the input within an amplitude factor \( aC(a) \) \((a^{r+1} \) in the example). In other words, the signal is identical to a scaled version of itself within an amplitude factor.

### 3. DISCRETE-TIME SELF-SIMILAR SIGNALS

#### 3.1. Self-Similarity
Two types of self-similar signals will be discussed in this paper: deterministic and stochastic.
Definition: A discrete-time sequence \( x(n) \) is deterministically self-similar or homogeneous with degree \( H \) if it satisfies the following relation:
\[
S_a\{x(n)\} = a^{-H}x(n)
\]
for any \( a > 0 \). A random process \( X(n) \) is said to be statistically self-similar with degree \( H \) if it satisfies the following equation
\[
S_{a,a}\{R_X(n, n')\} = a^{-2H}R_X(n, n')
\]
for any \( a > 0 \), where \( R_X(n, n') \) denotes the auto-correlation function of sequence \( X(n) \), and \( S_{a,a}\{x(m, n)\} \) for a 2-D function \( x(m, n) \) is defined in lines similar to that of \( S_a \). However, the scaling operation is applied on both \( m \) and \( n \) dimensions.

3.2. Discrete-Time Homogeneous Signal

As mentioned in section 2.2, the time sequence corresponding to inverse Fourier transform of function \([f(\omega)]^r\), if exists, satisfies (10) with \( H = -(r + 1) \). Thus, by choosing a function \([f(\omega)]^r\) which is absolutely integrable in \(-\pi \) to \( \pi \), we can derive a class of discrete-time homogeneous functions. This class of homogeneous functions could provide a model for discrete-time self-similar process in practice. They also serve as eigen-functions of the discrete-time scaling operator previously defined.

As we know, the class of continuous-time, regular, homogeneous functions such as \( f(t) = 1 \) is limited. Truly continuous homogeneous signals corresponding to the spectrum \( \Omega' \) do not exist because it is not a valid Fourier spectrum. In our formulation of discrete-time self-similar functions, we are able to derive purely discrete-time sequences as long as \([f(\omega)]^r\) defines a valid discrete-time Fourier spectrum. Non-trivial discrete-time homogeneous functions actually exist and can be derived in the following ranges of \( r \) parameter with respect to different mappings.

- Bilinear Transform. \(-1 < r < 1.\)
- \(1/\omega\)-Based Transform. \(-1 < r < 1.\)
- \(\log\)-Based Transform. \(r \neq -1, -2, -3, \ldots\)

Figure 2 shows some examples of discrete-time deterministic self-similar functions which are derived from discrete-time Fourier spectrum \([f(\omega)]^{1/2}\). \(f(\omega)\) is chosen as bilinear transform, \(1/\omega\)-based and \(\log\)-based transform respectively.

4. DISCRETE-TIME LINEAR SCALE-INVARIANT SYSTEMS AND SELF-SIMILAR FUNCTIONS

4.1. Discrete-Time Linear Scale-Invariant System

A linear scale-invariant (LSI) system is a linear system whose output is invariant to the scale changes of the input. Let }
4.2. Discrete-time Statistically Self-Similar Signal

As mentioned in [1, 3], most physical processes that exhibit statistical self-similarity are fundamentally non-stationary. The statistical properties of the signal change with time, but remain invariant with time scale. In this section we provide a model for such non-stationary self-similar random processes using the discrete-time LSI system. Our implementation in discrete-time domain is based on the following property of the discrete-time LSI system.

**Theorem:** If the input sequence of a discrete-time LSI system is discrete-time zero-mean white noise, the output sequence of the system is non-stationary and statistically self-similar which satisfies condition (11) with $H \neq -1 / 2$.

**Proof:** See [6].

Hence we can construct a non-stationary self-similar random signal with parameter $H = -1$ by passing a discrete-time zero-mean white noise through the discrete-time LSI system. By passing the signal thus obtained through the system again, a non-stationary self-similar random signal with parameter $H = -2$ is then acquired. Following this scenario, we are able to formulate a non-stationary self-similar random process with parameter $H$ being an arbitrary negative integer. Note that the choice of the one dimensional kernel $h(k)$ in our discrete-time LSI system is essentially arbitrary. We can choose a specific kernel $h(k)$ so that the output of the system exhibits the properties of the studied physical self-similar processes. As there is no restriction on the length of the kernel, a rich class of existing FIR or IIR filters can be applied to model the behavior of a large variety of self-similar random processes in practice.

Figure 4 demonstrates the effect of passing a discrete-time zero-mean white noise through a discrete-time LSI system. The output is a discrete-time stochastic self-similar signal. As is known, if the system output is wide-sense stationary, the 2-D plot of auto-correlation function of the output signal will consist of a series of diagonal straight contour. The auto-correlation plot in Figure 4 clearly demonstrate the non-stationary property of the output signal.

5. CONCLUSION

In this paper we present a novel model for discrete-time self-similar processes based on a new discrete-time continuous-dilation operation. This model can be viewed as an approach to study the properties of a uniformly sampled self-similar signal. The discrete-time LSI system based on this discrete-time scaling operation provides a potential tool for the analysis and simulation of natural self-similar processes because of its scale-invariant property and flexibility in the choice of one dimensional kernel. It can be emphasized that the study of discrete-time LSI systems deserves further investigation and could contribute to deeper insight into fractals or wavelet applications.

Figure 4: Simulation of passing a zero-mean white noise through discrete-time LSI system. The 6-tap Hanning window is used as 1-d kernel for the discrete-time LSI system. (a) system input (b) system output (c) auto-correlation function of the output (d) contour plot of the auto-correlation function of the output

6. REFERENCES


