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Abstract—In the framework of automated manufacturing systems (AMS), Petri nets are widely used to model, analyze, and control them. Resolving deadlocks is of paramount significance because their emergence may zero a system's throughput. Supervisory control technique is the most widely adopted method to resolve them. A control policy can be converted into satisfying a set of inequalities, each of which corresponds to a siphon in a Petri net structure. The number of siphons can be exponential in the worst case, so does the number of inequalities. Taking into account the independent and dependent inequalities, this paper proposes a method to remove all the dependent inequalities while preserving only the independent ones. This method can significantly reduce the size of a supervisory controller. Examples are presented to illustrate the effectiveness and efficiency of this method.

Note To Practitioners—Owing to its importance in practice, deadlock arouses interest. Various resolution methods have been proposed by researchers and practitioners. The recent advance is to associate various sensors and actuators to certain pre-determined working nodes for the sake of management and coordination of limited resources among concurrent operations. This uses a supervisory controller to a plant model. Nevertheless, existing approaches suffer from structural complexity due to their language-based paradigm. It guarantees that all the processes can be executed to their termination [1]–[5], [12], [13], [16], [17], [19]–[21], [24]–[26], [31]. Sophisticated software is involved to diagnose the malfunction, inspect the product quality, and monitor a production line. In a fully automated facility, any human activity is excluded to minimize the cost. Its dynamics is featured by a series of activities driven by the asynchronous occurrences of discrete events. Differential and difference equations become inapplicable despite their wide application to time-driven systems [23], [24], [28], [38]. Petri nets prove to be one of the most powerful and popular mathematical tools to tackle such event-driven systems owing to their formalism and compactness [3], [4], [6], [7], [9]–[11], [14], [16], [22], [27], [29], [30], [32]–[35], [37].

To fulfill specifications, either by intrinsic default or by external imposition, feedback control technique must be developed for such systems [28], [38]. A plant model that induces undesirable behavior must be controlled [28], [38]. In order to change its original behavior, one must design a supervisor, thereby leading to a so-called supervisory control technique (SCT). In the spirit of SCT, a supervisor must constrain the behavior of systems into the legal or admissible domain by forbidding their illegal one [38]. As an elegant supervisor, a premise is the separation between the plant model from the supervisor. Some formal methods, such as automata, are of difficulty to achieve so owing to their language-based paradigm. Petri nets prove to be promising as they can represent both of them in a compact and split way [23], [24], [28], [38].

Unlike automata, Petri nets provide abundant structure information with regard to a modeled system. Through capturing such information, one can synthesize the supervisor in a linear-algebraic way. In [15], the concept of generalized mutual exclusion constraints (GMEC) is proposed to limit the weighted sum of tokens in a subset of places. Each GMEC corresponds to a linear supervisory specification which can be implemented by a control place (monitor, in short) and its related flows to some controllable transitions. In terms of SCT, each GMEC determines a set of forbidden states while each monitor specifies the control mechanism deciding which transition to fire and which one to forbid at each state. An optimal supervisor should avoid the occurrence of all these forbidden states without intervening other good ones.

In the cases of modeling an AMS with Petri nets, it is not surprising that the feature of liveness is critical because it guarantees that all the processes can be executed to their termination [1]–[5], [12], [13], [16], [17], [19]–[21], [24]–[26], [31], [33]. Siphons are a structural object in Petri nets to characterize these deadlock states. To avoid such states, siphons cannot be undermarked [2], [3], [12], [17], [25], [26]. Alternatively, they must be always sufficiently marked during the system evolution, which leads to a typical GMEC problem.
Unfortunately, the number of siphons grows exponentially with regard to the net size. This implies that the number of monitors can be astronomical even for a moderately-sized system. Using inequality analysis, this paper proposes a method to identify and remove the redundant inequalities corresponding to certain GMEC. As a result, the number of monitors can decrease significantly, thus leading to a reduced liveness-enforcing supervisor.

Our approach makes key contributions in the following respects. First, it is applicable to any Petri net models where the GMEC technique can be used. For clarification, our experiments are conducted on some special classes of Petri nets; however, there is actually no limitation upon its applicability. Second, the behavior of the controlled system can be optimal if one removes only the redundant inequalities without influencing the others. Third, it requires only a procedure to solve a list of linear homogeneous systems, which proves to be polynomial in its computational complexity.

This paper is structured as follows. Section II reviews the basic definitions and notations of Petri nets used throughout this paper. Section III is devoted to a special class of Petri nets. Some novel approach is presented based upon their properties for the supervisory control purpose. In Section IV, with the aid of inequality analysis, our method is proposed to distinguish independent and dependent inequalities and show their use in simplifying a supervisor. Section V illustrates an example to verify the effectiveness of our approach. Section VI concludes the paper.

II. PRELIMINARIES

A Petri net is \( \mathcal{N} = (P, T, F, W) \) where \( P \) is a set of places, \( T \) is a set of transitions, \( F \subseteq (P \times T) \cup (T \times P) \) is a set of directed arcs, and \( W : (P \times T) \cup (T \times P) \to \mathbb{N} = \{0, 1, 2, \ldots\} \) such that \( P \cup T \neq \emptyset, P \cap T = \emptyset, \) \( W(x, y) = 0 \) if \( (x, y) \notin F \). A Petri net is said to be safe if \( \forall x, y \in P \cup T : W(x, y) \neq 0 \Rightarrow W(y, x) = 0. \) The preset of a node \( x \in P \cup T \) is defined as \( \bullet x = \{ y \in P \cup T \mid (y, x) \in F \}. \) Its postset \( \bullet^* x = \{ y \in P \cup T \mid (x, y) \in F \}. \) \( \mathcal{N} \) is a state machine if \( W : F \to \{1\} \) and \( \forall \tau \in T, [\tau] = [\star] = 1. \) It is a marked graph if \( W : F \to \{1\} \) and \( \forall \tau \in P, [\tau] = [\star] = 1. \) Its incidence matrix \( [N^+] = [W(p_i, t_j)] \) and output one \( [N^+] = [W(t_j, p_i)]. \) Its incidence matrix \( [N] = [N^+] - [N^-]. \) \( [N^+] \) (resp., \( [N^-], [N^*] \)) is the \( i \)-th row of \([N]\) (resp., \( [N^+], [N^-] \)).

A marking of \( \mathcal{N} \) is a mapping \( M : P \to \mathbb{N}. \) \( (N, M_0) \) is a net system with an initial marking \( M_0, t \) is enabled at \( M, \) denoted by \( M \models t \). If \( \forall \tau \in \bullet^* x, M(p) \geq W(p, t) \), \( M \) is reachable from \( M_0, \) denoted by \( M \models [\sigma] M' \), if there exists a firing sequence \( \sigma = \{ t_1 t_2 \ldots t_n \} \) such that \( M \models [t_1] M_1 \models [t_2] M_2 \ldots \models [t_n] M'. \) \( \sigma \) is a \( |T| \)-dimensional firing count vector where \( \sigma(t) \) states the number of \( t \)'s appearances in \( \sigma. \) The set of all markings reachable from \( M_0 \) is denoted by \( R(N, M_0). \) \( (N, M_0) \) is bounded if \( \exists \sigma \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}, \forall M \in R(N, M_0), \forall \tau \in P, M(p) \leq k, \) \( M_0 \) is live if \( \exists M \in R(N, M_0) \) where \( \forall \tau \in T, M(p) \leq k. \) \( M_0 \) is deadlock-free if \( \forall M \in R(N, M_0), \exists \tau \in T, M \not\models t. \) It is livelock if it is deadlock-free and \( \exists \tau \in T \) so that \( t \) is dead. \( (N, M_0) \) is live if \( \forall \tau \in T, M \models t \) is live under \( M_0. \)

A \( P_\cdot \) (resp., \( T_\cdot \)) vector is a column vector \( I : P \) (resp., \( J : T \)) indexed by \( P \) (resp., \( T \)), where \( P \) is the set of integers. A \( P \)-vector \( I \neq 0 \) becomes a \( P \)-invariant if \( [N]^T \cdot I = 0, \) where \( 0 \) means a vector of zeros. A \( P \)-invariant is called a \( P \)-semimififf \( I \geq 0, ||I|| = \{ p \in P \mid (I(p) \neq 0) \} \) is called the support of \( I. \) For economy of space, \( \sum_{p \in P} M(p) \cdot p \) (resp., \( \sum_{p \in P} I(p) \cdot p, \sum_{t \in T} J(t) \cdot t \) is used to denote vector \( M \) (resp., \( I, J \)). \( (N, M_0) \) is conservative (resp., consistent) if \( \exists \exists I > 0 \) (resp., \( \exists \exists J > 0 \) so that \( I^T \cdot [N] = 0^T \) (resp., \( [N] \cdot J^T \) = 0). A circuit is an ordered set \( \{ x_1, x_2, \ldots, x_n \} \) such that: \( 1) \{ x_1, x_2, \ldots, x_n \} \subseteq P \cup T, 2 \forall i \in N_{n-1} = [1, 2, \ldots, n-1], x_{i+1} \in x_i^*, 3 \forall \{ i, j \} \subseteq N_{n} \) exist, \( i, j \not\in \{ 1, n \}, x_i \neq x_j, 4) x_1 = x_n. \)

A nonempty set \( S \subseteq P \) (resp., \( Q \subseteq P \)) is a siphon (resp., trap) if \( S \subseteq S^* \) (resp., \( Q \subseteq Q^* \)). A strict minimal siphon is a siphon containing neither other siphon nor trap. \( (p) \) indicates the number of tokens in \( p \) at \( M. \) \( p \) is marked by \( M \) if \( (p) > 0. \) The sum of tokens in \( S \) is denoted by \( M(S) \), where \( M(S) = \sum_{p \in S} M(p). \) A subset \( S \subseteq P \) is marked by \( M \) if \( M(S) > 0. \) A siphon is undermarked if \( \exists \tau \in S^* \) can fire.

III. PETRI NET MODELING OF AMS

For better understanding, we focus throughout this paper on a special class of Petri nets, namely, System of Sequential Systems with Shared Resources (\( S^4R \)). Nevertheless, this does not necessarily mean the applicability limitation of our proposed method. In fact, it can be used in more general systems without an extension. In such kind of systems, various job types are modeled by state machines. The availability of various resources is modeled by resource places. Since there is no special limitation upon the resource quantity and types at each operation stage, \( S^4R \) models a general resource allocation mechanism. Although job routes in \( S^4R \) are constrained by a state machine, this does not mitigate its modeling capability because any net model can be converted to a state machine with its reachability graph. Fig. 1 shows the conversion process from a marked graph to a state machine, where the net in Fig. 1(a) denotes a marked graph while the net in Fig. 1(b) denotes a state machine. Compared with the latter, the former is assumed to be able to model operations like disassembly and assembly. Specifically, places of net in Fig. 1(b), i.e., \( p_1, p_2, p_3, p_4, \) and \( p_5, \) correspond to the markings of the net in Fig. 1(a), i.e., \( M_0 = p_1, M_1 = p_2 + p_3, M_2 = p_3 + p_4, M_3 = p_2 + p_5, \) and \( M_4 = p_4 + p_5. \) Specifically, an AMS can be partitioned into a set of resource types \( R = \{ R_i, i = 1, 2, \ldots, L \} \) and a set of process types \( J = \{ J_i, j = 1, 2, \ldots, K \}. \) Every resource type \( R_i \) is further characterized by
its capacity $C_i \in \mathbb{N}$. Processing requirements of process type $\mathcal{F}_j$ are defined by a set of concurrent or sequential stages. Each process stage, say $k$, modeled by a place $p_{jk}$ is associated with a conjunctive resource requirement, expressed by an $L$-dimensional vector $a_{p_{jk}}$ with $a_{p_{jk}}[i]$, $i \in \mathbb{N}_L = \{1, 2, \ldots, L\}$, indicating how many units of resource $R_i$ are required to support the execution of the stage denoted by $p_{jk}$. For systems whose each process can be converted to state machines, $S^4R$ can well handle them. Nevertheless, for more complex scenarios, such as free choice multiple reentrant flowlines, they are beyond $S^4R$’s description capability and deserve further investigation. Also, our method is not appropriate to tackle with systems exhibiting fast reconfiguration and unliable resources.

A. $S^4R$ Models

**Definition 1**: An $S^4R$ is a strongly-connected generalized pure Petri net $N = (P, T, F, W)$ where: 1) $P = P_0 \cup P_A \cup P_R$ is a partition such that: a) $P_0$, $P_A$, and $P_R$ are called idle, operation (or activity), and resource places, respectively; b) $P_0 = \bigcup_{i \in \mathbb{N}_K} \{p_{0i}\}$; c) $P_A = \bigcup_{i \in \mathbb{N}_K} P_{Ai}$, where for each $i \in \mathbb{N}_K$, $P_{Ai} \neq \emptyset$, and for each $i$, $j \in \mathbb{N}_K$, $i \neq j$, $P_{Ai} \cap P_{Aj} = \emptyset$; and d) $P_R = \{r_1, r_2, \ldots, r_n\}$, $n > 0$. 2) $T = \bigcup_{i \in \mathbb{N}_K} T_i$, where for each $i \in \mathbb{N}_K$, $T_i \neq \emptyset$, and for each $i$, $j \in \mathbb{N}_K$, $i \neq j$, $T_i \cap T_j = \emptyset$. 3) For each $i \in \mathbb{N}_K$, subnet $N_i = N|_{\{p_{0i}\} \cup P_{Ai} \cup T_i}$ is a strongly connected state machine such that every cycle contains $p_{0i}$. 4) For each $r \in P_R$, $\exists$ a unique minimal $P$-semiflow $X_r \in \mathbb{N}[P]$ such that $\{r\} = \|X_r\| \cap P_R$, $P_0 \cap \|X_r\| = \emptyset$, $P_A \cap \|X_r\| = \emptyset$, and $X_r(r) = 1$ where $\mathbb{N}[P]$ means $|P|$-dimensional vectors whose each component belongs to $\mathbb{N}$. 5) $P_A = \bigcup_{r \in P_R} \{\|X_r\| \setminus \{r\}\}$.

In $N$, $\forall p_{0i} \in P_0$, $M_0(p_{0i})$ indicates the upper bound of the maximum number of products that are allowed to be concurrently manufactured in a process initialized by $p_{0i}$. $\forall p \in P_A$, $M(p) > 0$ means ongoing operations modeled by $p$. $\forall r \in P_R$, $M_0(r)$ denotes the capacity of a resource $r$. From Definition 1, $S^4R$ is evidently conservative and consistent. $S^4R$ can model a set of concurrently-progressing types of parts. For products with the same type, they share the same processing route. The whole model is a composition of different processes through their sharing resources. In $S^4R$, each process exhibits routing flexibility as well as each stage allows multiple-resource acquisition. For more details, please refer to [31] and [33].

**Definition 2**: $M_0$ is an acceptable initial marking in $N$ if (1) $M_0(p_{0i}) \geq 1$, $\forall p_{0i} \in P_0$; (2) $M_0(p) = 0$, $\forall p \in P_A$; and (3) $\forall r \in T$, $M_0(r) \geq X_r(r)$, $\forall r \in P_R$, $\forall p \in P_A$.

Given an arbitrary marking $M \in R(N, M_0)$, a transition $t$ is process-enabled if $M^*(t \cap P_A) > 0$. Note that $M^*(t \cap P_A) = 1$ by definition. Correspondingly, $t$ is resource-enabled by $\forall r \in t \cap P_R$ if $M(r) \geq X(r, t)$. In the rest of this paper, $(N, M_0)$ is an acceptable marked $S^4R$.

**Definition 3**: Let $r \in P_R$ be a resource place in $(N, M_0)$. The set of holders of $r$ is the support of a minimal $P$-semiflow $X_r$ without $r$, i.e., $H(r) = \|X(r)\| \setminus \{r\}$. Clearly, $H(r)$ contains only operation places due to $\|X(r)\| \cap P_R = \{r\}$.

Let $S_R = S \cap P_R$ and $S_A = S \cap P_A$.

**Definition 4**: Let $S$ be a siphon that can be undermarked in $(N, M_0)$. Token takers, denoted by $\bar{H}(S)$, are the places that correspond to the holders of the resources in $S$ but do not belong to $S$.

Suppose $H_{SR} = \bigcup_{r \in S_R} H(r)$. We have $\bar{H}(S) = H_{SR} \setminus S = (H_{SR} \setminus P_A) \setminus S_A$. The following condition for liveness of $S^4R$ is presented in [33]. $(N, M_0)$ is live if $\exists M \in R(N, M_0)$ and an undermarked siphon $S$ such that 1) $\forall r \in S_R$, $M(r) < W(r, t)$; 2) $\forall p \in S_A$, $M(p) = 0$; and 3) $\forall p \in \bar{H}(S)$, $M(p) > 0$.

Thanks to their special structure, $S^4R$ can describe AMS in which each product is manufactured via sequential and/or concurrent manufacturing processes. It is composed of a set of subsets $N_i$, $i \in \mathbb{N}_K$, which are in one-to-one correspondence with a product and its related manufacturing processes. More specifically, each $N_i$ can be decomposed into an acyclic graph and an idle place $p_{0i}$. The operations together with their interactions required by a process are represented by the activity places and transitions involved in the respective acyclic graph of $N_i$. A set of activity places with a same ingoing transition correspond to the initialization of a number of concurrently executed processes, while the ones with a same outgoing transition correspond to an assembly operation. The initial marking of an idle place $p_{0i}$ corresponds to the number of products that are allowed in the system at a time. As a convention, $p_{0i}$ is also designated as the final destination of all finished process instances to model repetitive production. Places in $P_R$ are used to model various resource types. Their marking during the evolution of a Petri net corresponds to the number of available resources in the modeled AMS. In particular, their initial markings define the capacities of the corresponding resource types.

In the sequel, when mentioning $(N, M_0)$, we refer to an acceptably marked $S^4R$. Fig. 2 shows two typical $S^4R$ nets. Obviously, these two Petri nets share the same process nets whereas their resource allocation mechanisms are different. The former allows only single unit of resource acquisition at each operation stage whereas the latter allows multiple ones. Therefore, the latter is more general. Owing to its popularity,
the former is presented as a benchmark.

Under the assumption that \( P_0 = \{ p_1, p_7 \}, P_{A_1} = \{ p_2 - p_6 \}, P_{A_2} = \{ p_8 - p_{10} \}, P_R = \{ p_{11} - p_{13} \} \), Fig. 2 shows two \( S^R \) representing AMS consisting of five resource types \( R_1 - R_5 \) with capacities \( C_1 = C_3 = 2, C_2 = C_4 = C_5 = 1 \), and supporting two job types \( J_1 \) and \( J_2 \). Job type \( J_1 \) (resp., \( J_2 \)) is defined by the set of partially ordered job stages \{ \( p_r - p_{10} \) \} (resp., \{ \( p_r - p_{10} \) \}). The conjunctive resource requirements associated with various job stages in the net in Fig. 2(a) (resp., Fig. 2(b)) are as follows: \( a_{p_2} = p_4, a_{p_3} = p_5, a_{p_4} = p_{11}, a_{p_5} = p_{12} \), \( a_{p_6} = p_{13}, a_{p_7} = p_{13}, a_{p_8} = p_{12}, \) and \( a_{p_{10}} = p_{11} \) (resp., \( a_{p_2} = p_{14}, a_{p_3} = p_{15}, a_{p_4} = 2p_{11}, a_{p_5} = 2p_{11} + p_{12}, a_{p_6} = p_{13}, a_{p_7} = p_{13}, a_{p_8} = 2p_{12}, \) and \( a_{p_{10}} = p_{11} + 2p_{12} \)).

The \( P \)-semiflows corresponding to the resources are: \( I_1 = p_4 + p_{10} + p_{11}, I_2 = p_5 + p_8 + p_{12}, I_3 = p_6 + p_{14}, I_4 = p_2 + p_{14}, \) and \( I_5 = p_3 + p_{15} \). (resp., \( I_1 = 2p_4 + 2p_{14} + p_{11} + p_{12}, I_2 = p_5 + 2p_9 + 2p_{10} + p_{12}, I_3 = p_6 + p_8 + p_{13}, I_4 = p_2 + 2p_{14}, \) and \( I_5 = p_3 + p_{15} \)).

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Evidently, a siphon can be prevented from being under-marked by simply applying a specification $I^T \cdot M \leq M_0(S_R) - \sum_{p \in S} I(p) \cdot (\max_{p^*} - 1) - 1$ to the net. For the sake of brevity, we assume a scalar $b = M_0(S_R) - \sum_{p \in S} I(p) \cdot (\max_{p^*} - 1) - 1$ in the sequel.

Based on Corollary 1, each siphon can be $\max$-controlled through a GMEC denoted by a pair $(l, b)$. The latter can be easily identified through the structural analysis of an $S^4$R system. An $S^4$R system becomes live when all siphons are $\max$-controlled with $(l, b)$. For more detail, please refer to [2], [31], [33]. In the sequel, we assume all siphons are produced with an enumeration method.

Take the net shown in Fig. 2(a) as an example. $S = \{p_5, p_{10} - p_{12}\}$ is a siphon. Evidently, we have $gs = p_4 + p_5 + p_9 + p_{10} + p_{11} + p_{12}$, $hs = p_4 + p_9 + p_c$, and $Is = gs - hS = p_5 + p_{10} + p_{11} + p_{12} - p_c$. As a result, we have $\vartheta S = p_5 + p_{10} + p_{11} + p_{12}$ and $lS = p_4 + p_9$. $M_0(S_R) = M_0(p_{11}) + M_0(p_{12}) = 2 + 1 = 3$. $\sum_{p \in S} I(p) \cdot (\max_{p^*} - 1) = (\max_{p^*} - 1) + (\max_{p^*} - 1) + (\max_{p^*} - 1) + (\max_{p^*} - 1) = 1 - 1 + 1 - 1 + 1 - 1 = 0$. Therefore, $b = M_0(S_R) - \sum_{p \in S} I(p) \cdot (\max_{p^*} - 1) = 3 - 0 - 1 = 2$. After a pair $(l, b)$ is obtained, a monitor $p_c$ along with its outgoing and ingoing arcs can be calculated with the technique in [28]. Tables I and II show the corresponding $S$, $gs$, $hS$, $Is$, $\vartheta$, $l$, and $b$ in the Petri nets shown in Figs. 2(a) and (b), respectively.

### IV. Supervisor Simplification via AMS Analysis

The above analysis shows that the liveness supervision in the framework of Petri nets can be converted to the satisfaction of a set of inequalities. These inequalities constitute a linear system that restricts the behavior of the plant model. However, one must notice that such a set of inequalities is not minimal. In other words, some inequalities may be redundant, implying that some inequalities are dependent on the others. As one inequality corresponds to one monitor, the removal of those dependent ones can reduce the size of the final supervisor. Therefore, it is attractive to identify a way to remove them while preserving the independent ones. The separation between them might not be unique. Some selection techniques should be developed to properly make their identification and distinction.

### A. Identification of Independent and Dependent Inequalities

Suppose $L = [l_1, l_2 \ldots l_n]$ and $B = [b_1, b_2 \ldots b_n]^T$. $L^T \cdot M \leq B$ means $n$ GMECs. Among them, some are dependent on others.

**Definition 10:** Let $L^T \cdot M \leq B$ be a set of inequalities, $M = \{M[i]^T : M \leq b_i, \forall i \in \mathbb{N}_n\}$, and $M_{\mathbb{N}_n \setminus \{k\}} = \{M[i]^T : M \leq b_i, \forall i \in \mathbb{N}_n - \{k\}\}$. $L_k^T \cdot M \leq b_k$ is said to be dependent on other inequalities iff $M = M_{\mathbb{N}_n \setminus \{k\}}$.

**Proposition 3:** An inequality $L_k^T \cdot M \leq b_k$ is dependent on the others if $\min\{b_k - L_k^T \cdot M, M \in M_{\mathbb{N}_n \setminus \{k\}}\} > 0$.

**Proof:** For the necessary part, we have $M = M_{\mathbb{N}_n \setminus \{k\}}$ according to Definition 10. Since $L_k^T \cdot M \leq b_k$ holds in the space determined by $M$ according to the hypothesis of this proposition, it is apparent that $L_k^T \cdot M \leq b_k$ holds in $M_{\mathbb{N}_n \setminus \{k\}}$. This means that $\min\{b_k - L_k^T \cdot M, M \in M_{\mathbb{N}_n \setminus \{k\}}\} > 0$.

For the sufficient part, we have $L_k^T \cdot M \leq b_k$ because of $\min\{b_k - L_k^T \cdot M, M \in M_{\mathbb{N}_n \setminus \{k\}}\} > 0$. This means $M_{\mathbb{N}_n \setminus \{k\}}$ implicitly ensures $L_k^T \cdot M \leq b_k$. Moreover, the combination of $M_{\mathbb{N}_n \setminus \{k\}}$ and $L_k^T \cdot M \leq b_k$ is equivalent to $M$. As a result, we have $M = M_{\mathbb{N}_n \setminus \{k\}}$.

Proposition 3 implies that the dependence of one inequality upon others can be determined by solving a mathematical programming problem:

$$\min b_k - L_k^T \cdot M$$

subject to

$$L_k^T \cdot M \leq b_i, \quad i \in \mathbb{N}_n \setminus \{k\}$$

**Corollary 2:** An inequality $L_k^T \cdot M \leq b_k$ is independent iff $\min\{b_k - L_k^T \cdot M, M \in S_{\mathbb{N}_n \setminus \{k\}}\} < 0$.

**Proof:** Owing to its duality to Proposition 3, this statement holds obviously.

Consider the three inequalities in Fig. 2(a). To verify one’s dependency upon the other two, we can have the following three mathematical programming formulations.

For $M(p_4) + M(p_9) \leq 2$, we have

$$\min 2 - M(p_4) - M(p_9)$$

subject to

$$M(p_5) + M(p_8) \leq 2$$

$$M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4$$

### TABLE I

<table>
<thead>
<tr>
<th>$i$</th>
<th>$S_i$</th>
<th>$g_S$</th>
<th>$h_S$</th>
<th>$l_i$</th>
<th>$b_i$</th>
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<tbody>
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<td>${p_5, p_{10}, p_{11}, p_{12}}$</td>
<td>$p_4 + p_5 + p_9 + p_{10} + p_{11} + p_{12}$</td>
<td>$p_4 + p_9 + p_c$</td>
<td>$p_4 + p_9$</td>
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<tr>
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<td>$p_5 + p_8 + p_c$</td>
<td>$p_5 + p_8$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>${p_6, p_{10}, p_{11}, p_{12}, p_{13}}$</td>
<td>$p_4 + p_5 + p_6 + p_9 + p_{10} + p_{11} + p_{12} + p_{13}$</td>
<td>$p_4 + p_5 + p_8 + p_c$</td>
<td>$p_4 + p_5 + p_8 + p_9$</td>
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### TABLE II

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<th>$g_S$</th>
<th>$h_S$</th>
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<th>$b_i$</th>
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</thead>
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<tr>
<td>2</td>
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<td>$p_5 + p_8 + p_c$</td>
<td>$p_5 + p_8$</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>${p_6, p_{10}, p_{11}, p_{12}, p_{13}}$</td>
<td>$2p_4 + p_5 + p_9 + 2p_{10} + p_{11} + 2p_{12} + p_{13}$</td>
<td>$2p_4 + 3p_5 + p_8 + 2p_9 + p_c$</td>
<td>$2p_4 + 3p_5 + p_8 + 2p_9$</td>
<td>7</td>
</tr>
</tbody>
</table>
For $M(p_2) + M(p_8) \leq 2$, we have
\[
\min 2 - M(p_5) - M(p_8)
\]
subject to
\[
M(p_4) + M(p_9) \leq 2
\]
\[
M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4
\]
For $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4$, we have
\[
\min 4 - M(p_4) - M(p_5) - M(p_8) - M(p_9)
\]
subject to
\[
M(p_4) + M(p_8) \leq 2
\]
\[
M(p_5) + M(p_8) \leq 2
\]
Their solutions are $-2$, $-2$, and $0$, respectively. This implies that $M(p_4) + M(p_9) \leq 2$ and $M(p_5) + M(p_8) \leq 2$ are independent of the remaining inequalities whereas $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4$ is dependent on $M(p_4) + M(p_9) \leq 2$ and $M(p_5) + M(p_8) \leq 2$. However, for the first two sets of formulations, we can substitute $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4$ by $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 2$. Note that when the latter is satisfied, the former is of course satisfied. The objectives become 0, which implies that $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 2$ and $M(p_4) + M(p_8) \leq 2$ can become dependent constraints if we reduce the right-hand scalar of their independent one, i.e., $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 2$. In general, the right-hand-side scalar reduction results a more restrictive constraint that realizes the original one in a less permissive way. On the other hand, this can be well used to reduce the size of a supervisor at the likely sacrifice of behavior permissiveness.

According to the above statement, we need to establish and solve a set of mathematical programming formulations to verify the dependency of each inequality with respect to others. As known, it can be quite time-consuming to solve them. Moreover, in the case that the dependency of some inequalities does not hold, one may resort to decreasing the right-hand scalars with regard to the independent inequalities. Nevertheless, there is no unified principle to follow. In the following, we present a quite efficient method to tackle such issue.

**Theorem 2:** Let $L^T \cdot M \leq B$ be a set of inequalities and $k \in \mathbb{N}_n$, $l^T_k \cdot M \leq b_k$ be dependent on the others if there exist $n - 1$ nonnegative coefficients $\alpha_i, i \in \mathbb{N}_n \setminus \{k\}$ such that $l_k \leq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot l_i$ and $b_k \geq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot b_i$.

**Proof:** For the necessary part, we assume that $l^T_k \cdot M \leq b_k$ is dependent on the others. According to Proposition 3, we have $\exists \{b_k - l^T_k \cdot M, M \in \mathcal{M}_{\mathbb{N}_n \setminus \{k\}}\} \geq 0$. Based on the duality theorem, this is equivalent to the statement $\max\{l^T_k \cdot M - b_k, M \in \mathcal{M}_{\mathbb{N}_n \setminus \{k\}}\} \leq 0$. This implies that there exists an optimal solution $M^*$ such that $l^T_k \cdot M^* \leq b_k$. The dual mathematical programming is $\min\{\sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot b_i, \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot l_i \geq l_k, \forall i \in \mathbb{N}_n \setminus \{k\}\}$ such that $\sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot b_i = l^T_k \cdot M \leq b_k, \sum_{i \in \mathbb{N}_n \setminus \{k\}} \geq l_k$.

For the sufficient part, we have $l_k \leq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot l_i$ and $b_k \geq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot b_i$ according to the hypothesis. For any $M \geq 0$, we have $l^T_k \cdot M \leq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot l^T_i \cdot M \leq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot b_i$. According to Definition 10, we know that $l^T_i \cdot M \leq b_i$ is dependent on other inequalities, i.e., $l^T_i \cdot M \leq b_i$, where $i \in \mathbb{N}_n \setminus \{k\}$.

In Fig. 2(a), $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4$ is dependent on the other two, i.e., $M(p_4) + M(p_9) \leq 2$ and $M(p_5) + M(p_8) \leq 2$. This is because $(M(p_4) + M(p_9)) + (M(p_5) + M(p_8)) \leq (M(p_4) + M(p_5) + M(p_9) + M(p_8))$ and $4 \geq 2 + 2$. $M(p_4) + M(p_9) \leq 2$ is not dependent on $M(p_5) + M(p_8) \leq 2$ and $M(p_4) + M(p_5) + M(p_8) \leq 4$. This is because, despite the fact that $M(p_4) + M(p_5) + M(p_8) + M(p_9)$, we have $2 \leq 4$. An important and interesting issue is that we can replace the scalar 4 with 2 so that $2 \leq 2$ holds, which makes $M(p_4) + M(p_9) \leq 2$ dependent on $M(p_5) + M(p_8) \leq 2$ and $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 4$. The same analysis applies to the dependency of $M(p_4) + M(p_9) \leq 2$ on the other.

In Fig. 2(b), $2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9) \leq 7$ is independent of the other two, i.e., $2M(p_4) + 2M(p_9) \leq 4$ and $M(p_5) + M(p_8) \leq 4$. This is because, although $(2M(p_4) + 2M(p_9)) + 3M(p_5) + M(p_8)) \geq 2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9)$, we have $7 \leq 3 \times 4 + 3$. To make $2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9) \leq 7$ depend, we can decrease 4 to 1 such that $7 \leq 3 \times 1 + 3$. $2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9) \leq 3$ is not dependent on $M(p_5) + M(p_8) \leq 4$ and $2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9) \leq 7$. This is because, despite the fact that $2p_4 + 2p_9 \leq 2p_4 + 3p_5 + p_8 + p_9$, we have $3 \leq 7$. Similarly, we can decrease the right-hand scalar 7 to 3 such that $3 \leq 3$ holds, which makes $2M(p_4) + 3M(p_9) \leq 3$ dependent on $M(p_5) + M(p_8) \leq 4$ and $2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9) \leq 3$. The same analysis applies to the dependency of $M(p_5) + M(p_9) \leq 4$ on the other.

**B. Analysis of Independent and Dependent Inequalities**

It is noticed that the above analysis presents a simple procedure to derive the dependency relationship between one inequality and the others. An interesting issue is that the independent inequalities are actually not unique. As shown by the examples, two independent inequalities may result when different sets of inequalities are determined as the dependent ones. With the aid of a multisite technique, we herein present some structural analysis of these inequalities and their dependency relationship.

**Proposition 4:** Let $L^T \cdot M \leq B$ be a set of inequalities and $k \in \mathbb{N}_n$. If $l^T_k \cdot M \leq b_k$ is dependent on other inequalities, $\|l_k\| \subseteq \{l_i \mid i \in \mathbb{N}_n \setminus \{k\}\} \|l_i\|$.

**Proof:** We can prove it by contradiction. Suppose that $l_k$ is dependent on $l_i$, where $i \in \mathbb{N}_n \setminus \{k\}$ and $\exists p \in \|l_k\|$ so that $t \notin \{l_i \mid i \in \mathbb{N}_n \setminus \{k\}\} \|l_i\|$. Then, we have $l_k(p) \geq 1$ and $\sum_{i \in \mathbb{N}_n \setminus \{k\}} l_i(p) = 0$. As a result, we have $l_k(p) > \sum_{i \in \mathbb{N}_n \setminus \{k\}} l_i(p)$, leading to the fact that $l_k \notin \{l_i \mid i \in \mathbb{N}_n \setminus \{k\}\}$ is true. Both $M(p_4) + M(p_9) \leq 2$ and $M(p_5) + M(p_9) \leq 2$ are dependent on $M(p_4)$.
+ M(p5) + M(p8) + M(p9) \leq 4 because \{p4, p9\} \subseteq \{p4, p5, p8, p9\} and \{p5, p8\} \subseteq \{p4, p5, p8, p9\}. M(p4) + M(p5) + M(p8) + M(p9) \leq 4 is not dependent on either M(p4) + M(p9) \leq 2 or M(p5) + M(p8) \leq 2 because \{p4, p5, p8, p9\} \not\subseteq \{p4, p9\} and \{p4, p5, p8, p9\} \not\subseteq \{p5, p8\}. Nevertheless, M(p4) + M(p5) + M(p8) + M(p9) \leq 4 is dependent on the union of M(p4) + M(p9) \leq 2 and M(p5) + M(p8) \leq 2 because \{p4, p5, p8, p9\} \subseteq \{p4, p9\} \cup \{p4, p5, p8, p9\}. To be concise, the analysis upon the Petri net model in Fig. 2(b) is omitted owing to its similarity to the case in Fig. 2(a).

**Theorem 3:** Let \( L^T \cdot M \leq B \) be a set of inequalities and \( k \in \mathbb{N}_n \), \( T_k \cdot M \leq b_k \) is dependent on other inequalities iff there exist \( n - 1 \) nonnegative coefficients \( \alpha_i, i \in \mathbb{N}_n \setminus \{k\} \) such that \( \|l_k\| = \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot \|l_i\| \) and \( b_k \geq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot b_i \).

**Proof:** This proof is a multisets version of the proof of Theorem 2. Owing to their similarity, it is omitted.

### C. Supervisor Simplification

Suppose \( \mathcal{L} = \{\|l_i\|\}, i \in \mathbb{N}_n \). From the above analysis, the key issue is to identify \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_m\} \subseteq \mathbb{N}_n \) leading to two sets, i.e., \( \mathcal{L}_2 = \{\|l_i\|\} \subseteq \mathcal{L} \) and \( \mathcal{L}_D = \mathcal{L} \setminus \mathcal{L}_2 \) such that: 1) \( \forall \omega_k \in \Omega, \exists \alpha_k \geq 0 \) such that \( l_{\omega_k} = \sum_{i \in \Omega(\omega_k)} \alpha_i \cdot l_i \); and 2) \( \forall \mu \in \mathbb{N}_n \setminus \Omega, \exists \alpha_\mu \geq 0 \), \( l_{\mu} \leq \sum_{i \in \mathbb{N}_n \setminus \{k\}} \alpha_i \cdot l_i \). Obviously, \( \mathcal{L}_D \) is not unique.

**Definition 11:** Let \( \omega \in \Omega, j = \mathbb{N}_n \setminus \Omega, \{\|l_{\omega_1}\|, \|l_{\omega_2}\|, \ldots, \|l_{\omega_j}\|\} \) is called a \( \text{max-} \mathcal{L}_2 \) (resp., \( \text{min-} \mathcal{L}_2 \)) of \( \mathcal{L} \) if \( \forall i, j, \|l_{\omega_i}\| \geq \|l_{\omega_j}\| \) (resp., \( \|l_{\omega_i}\| \leq \|l_{\omega_j}\| \)).

As opposed to \( \mathcal{L} \), we have \( \mathcal{B} = \{b_i\}, i \in \mathbb{N}_n \), \( \mathcal{B}_i = \{b_i\}, i \in \Omega \), and \( \mathcal{B}_D = \{b_i\}, i \in \mathbb{N}_n \setminus \Omega \). \( L_{\mathcal{B}}, B_{\mathcal{B}}, \) and \( B_i \) denote vectors while their corresponding multisets are represented by \( L_{\mathcal{B}}, B_{\mathcal{B}}, \mathcal{B}_i \).

Take the Petri net model in Fig. 2(a) as an example. \( M(p1) + M(p5) + M(p8) + M(p9) \leq 4 \) is a \( \text{max-} \mathcal{L}_2 \) while \( M(p4) + M(p9) \leq 2 \) and \( M(p5) + M(p8) \leq 2 \) are a \( \text{min-} \mathcal{L}_2 \).

Therefore, to reduce the supervisor size, our first step is to identify these dependent inequalities, which can be realized by the following algorithm.

**Algorithm 1:** Identification of Independent and Dependent Inequalities

**Input:** \( L^T \cdot M \leq B \)

**Output:** Independent Inequalities \( L^T_2 \cdot M \leq B_{\mathcal{B}} \) and Dependent Inequalities \( L^T_D \cdot M \leq B_{\mathcal{B}} \) with \( L_{\mathcal{B}} = [l_{\omega_1}, l_{\omega_2}, \ldots, l_{\omega_m}], L_{\mathcal{B}} = [l_{\mu_1}, l_{\mu_2}, \ldots, l_{\mu_m}] \), and \( B_{\mathcal{B}} = [b_1, b_2, \ldots, b_{\mathcal{B}_i}] \) where \( \{\mu_1, \mu_2, \ldots, \mu_{m-n}\} \subseteq \mathbb{N}_n \setminus \{\omega_1, \omega_2, \ldots, \omega_m\} \).

1. \( i = 1, m = 1, L_{\mathcal{B}} := \emptyset, L_{\mathcal{D}} := L, B_{\mathcal{B}} := \emptyset, \) and \( B_{\mathcal{D}} := B \).
2. Arrange all the elements in \( L \) according to the ascending (resp., descending) order of \( \|l_i\| \) for the \( \text{min-} \mathcal{L}_2 \) (resp., \( \text{max-} \mathcal{L}_2 \) ), respectively;
3. while \( (i \leq |L|) \) do
4. Check whether \( \exists \alpha_j \geq 0 \) so that \( l_i \leq \sum_{j=1}^{m} \alpha_j \cdot l_{\omega_j} \). If so, go to Step 5; otherwise, go to Step 6.
5. \( m := m + 1, L_{\mathcal{B}} := L_{\mathcal{B}} \cup \{l_i\}, L_{\mathcal{D}} := L_{\mathcal{D}} \setminus \{l_i\}, B_{\mathcal{B}} := B_{\mathcal{B}} \cup \{b_i\}, \) and \( B_{\mathcal{D}} := B_{\mathcal{D}} \setminus \{b_i\} \);
can achieve the maximal permissiveness with a structurally simplified supervisor, thus avoiding astronomical number of monitors. Most of enforcing supervisor can be synthesized in a quite compact system is controlled by the first two inequalities or all the Fig. 4. The diagram block of an AMS.

<table>
<thead>
<tr>
<th>Method</th>
<th>$t$</th>
<th>$M_0(p_c)$</th>
<th>$p_c$</th>
<th>$p_o^*$</th>
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<tr>
<td>Our Approach</td>
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<td>${t_5, t_{10}}$</td>
<td>${t_3, t_8}$</td>
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<td>Approach in [26]</td>
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<td>1</td>
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<td>${t_6, t_9}$</td>
<td>${t_5, t_8}$</td>
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**TABLE IV**

<table>
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<th>Method</th>
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<tr>
<td>Other Approach</td>
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<td></td>
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<td>${2t_6, 2t_{10}}$</td>
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</tbody>
</table>

For both nets in Fig. 2, the method in [2] leads to 3 monitors, namely, $p_{c1} - p_{c2}$ since there are 3 siphons. These monitors constitute supervisors as shown in Tables III and IV, respectively.

In Fig. 2(a), among 3 siphons, 2 of them, are found to be elementary siphons according to [26]. Thus, 2 monitors are necessary to constitute a supervisor, as shown in Table III. In Fig. 2(b), no siphons are found to be elementary siphons. Therefore, the supervisor remains the same as in [2] and is shown in Table IV.

By contrast, our approach shows that only one monitor is necessary for each Petri net because inequalities $M(p_4) + M(p_5) + M(p_8) + M(p_9) \leq 2$ and $2M(p_4) + 3M(p_5) + M(p_8) + 2M(p_9) \leq 3$ prove to be independent in Figs. 2(a) and (b), respectively. It shows that our approach can reduce the supervisor size more compared to approaches in [2] and [26]. For all supervisors, the net liveness can be achieved after their outgoing transitions move to $p_o^*$ as done in [12].

**V. ILLUSTRATIVE EXAMPLE**

Fig. 4 shows the block diagram of an AMS where three product types, i.e., $J_1 - J_3$, are manufactured. It is composed of three robots $R_1 - R_3$ and four machines $M_1 - M_4$. Each robot can hold two products. Each machine can deal with four products at a time. There are three loading buffers $I_1 - I_3$ and three unloading buffers $O_1 - O_3$ to load and unload the AMS. The action area for $R_1$ is $I_1$, $O_1$, $M_1$, $M_2$, $M_3$, and $M_4$; for robot $R_2$ is $I_2$, $O_3$, $M_1$, and $M_3$; for robot $R_3$ is $I_3$, $O_2$, $M_2$, and $M_4$. By these resources, products $J_1 - J_3$ can be concurrently manufactured. Every arriving raw product belongs to one of these three products. According to the predefined routes, a raw product $J_1$ is taken from $I_1$ by $R_1$. After being processed by $M_2$, it is moved to $O_1$ by $R_1$. A raw product $J_2$ is taken from $I_2$ by $R_2$. Two flexible routes are available for its further treatment. First, it is manufactured in $M_1$ and then moved to $M_2$ by $R_1$. Second, it is manufactured...
in \(M_3\) and then moved to \(M_2\) by \(R_1\). After its process in either \(M_2\) or \(M_5\), it is moved to \(O_2\) by \(R_3\) so that a final product of \(J_2\) is obtained. A raw product of \(J_3\) is taken from \(I_3\) by \(R_3\). After being processed by \(M_4\), it is moved to \(M_3\) by \(R_1\). After that, it is moved to \(O_3\) by \(R_2\). Noticeably, two copies of resources are required when \(J_2\) is processed by \(M_1\) or \(J_3\) is processed by \(M_4\). In Fig. 4, resource allocation is indicated by a directed arc as labeled by \(t_j, j \in \mathbb{N}_{20}\). A number across the arc indicates the resource quantity while the default value is one.

Fig. 5 shows the net model of this AMS, which allows multiple resource acquisitions and flexible routes. The system is an \(S^4R\) where \(P_0 = \{p_1, p_5, p_{14}\}, P_A = \{p_2-p_4\}, P_{A_2} = \{p_6-p_{13}\}, P_{A_3} = \{p_{15} - p_{19}\}, P_R = \{p_{20} - p_{26}\}, t_0 = t_1, t_2 = t_5, \) and \(t_3 = t_{15}\). Places \(p_{20} - p_{26}\) denote \(M_1, M_2, R_1, R_2, M_3, R_4, \) and \(R_5\), respectively. Initially, it is assumed that there are no parts in process. \(M(p_1) = M(p_2) = M(p_{19}) = 8\) represents that the maximum job instances that are allowed for part types \(J_1 - J_3\) at a time, respectively.

<table>
<thead>
<tr>
<th>(i)</th>
<th>(p_{ci})</th>
<th>(p^*_c)</th>
<th>(M(p_{ci}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>({t_6, t_{19}})</td>
<td>({t_5, t_{15}})</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>({t_4, t_{17}})</td>
<td>({t_5, t_{15}})</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>({t_3, t_{10}})</td>
<td>({t_5, t_{15}})</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>({t_6, t_{12}, t_{17}})</td>
<td>({t_5, t_{15}})</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>({t_7, t_{26}, t_{19}})</td>
<td>({t_{25}, t_{15}})</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>({t_6, t_{17}, t_{17}})</td>
<td>({t_5, t_{15}})</td>
<td>2</td>
</tr>
</tbody>
</table>

This net is deadlock-prone since some siphons can be eventually undermarked during system evolution. Our analysis shows that there are 21 siphons, i.e., \(S_1 = \{t_{13}, t_{16}, p_{25}, p_{26}\}, S_2 = \{p_4, p_{13}, p_{19} - p_{26}\}, S_3 = \{p_4, p_{13}, p_{19}, p_{22}, p_{24} - p_{26}\}, S_4 = \{p_4, p_{13}, p_{17}, p_{21}, p_{22}, p_{25}, p_{26}\}, S_5 = \{p_4, p_{11}, p_{12}, p_{19} - p_{25}\}, S_6 = \{p_4, p_{11}, p_{12}, p_{18}, p_{21}, p_{22}, p_{24}, p_{25}\}, S_7 = \{p_4, p_{11}, p_{12}, p_{21}, p_{22}, p_{25}\}, S_8 = \{p_4, p_{10}, p_{13}, p_{19} - p_{24}, p_{26}\}, S_9 = \{p_4, p_{10}, p_{13}, p_{18}, p_{21}, p_{22}, p_{24}, p_{26}\}, S_{10} = \{p_4, p_{10}, p_{11}, p_{19} - p_{24}\}, S_{11} = \{p_4, p_{10}, p_{11}, p_{18}, p_{21}, p_{22}, p_{24}\}, S_{12} = \{p_4, p_{10}, p_{13}, p_{17}, p_{21}, p_{22}, p_{25}\}, S_{13} = \{p_4, p_{10}, p_{13}, p_{17}, p_{21}, p_{22}\}, S_{14} = \{p_2, p_4, p_9, p_{13}, p_{19}, p_{20} - p_{26}\}, S_{15} = \{p_2, p_4, p_9, p_{12}, p_{19}, p_{20} - p_{25}\}, S_{16} = \{p_2, p_4, p_9, p_{13}, p_{18}, p_{22}, p_{24} - p_{26}\}, S_{17} = \{p_2, p_4, p_9, p_{13}, p_{17}, p_{22}, p_{25}\}, S_{18} = \{p_2, p_4, p_9, p_{13}, p_{17}, p_{22}, p_{25}\}, S_{19} = \{p_2, p_4, p_9, p_{12}, p_{17}, p_{22}, p_{25}\}, S_{20} = \{p_2, p_4, p_9, p_{10}, p_{19}, p_{20}, p_{22} - p_{24}\}, \) and \(S_{21} = \{p_2, p_4, p_9, p_{10}, p_{18}, p_{22}, p_{24}\}\).

According to our approach, these siphons lead to 21 inequalities as shown in Table V. Using Algorithms 1 and 2, the second inequality, \(i_2^T \cdot M \leq b_2\), is an independent inequality and also \(\text{max-}\)L\(_2\) after we decrease \(b_2\) from 19 to 4. This is because, \(\forall i \in \mathbb{N}_{21} \setminus \{2\}\), we have \(i_2^T \cdot M \leq b_2\) and \(b_i \geq b_2\). Based on this inequality, we can produce the corresponding monitor as shown in Fig. 6. Our analysis shows that the controlled system is live with 6970 reachable states. For the sake of brevity, \(p^*_c\) is assumed to be \(t_0\) as done in [12].

As \(\text{max-}\)L\(_2\) is quite conservative and the system behavior can be greatly restricted. We also can resort to \(\text{min-}\)L\(_2\), which shows a set of dependent inequalities, i.e., inequalities 1, 12, 13, and 19 - 21. One can easily verify that any other inequality is dependent on these ones. This set of inequalities produce monitors as shown in Table VI. The corresponding controlled system is live with 13960 reachable states.

VI. CONCLUSION

This work focuses on the synthesis of liveness enforcing supervisors of automated manufacturing systems allowing both...
Inequalities (i, M ≤ 6) | Independent
---|---
1 | M(p_1) + M(p_2) + 2M(p_10) ≤ 4
2 | M(p_1) + M(p_2) + M(p_8) + 2M(p_2) + M(p_1) + M(p_10) + M(p_11) + 2M(p_12) + M(p_17) + M(p_18) ≤ 17
3 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) + M(p_18) ≤ 17
4 | M(p_2) + M(p_3) + M(p_9) + M(p_10) + 2M(p_11) + M(p_17) ≤ 12
5 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) ≤ 12
6 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) ≤ 12
7 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) ≤ 8
8 | M(p_2) + M(p_3) + M(p_8) + 2M(p_2) + M(p_10) + M(p_11) + M(p_10) + M(p_17) + M(p_18) ≤ 16
9 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_11) + M(p_17) ≤ 14
10 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) ≤ 9
11 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) + M(p_18) ≤ 7
12 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) + M(p_18) ≤ 5
13 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) + M(p_18) ≤ 5
14 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) + M(p_18) ≤ 15
15 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) + M(p_17) + M(p_18) ≤ 13
16 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) + M(p_18) ≤ 10
17 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + 2M(p_11) ≤ 8
18 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) ≤ 6
19 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) ≤ 4
20 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) ≤ 10
21 | M(p_2) + M(p_3) + M(p_8) + M(p_10) + M(p_17) + M(p_18) ≤ 5

**Table V**

Generated inequalities corresponding to siphons

**Table VII**

Generated monitors for the net in Fig. 5 due to [3]

**References**


Flexible routes and multiple resource acquisition operations. Deadlocks are related to the emergence of a class of Petri net objects, namely undermarked siphons. To avoid their undermarkedness, the number of tokens in these siphons’ complementary places must be strictly limited, thus leading to a set of general mutual exclusive constraints. They can be distinguished as the independent and dependent ones. The dependent ones can be implicitly controlled after these independent ones are properly controlled. This technique can reduce the number of needed monitors. Numerical results show that our resulting supervisor is simple in its structure and can ensure more permissive behavior. The generalization of the research results in this paper will be conducted for more complex systems.

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