Inequalities and bounds for elliptic integrals

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Abstract

Computable lower and upper bounds for the symmetric elliptic integrals and for Legendre’s incomplete integral of the first kind are obtained. New bounds are sharper than those known earlier. Several inequalities involving integrals under discussion are derived.

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1. Introduction and definitions

Elliptic integrals play an important role in the fields of conformal mappings, astronomy, physics and engineering, to mention the most prominent ones. It is well known that they cannot be represented by the elementary transcendental functions. Therefore, there is a need for sharp computable bounds for the family of integrals under discussion.

The goal of this paper is to establish new bounds and inequalities for the standard symmetric elliptic integrals which have been studied extensively for several years by B.C. Carlson and his collaborators (see [12,13,15–18,20,32]) and other researchers (see [21–23]). All members of this family of integrals are homogeneous functions of two or three or four variables and they enjoy the symmetry in two or more variables. Other elliptic integrals discussed in this paper include Legendre integrals. They all can be expressed in terms of the symmetric elliptic integrals.
In what follows, we will assume that \(x, y, z\) are nonnegative numbers and that at most one of them is 0. The symmetric elliptic integral of the first kind is defined by

\[
R_F(x, y, z) = \frac{1}{2} \int_0^\infty \left[ (t + x)(t + y)(t + z) \right]^{-1/2} dt
\]

(see, e.g., [16, (1.1)]). Clearly \(R_F\) is symmetric in all variables, homogeneous of degree \(-\frac{1}{2}\) in \(x, y, z\) and satisfies \(R_F(x, x, x) = x^{-1/2}\).

Let \(p > 0\). The symmetric integral of the third kind

\[
R_J(x, y, z, p) = \frac{3}{2} \int_0^\infty \left[ (t + x)(t + y)(t + z) \right]^{-1/2}(t + p)^{-1} dt
\]

is symmetric in \(x, y, z, p\), homogeneous of degree \(-\frac{3}{2}\) in \(x, y, z, p\) and satisfies \(R_J(x, x, x, x) = x^{-3/2}\) (see, e.g., [16, (1.2)]). A degenerate case of \(R_J\) is the elliptic integral of the second kind

\[
R_D(x, y, z) = R_J(x, y, z, z) = \frac{3}{2} \int_0^\infty \left[ (t + x)(t + y) \right]^{-1/2}(t + z)^{-3/2} dt
\]

which is symmetric in \(x, y\) only. A completely symmetric integral of the second kind

\[
R_G(x, y, z) = \frac{1}{4} \int_0^\infty \left[ (t + x)(t + y)(t + z) \right]^{-1/2} \left( \frac{x}{t + x} + \frac{y}{t + y} + \frac{z}{t + z} \right) t dt
\]

is symmetric and homogeneous of degree \(\frac{1}{2}\) in its variables, satisfies \(R_G(x, x, x) = x^{1/2}\) and is well defined if any or all of \(x, y, z\) are 0 (see, e.g., [16, (1.5)]). All four integrals defined above are the incomplete integrals. Two complete symmetric integrals of the first and the second kind are defined as follows:

\[
R_K(x, y) = \frac{2}{\pi} R_F(x, y, 0)
\]

and

\[
R_E(x, y) = \frac{4}{\pi} R_G(x, y, 0)
\]

(see [15, (9.2-3)]).

An important elementary transcendental function used in this paper, denoted by \(R_C\), is the degenerate case of \(R_F\):

\[
R_C(x, y) = R_F(x, y, y) = \frac{1}{2} \int_0^\infty (t + x)^{-1/2}(t + y)^{-1} dt
\]

(\(x \geq 0, y > 0\)). It is known that

\[
R_C(x, y) = \begin{cases} 
(y - x)^{-1/2} \cos^{-1} \left( \frac{x}{y} \right)^{1/2}, & x < y, \\
(x - y)^{-1/2} \cosh^{-1} \left( \frac{x}{y} \right)^{1/2}, & x > y
\end{cases}
\]

(see [15, (6.9-15)]). Let us note that

\[
R_C(0, y) = \frac{\pi}{2y^{1/2}}.
\]
Other degenerate symmetric elliptic integrals which are used in this paper include
\[ j(x, y, z) = R_J(x, y, y, z) \]  
and
\[ d(x, y) = j(x, x, y) = R_D(y, y, x). \]
For \( x \geq 0, y > 0 \) and \( z > 0 \) both \( j \) and \( d \) can be expressed in terms of \( R_C \):
\[ j(x, y, z) = \begin{cases} \frac{3}{z-y} \left[ R_C(x, y) - R_C(x, z) \right], & y \neq z, \\ \frac{2(x-y)y}{x^{3/2}} \left[ x^{1/2} - y R_C(x, y) \right], & x \neq y = z, \\ x = y = z \end{cases} \]  
(see [23, (2.7)]) and
\[ d(x, y) = \begin{cases} \frac{3}{x-y} \left[ R_C(x, y) - x^{-1/2} \right], & x \neq y, \\ \frac{x^{-3/2}}{x^{-3/2}}, & x = y \end{cases} \]  
(see [23, (2.10)]).

Legendre’s incomplete integral of the first kind is defined as
\[ F(\phi, k) = \int_0^\phi (1 - k^2 \sin^2 \theta)^{-1/2} d\theta, \]  
(1.14)
\[ 0 < \phi \leq \pi/2, \quad k^2 \sin^2 \phi < 1 \] (see [15, (9.3-1)]). It is known that
\[ F(\phi, k) = R_F(c - 1, c - k^2, c), \]  
(1.15)
where \( c = (\sin \phi)^{-2} \) (see [16, (4.5)]). Legendre’s complete elliptic integral of the second kind
\[ E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{1/2} d\theta \]  
(0 < \( k < 1 \)) satisfies
\[ E(k) = \frac{\pi}{2} R_E(1 - k^2, 1) \]  
(1.16)
(see [15, (9.2–14)]).

This paper is a continuation of the earlier work [23] and is organized as follows. In Section 2 we recall definition of the \( R \)-hypergeometric functions. All elliptic integrals defined in this section admit representations in terms of these functions which are defined as integral averages of a power function. This convenient form of representing integrals under discussion is utilized to establish either logarithmic convexity or concavity of these integrals in their variables. Section 3 deals with bounds and inequalities for the incomplete symmetric integrals. New upper bounds for \( R_F, R_J \) and \( R_D \) are obtained. They are sharper than the corresponding bounds established in [23, Theorem 3.2]. Upper bounds for the difference and the quotient of two integrals are also included. Lower and upper bounds for \( R_F(x, y, A) \) (\( x > 0, y > 0, A = (x+y)/2 \)) are also derived. New bounds and inequalities for the complete integrals \( R_K \) and \( R_E \) are presented in Section 4. Bounds for Legendre’s incomplete integral \( F(\phi, k) \) are discussed in Section 5.
2. The $R$-hypergeometric functions and logarithmic convexity or concavity of symmetric integrals

In what follows, we shall employ notation and some definitions introduced in Carlson’s monograph [15]. The symbols $\mathbb{R}_+$ and $\mathbb{R}_>$ will stand for the nonnegative semi-axis and the set of positive numbers, respectively. For $b = (b_1, \ldots, b_n) \in \mathbb{R}_+^n$ and $X = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, the $R$-hypergeometric function of order $-a \in \mathbb{R}$ with parameters $b$ and variables $X$ is defined by

$$R_{-a}(b; X) = \int_{E_{n-1}} (u \cdot X)^{-a} \mu_b(u) \, du,$$  \hspace{1cm} (2.1)

where

$$E_{n-1} = \{(u_1, \ldots, u_{n-1}) : u_i \geq 0, \ 1 \leq i \leq n-1, \ u_1 + \cdots + u_{n-1} \leq 1\}$$

is the Euclidean simplex, $u = (u_1, \ldots, u_{n-1}, u_n)$, where $u_n = 1 - u_1 - \cdots - u_{n-1}$, $u \cdot X = u_1 x_1 + \cdots + u_n x_n$ is the dot product of $u$ and $X$,

$$\mu_b(u) = \frac{1}{B(b)} \prod_{i=1}^{n} u_i^{b_i-1}$$  \hspace{1cm} (2.2)

is the Dirichlet measure on $E_{n-1}$, $B(\cdot)$ stands for the multivariate beta function and $du = du_1 \cdots du_{n-1}$.

Function $R_{-a}$ is also called the Dirichlet average of the power function $t \rightarrow t^{-a}$ ($t > 0$) (see [15, Chapter 6]). We list below some elementary properties of $R_{-a}$:

(i) A vanishing $b$-parameter can be omitted along with the corresponding variable (see [15, Theorem 6.2-4]).

(ii) Permutation symmetry (symmetry in indices $1, \ldots, n$ which label the parameters and the variables). (See [15, Theorem 5.2-3].)

(iii) Equal variables can be replaced by a single variable if the corresponding parameters are replaced by their sum (see [15, Theorem 5.2-4].) In particular, $R_{-a}(x, \ldots, x) = x^{-a}$.

For $a > 0$, $R_{-a}$ admits another integral representation

$$R_{-a}(b; X) = \frac{1}{B(a, a')} \int_{0}^{\infty} t^{a'-1} \prod_{i=1}^{n} (t + x_i)^{-b_i} \, dt,$$  \hspace{1cm} (2.3)

where $a' = b_1 + \cdots + b_n - a > 0$ (see [15, (6.8-6)]).

Symmetric elliptic integrals defined in Section 1 are represented by the $R$-hypergeometric functions $R_{-a}$. We have [15, Chapter 9] and [18, (16)–(18)]

$$R_F(x, y, z) = R_{-1/2} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z \right), \quad R_C(x, y) = R_{-1/2} \left( \frac{1}{2}, 1; x, y \right),$$  \hspace{1cm} (2.4)

$$R_J(x, y, z, p) = R_{-3/2} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; x, y, z, p \right),$$

$$R_D(x, y, z) = R_{-3/2} \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x, y, z \right),$$  \hspace{1cm} (2.5)

$$R_G(x, y, z) = R_{1/2} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x, y, z \right).$$  \hspace{1cm} (2.6)
\[ R_K(x, y) = R_{-1/2} \left( \frac{1}{2}, \frac{1}{2}; x, y \right), \]  
\[ R_E(x, y) = R_{1/2} \left( \frac{1}{2}, \frac{1}{2}; x, y \right). \]

We will now deal with the logarithmic convexity and concavity of all integrals listed in (2.4)–(2.8). Recall that a function \( f : D \to \mathbb{R}_+ \) is said to be logarithmically convex (log-convex) if for all \( X, Y \in D \) the following inequality:

\[ f[\lambda X + (1 - \lambda)Y] \leq [f(X)]^\lambda [f(Y)]^{1-\lambda} \]

is satisfied for \( 0 \leq \lambda \leq 1 \) (see [27]). Clearly a log-convex function is convex. The following result will be utilized in the subsequent sections of this paper.

**Proposition 2.1.** Let \( b \in \mathbb{R}_+^n \) and let \( X \in \mathbb{R}_n^+ \). Then the \( R \)-hypergeometric function \( R_p(b; X) \) is log-convex in its variables if \( p < 0 \) and is concave if \( 0 < p < 1 \).

**Proof.** Logarithmic convexity of \( R_p(b; X) \) \((p < 0)\) in its variables is established in [26, Proposition 2.1]. In order to prove concavity of \( R_p(b; X) \) in \( X \), when \( 0 < p < 1 \), we use the inequality

\[ (r > 0, s > 0) \]

\[ \lambda r + (1 - \lambda)s \]

\[ \geq \lambda r^p + (1 - \lambda)s^p \]

\( (r > 0, s > 0) \) together with (2.1) to obtain

\[ R_p(b; \lambda X + (1 - \lambda)Y) = \int_{E_{n-1}} [u \cdot (\lambda X + (1 - \lambda)Y)]^p \mu_b(u) \, du \]

\[ \geq \int_{E_{n-1}} [\lambda(u \cdot X)^p + (1 - \lambda)(u \cdot Y)^p] \mu_b(u) \, du \]

\[ = \lambda R_p(b; X) + (1 - \lambda)R_p(b; Y), \]

where \( Y \in \mathbb{R}_n^+ \) and \( 0 \leq \lambda \leq 1 \). The proof is complete. \( \square \)

**Corollary 2.2.** As the functions of their variables the elliptic integrals \( R_F, R_C, R_J, j, R_D, d, \) and \( R_K \) are log-convex while the integrals \( R_G \) and \( R_E \) are concave.

**Proof.** This is an immediate consequence of Proposition 2.1 and the formulas (2.4)–(2.8), (1.10) and (1.11). \( \square \)

### 3. Incomplete symmetric integrals

We begin this section by proving new upper bounds for the integrals \( R_F, R_J \) and \( R_D \) (see Theorem 3.2). They are sharper than the corresponding bounds derived in [23, (3.3), (3.4), (3.6)].

We need the following.

**Lemma 3.1.** Let \( f \) and \( g \) be nonnegative functions defined on the interval \([c, d]\) with \( g(t) \neq 0 \) for all \( c \leq t \leq d \). Assume that both functions \( f/g \) and \( fg \) are integrable on \([c, d]\). Then the following inequality:

\[ \int_c^d f(t) \, dt \leq \left[ \int_c^d \frac{f(t)}{g(t)} \, dt \int_c^d f(t)g(t) \, dt \right]^{1/2} \quad (3.1) \]

holds true.
Proof. We use the Cauchy–Schwarz inequality for integrals to obtain
\[
\int_{c}^{d} f(t) \, dt = \int_{c}^{d} \left[ \frac{f(t)}{g(t)} \right]^{1/2} \left[ f(t) g(t) \right]^{1/2} \, dt \\
\leq \left[ \int_{c}^{d} \frac{f(t)}{g(t)} \, dt \int_{c}^{d} f(t) g(t) \, dt \right]^{1/2}.
\]
\[\square\]

We are in a position to prove the following.

Theorem 3.2. Let \(x, y, z, p\) be positive numbers and let \(A = \frac{x + y}{2}\). Then
\[
R_{F}(x, y, z) \leq \left[ \frac{1}{2} \left( R_{C}(z, x) + R_{C}(z, y) \right) R_{C}(z, A) \right]^{1/2}, \tag{3.2}
\]
\[
R_{J}(x, y, z, p) \leq \left[ \frac{1}{2} \left( j(z, p, x) + j(z, p, y) \right) j(z, p, A) \right]^{1/2}, \tag{3.3}
\]
and
\[
R_{D}(x, y, z) \leq \left[ \frac{1}{2} \left( d(z, x) + d(z, y) \right) d(z, A) \right]^{1/2}, \tag{3.4}
\]
where the functions \(R_{C}, j\) and \(d\) are defined in (1.7), (1.10) and (1.11), respectively.

Proof. In order to establish (3.2) we use Lemma 3.1 with \(c = 0, d = \infty\), \(f(t) = \left[ (t + x)(t + y)\right]^{-1/2}(t + z)^{-1}\) and \(g(t) = \left[ (t + x)(t + y)\right]^{1/2}(t + A)^{-1}\) to obtain
\[
I := \int_{0}^{\infty} f(t) \, dt \\
\leq \left[ \int_{0}^{\infty} (t + A)[(t + x)(t + y)]^{-1}(t + z)^{-1/2} \, dt \int_{0}^{\infty} (t + z)^{-1/2}(t + A)^{-1} \, dt \right]^{1/2}.
\]
Using the partial-fraction decomposition
\[
\frac{t + A}{(t + x)(t + y)} = \frac{1}{2} \left( \frac{1}{t + x} + \frac{1}{t + y} \right), \tag{3.5}
\]
we obtain
\[
I \leq \left[ \frac{1}{2} \left( \int_{0}^{\infty} (t + z)^{-1/2}(t + x)^{-1} \, dt + \int_{0}^{\infty} (t + z)^{-1/2}(t + y)^{-1} \, dt \right) \right]^{1/2} \cdot \int_{0}^{\infty} (t + z)^{-1/2}(t + A)^{-1} \, dt. \tag{3.6}
\]
Multiplying both sides of (3.6) by \(\frac{1}{2}\) and next using (1.1) and (1.7) we obtain assertion (3.2). For the proof of (3.3) we use Lemma 3.1 again with \(c\) and \(d\) as above, \(f(t) = \left[ (t + x)(t + y)(t + z)\right]^{-1/2}(t + p)^{-1}\), and \(g(t)\) as defined earlier in this proof. Making use of (3.5) we have
\[
\frac{f(t)}{g(t)} = \frac{1}{2} \left[ (t + z)^{-1/2}(t + x)(t + p)\right]^{-1} + (t + z)^{-1/2}(t + y)(t + p)^{-1}
\]
and
\[
f(t)g(t) = (t + z)^{-1/2}(t + p)(t + A)^{-1}.
\]
Substituting these expressions into (3.1) and making use of (1.2) and (1.10) we obtain the desired result (3.3). The upper bound (3.4) is a special case of (3.3). Recall that \( R_D(x, y, z) = R_J(x, y, z, z) \) (see (1.3)). The proof is complete. □

Numerical experiments support the following.

**Conjecture.** Let \( x > 0, y > 0 \) and \( z \geq 0 \). Then
\[
\left[ \frac{1}{2} \left( g(z, x) + g(z, y) \right) g(z, A) \right]^{1/4} \leq R_G(x, y, z),
\]
where \( A = \frac{x + y}{2} \) and
\[
g(x, y) = R_G(x, y, y) = \begin{cases} x^{1/2} + yR_C(x, y), & x \neq y, \\ x^{1/2}, & x = y \end{cases}
\]
(see [23, (2.11)]).

Before we state and prove the next result, let us introduce more notation. In what follows, the letters \( \beta \) and \( \delta \) will stand for the roots of the Chebyshev polynomial \( T_2(t) = 8t^2 - 8t + 1 \) on \([0, 1]\), i.e., \( \beta = \frac{1 - 2^{-1/2}}{2} \) and \( \delta = 1 - \beta = \frac{1 + 2^{-1/2}}{2} \). For \( x > 0 \) and \( y > 0 \) we define
\[
u = \beta x + \delta y, \quad v = \beta y + \delta x.
\]
(3.7)

Our next result reads as follows.

**Theorem 3.3.** Let \( x, y, z \) be positive numbers. Then
\[
R_C(z, A) \leq \frac{1}{2} \left[ R_C(z, u) + R_C(z, v) \right] \leq R_F(x, y, z) \leq \left[ R_C(z, x)R_C(z, y) \right]^{1/2},
\]
and
\[
d(z, A) \leq \frac{1}{2} \left[ d(z, u) + d(z, v) \right] \leq R_D(x, y, z) \leq \left[ d(z, x)d(z, y) \right]^{1/2},
\]
where \( A = \frac{x + y}{2} \).

**Proof.** The first inequality in (3.8) is an immediate consequence of the fact that \( R_C \) is log-convex and hence convex in its variables. It follows from (3.7) that \( \frac{u + v}{2} = A \). The second inequality in (3.8) is established in [23, (3.3)]. For the proof of the third inequality in (3.8) we use (1.1) and the Cauchy–Schwarz inequality for integrals to obtain
\[
R_F(x, y, z) = \int_0^\infty \frac{(1/2)^{1/2}}{(t + z)^{1/4}(t + x)^{1/2}} \frac{(1/2)^{1/2}}{(t + z)^{1/4}(t + y)^{1/2}} dt
\]
\[
\leq \left( \frac{1}{2} \int_0^\infty \frac{dt}{(t + z)^{1/2}(t + x)} \right)^{1/2} \left( \frac{1}{2} \int_0^\infty \frac{dt}{(t + z)^{1/2}(t + y)} \right)^{1/2}
\]
\[
= \left[ R_C(z, x)R_C(z, y) \right]^{1/2},
\]
where in the last step we have applied formula (1.7). The first inequality in (3.9) is a consequence of convexity of the function $d(z, \cdot)$ while the second one is proven in [23, (3.6)]. For the proof of the third inequality in (3.9) we apply the Cauchy–Schwarz inequality to (1.3) to obtain

$$RD(x,y,z) = \int_0^\infty \left( \frac{3}{2} \int_0^\infty \frac{dt}{(t+z)^{3/2}(t+x)} \right)^{1/2} \left( \frac{3}{2} \int_0^\infty \frac{dt}{(t+z)^{3/2}(t+y)} \right)^{1/2} \leq \left[d(z,x)d(z,y)\right]^{1/2}. $$

In the last step we have used (1.11) and (1.3). This completes the proof. □

By use of the same method as in the proof of the last theorem one can show, using the first inequality in [23, (3.4)], (1.2) and (1.10) that

$$j(z,p,A) \leq \frac{1}{2} [j(z,p,u) + j(z,p,v)] \leq R_F(x,y,z) \leq \left[ j(z,p,x) j(z,p,y) \right]^{1/2}. $$

This implies (3.9) because of (1.3) and (1.11). We omit further details.

We shall establish now inequalities involving $R_F$ and $R_D$. Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ and let $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The symbol $Y_i (i = 1, 2, 3)$ will stand for the vector obtained from $Y$ by moving $y_i$ to the third position. Thus, $Y_1 = (y_2, y_3, y_1)$, $Y_2 = (y_1, y_3, y_2)$, etc.

**Theorem 3.4.** Let

$$s = \frac{1}{6} \sum_{i=1}^3 (x_i - y_i)R_D(Y_i).$$

Then

$$R_F(Y) - R_F(X) \leq s \quad (3.10)$$

and

$$R_F(Y) \ln \left[ \frac{R_F(Y)}{R_F(X)} \right] \leq s. \quad (3.11)$$

**Proof.** We shall utilize a well-known result for the convex functions. Let $f : C \to \mathbb{R}$ ($C$—a convex subset of a Euclidean space) be a convex function on the interior of $C$ with continuous partial derivatives of order one. Then

$$f(X) - f(Y) \geq (X - Y) \cdot \Delta f(Y) \quad (3.12)$$

holds for all $X, Y \in C$ (see [27,30]). Here $\Delta f$ stands for the gradient of $f$. Using (1.1) and (1.3) one obtains

$$\Delta R_F(Y) = -\frac{1}{6} \left[ R_D(Y_1), R_D(Y_2), R_D(Y_3) \right]. \quad (3.13)$$

Since $R_F$ is convex in its variables, inequality (3.10) follows from (3.12), with $f = R_F$, and from (3.13). If the function $f$ is log-convex on $\text{Int}(C)$, then (3.12) implies, on replacing $f$ by $\ln f$, that

$$f(Y) \ln \left[ \frac{f(X)}{f(Y)} \right] \geq (X - Y) \cdot \Delta f(Y). \quad (3.14)$$

Inequality (3.11) follows from (3.14), with $f = R_F$, and (3.13). The proof is complete. □
Corollary 3.5. Let $Y \in \mathbb{R}^3_+$ and let $a \geq 0$. With $Y + a := (y_1 + a, y_2 + a, y_3 + a)$ the following inequalities:

$$RF(Y) - RF(Y + a) \leq \frac{a}{6} (y_1 y_2 y_3)^{-3/10} \left( y_1^{-3/5} + y_2^{-3/5} + y_3^{-3/5} \right)$$

(3.15)

and

$$RF(Y) \ln \left[ \frac{RF(Y)}{RF(Y + a)} \right] \leq \frac{a}{2} (y_1 y_2 y_3)^{-1/2}$$

(3.16)

are valid.

Proof. Inequality (3.15) follows from (3.10), with $X = Y + a$, and from

$$RD(Y_i) \leq (y_1 y_2 y_3)^{-3/10} y_i^{-3/5}$$

($i = 1, 2, 3$) where the last bound is obtained from

$$RD(x, y, z) \leq (xyz)^{-3/10},$$

$x > 0, y > 0, z > 0$ (see [12, (2.5)]). For the proof of (3.16) we use (3.11), with $X = Y + a$, and apply the formula

$$\sum_{i=1}^{3} RD(Y_i) = 3(y_1 y_2 y_3)^{-1/2}$$

which follows from [15, (5.9-5)] and [15, (6.6-5)] with $t = -\frac{3}{2}$ and $b = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. □

The elliptic integral $RF(x, y, A)$ ($x > 0, y \geq 0, A = \frac{x+y}{2}$) is often called the general case of the first lemniscate constant and is associated with the lemniscatic mean $LM(x, y)$ of $x$ and $y$ as follows [24]:

$$RF(x, y, A) = \left[ LM(A, G) \right]^{-1/2},$$

(3.17)

where $G = (xy)^{1/2}$ is the geometric mean of $x$ and $y$. Recall that the mean $LM(x, y)$ is the common limit

$$LM(x, y) = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n$$

of two sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$, where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = (x_{n+1} x_n)^{1/2}$$

(3.18)

$n \geq 0$ (see [14]). Lower and upper bounds for $RF(x, y, A)$ are obtained in the following.

Theorem 3.6. Let $x > 0, y > 0 (x \neq y)$ and let the sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ be defined in (3.18) with $x_0 = A$ and $y_0 = G$. Then for every $n \geq 0$,

$$\left( \frac{5}{3x_n + 2y_n} \right)^{1/2} < RF(x, y, A) < (x_n^{3/2} y_n^{3/2})^{-1/10}.$$

(3.19)
In particular,
\[
\left( \frac{5}{3A + 2G} \right)^{1/2} < R_F(x, y, A) < (A^3 G^2)^{-1/10}.
\]

(3.20)

Proof. It has been shown in [24] that
\[
(x_n^3 y_n^2)^{1/5} < LM(x, y) < \frac{3x_n + 2y_n}{5} \quad (x \neq y)
\]
for all \( n \geq 0 \). Letting \( x := A \) and \( y := G \) and next utilizing (3.17) we arrive at (3.19). Bounds (3.20) follow from (3.19) by letting \( n = 0 \). The proof is complete. \( \Box \)

4. Complete symmetric integrals

The goal of this section is to establish bounds for the complete symmetric integrals \( R_K \) and \( R_E \) which are defined in (1.5) and (1.6), respectively. Also, some inequalities involving these integrals are included.

Some of these bounds are expressed in terms of the logarithmic mean or a power mean of two positive numbers. For the reader’s convenience we recall definitions of these means. In what follows we will always assume that \( x > 0 \) and \( y > 0 \) and write \( G \) and \( A \) for the geometric mean and the arithmetic mean of \( x \) and \( y \). The logarithmic mean of order 1 of \( x \) and \( y \) is defined by

\[
L(x, y) \equiv L = \begin{cases} 
\frac{x - y}{\ln x - \ln y}, & x \neq y, \\
\frac{x}{y}, & x = y.
\end{cases}
\]

(4.1)

The logarithmic mean of order \( p \in \mathbb{R} \) of \( x \) and \( y \) is denoted by \( L_p(x, y) \) and defined as

\[
L_p(x, y) = \begin{cases} 
[L(x^p, y^p)]^{1/p}, & p \neq 0, \\
G, & p = 0.
\end{cases}
\]

(4.2)

The power mean \( A_p(x, y) \) of order \( p \in \mathbb{R} \) of \( x \) and \( y \) is defined by

\[
A_p(x, y) = \begin{cases} 
\left( \frac{x^p + y^p}{2} \right)^{1/p}, & p \neq 0, \\
G, & p = 0.
\end{cases}
\]

(4.3)

It is well known that both means \( L_p \) and \( A_p \) increase with an increase in \( p \). We shall also use celebrated Gauss’ arithmetic-geometric mean \( AGM(x, y) \) which is the iterative mean, i.e., it is a common limit

\[
AGM(x, y) \equiv AGM = \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n,
\]

where now the sequences \( \{x_n\}_{0}^{\infty} \) and \( \{y_n\}_{0}^{\infty} \) are defined as follows:

\[
x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = (x_n y_n)^{1/2},
\]

\( n \geq 0 \) (see, e.g., [15, (6.10-6)])

Our first result reads as follows.
Theorem 4.1. Let \( x > 0, y > 0 \) (\( x \neq y \)). Then
\[
\frac{1}{L_{3/2}(A, G)} < RK(x^2, y^2) < \frac{1}{L(A, G)}.
\] (4.4)

**Proof.** The following result
\[
L(x, y) < AGM(x, y) < L_{3/2}(x, y)
\] (4.5)
is known. The first equality in (4.5) is due to Carlson and Vuorinen [19] while the second one is established in [10, Proposition 2.7]. To complete the proof of (4.4) we let in (4.5) \( x := A, y := G \) and next apply
\[
AGM(A, G) = AGM(x, y) = \frac{1}{RK(x^2, y^2)},
\] (4.6)
where the first equality in (4.6) is the invariance property of the Gauss mean while the second one is given in [15, (6.10-8)]. The proof is complete. □

Weaker and simpler bounds for \( RK \):
\[
(AL)^{-1/2} < RK(x^2, y^2) < L^{-1}
\] (4.7)
follow from (4.5), with \( L_{3/2}(x, y) \) replaced by \( L_2(x, y) = (AL)^{1/2} \), and from (4.6). Let us note that \( L < L(A, G) \) (see [25, Theorem 3.1]).

In what follows, we will write \( H(x, y) \equiv H = \frac{2\sqrt{xy}}{x+y} \) and \( Q(x, y) \equiv Q = (\frac{x^2+y^2}{2})^{1/2} \) for the harmonic and the root-mean-square means, respectively.

Proposition 4.2. Let \( x > 0, y > 0 \) (\( x \neq y \)). Then
\[
RK(x^2, y^2) < (HQ)^{-1/2}.
\]

**Proof.** We substitute \( x := x^2, y := y^2 \) and \( z = 0 \) into (3.2) to obtain
\[
RF(x^2, y^2, 0) < \left[ \frac{1}{2} (RC(0, x^2) + RC(0, y^2)) RC\left(0, \frac{x^2 + y^2}{2}\right) \right]^{1/2}.
\]
Making use of (1.5) and (1.9) we obtain the desired result. □
the power mean is a homogeneous function of degree 1 and its variables while $R_E$ is homogeneous of degree $\frac{1}{2}$ one obtains (4.8).

A new upper bound for $R_E$ is established in the following.

**Theorem 4.3.** Let $x > 0$, $y > 0$ ($x \neq y$) and let $\beta$ and $\delta$ have the same meaning as in (3.7). Then

$$R_E(x, y) < \frac{1}{2}(\sqrt{\beta x + \delta y} + \sqrt{\beta y + \delta x}).$$

(4.10)

**Proof.** Using (2.8), (2.1) and (2.2) we have

$$R_E(x, y) = \frac{1}{\pi} \int_0^1 [(1 - u)u]^{-1/2}[(1 - u)x + uy]^{1/2} du.$$  

(4.11)

We apply the two-point Gauss–Chebyshev quadrature formula with the remainder (see [7, Theorem 5.3]) to the integral in (4.11) to obtain

$$R_E(x, y) = \frac{1}{2} \left[f(\beta) + f(\delta)\right] + E_G,$$

(4.12)

where $f(u) = [(1 - u)x + uy]^{1/2}$, $E_G = \text{const.} f^{(4)}(\xi)$, const. $> 0$ and $0 < \xi < 1$. Since

$$f^{(4)}(u) = -\frac{15}{16}(y - x)^4[1 - u) x + uy]^{-7/2} < 0$$

for $0 < u < 1$, $E_G < 0$. This in conjunction with (4.12) gives the desired result (4.10). □

We close this section with the following.

**Theorem 4.4.** Let $x > 0$, $y > 0$ ($x \neq y$). Then

$$A_{3/2}(A, G) < \frac{1}{2}[R_E(x^2, y^2) + xyR_K(x^2, y^2)] < A_1(A, G)$$

(4.13)

and

$$\left(\frac{4}{3x + y}\right)^{1/2} < \frac{R_E(x, y) - yR_K(x, y)}{x - y} < (x^3y)^{-1/8}.$$  

(4.14)

**Proof.** In order to prove (4.13) we use (4.8) with $x$ replaced by $A$ and $y$ replaced by $G$. Application of the Landen transformation

$$R_E(A^2, G^2) = \frac{1}{2}[R_E(x^2, y^2) + xyR_K(x^2, y^2)]$$

(see [15, Example 9.5-2]) gives assertion (4.13). In order to establish (4.14) we apply [15, (8.3-2)] to $R_D(0, x, y)$ and use (2.5) to obtain

$$\frac{4}{3\pi} R_D(0, x, y) = R_{-3/2} \left(\frac{1}{2}, \frac{3}{2}; x, y\right).$$

This in conjunction with

$$\frac{4}{3\pi} R_D(0, x, y) = \frac{2}{y(x - y)}[R_E(x, y) - yR_K(x, y)]$$
(see [16, (2.40)]) gives
\[ \frac{2}{y(x - y)} \left[ R_E(x, y) - y R_K(x, y) \right] = R_{-3/2} \left( \frac{3}{2}; x, y \right). \] (4.15)

The bounds for the right-hand side of (4.15) are obtained using [12, (2.5)]. The result is
\[ \frac{1}{y} \left( \frac{4}{3x + y} \right)^{1/2} < R_{-3/2} \left( \frac{1}{2}, \frac{3}{2}; x, y \right) < \frac{1}{y(x^3 y)^{-1/8}}. \]

Combining this with (4.15) yields (4.14). The proof is complete. \( \Box \)

5. Bounds for Legendre’s incomplete integral \( F \)

We shall assume that the amplitude \( \phi \) and the modulus \( k \) will satisfy \( 0 < \phi < \frac{\pi}{2} \) and \( 0 < k < 1 \). The following bounds for the incomplete integral of the first kind are established in [11, (4.5)]
\[ L_1 < F(\phi, k) < U_1, \] (5.1)

where
\[ L_1 = \frac{\tanh^{-1}(\sigma \sin \phi)}{\sigma}, \quad U_1 = \frac{1}{2} \left( \tanh^{-1}(\sin \phi) + \frac{\tanh^{-1}(k \sin \phi)}{k} \right) \] (5.2)

and \( \sigma^2 = \frac{1 + k^2}{2} \). We shall establish refinements of (5.1). In what follows the symbols \( \beta \) and \( \delta \) will have the same meaning as in (3.7).

**Theorem 5.1.** The following inequalities:
\[ L_1 < L_2 < F(\phi, k) < U_2 < U_1, \] (5.3)

where
\[ L_2 = \frac{1}{2} \left[ \frac{\tanh^{-1}(\mu \sin \phi)}{\mu} + \frac{\tanh^{-1}(v \sin \phi)}{v} \right] \] (5.4)

\( \mu = (\beta + \delta k^2)^{1/2}, \ v = (\delta + \beta k^2)^{1/2} \) and
\[ U_2 = \left[ \frac{\tanh^{-1}(\sin \phi) \tanh^{-1}(k \sin \phi)}{k} \right]^{1/2} \] (5.5)

are valid.

**Proof.** For the proof of the first three inequalities in (5.3) we utilize (3.8) and (1.15) with \( x = c - 1 \), \( y = c - k^2 \) and \( z = c \), where \( c = (\sin \phi)^{-2} \). Making use of
\[ R_C(1, 1 - a^2) = \frac{\tanh^{-1} a}{a}, \] (5.6)
$|a| < 1$ (see [15, Example 6.9-16]) and taking into account that $R_C$ is homogeneous of degree $-\frac{1}{2}$ in its variables we obtain

$$R_C(z, A) = L_1, \quad \frac{1}{2}[R_C(z, u) + R_C(z, v)] = L_2,$$

$$RF(c - 1, c - k^2, c) = F(\phi, k), \quad [R_C(z, x)R_C(z, y)]^{1/2} = U_2.$$

This gives the first four members of (5.3). The last inequality $U_2 < U_1$ follows from the inequality of the arithmetic and geometric means applied to (5.5) and the second formula in (5.2). The proof is complete. □

Bounds for Legendre’s complete elliptic integrals $K$ and $E$ have been obtained in several papers (see [1–6,8,9,28,29,31]). Applying formulas

$$K(k) = \frac{\pi}{2} R_K(1 - k^2, 1)$$

(see [15, (9.2-14)]) and (1.16) to the bounds and inequalities discussed in Section 4 of this paper and to [23, Section 4] one obtains several new results involving integrals $K$ and $E$. We omit further details.

References