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CORRIGENDUM

Order Independence for Iterated Weak Dominance[1]

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In Marx and Swinkels (1997), we identify a condition on a normal form game, *transference of decisionmaker indifference* (TDI), under which the order of elimination by weak dominance does not matter. That is, regardless of the order in which weakly dominated strategies are removed, any two *full reductions*—games in which no further removals are possible—are the same up to the addition or removal of redundant strategies and the renaming of equivalent strategies. TDI requires that if, for a given pure strategy profile s_{-i} for the other players, player i is indifferent between two of his pure strategies s_i and t_i, then the other players are indifferent between (s_i, s_{-i}) and (t_i, s_{-i}) as well. We also define *nice weak dominance* (NWD) and explore its implications. A weak dominance is *nice* if TDI is satisfied for the pair of strategies involved.

[1]We thank Ray Deneckere for pointing out the error in our paper and Larry Samuelson for useful advice.

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1 \ 2	l	r
t	2,0	2,0
m	2,3	0,1
b	2,3	1,4

G

1 \ 2	r
t	2,0

R_1

1 \ 2	l
t	2,0
m	2,3

R_2

1 \ 2	l
t	2,0
b	2,3

T

FIGURE 1

Theorem 1 and Corollary 1 of the paper are correct. That is, under TDI, order under weak dominance does not matter. And, even in games not satisfying TDI, order under nice weak dominance does not matter. Theorem 2 and Corollary 2, for the mixed strategy case, are also correct. The discussions of backward induction and complexity also remain valid.

However, Proposition 1 and the analogous Proposition 2 for mixed strategies are not correct as stated. Proposition 1 claims that nice weak dominance sets an upper bound for any full reduction by weak dominance. That is, it claims that any full reduction by weak dominance is a subset of any reduction by nice weak dominance, once again up to redundancies and renaming.

We are grateful to Ray Deneckere for pointing out that this is false and for providing the following counterexample. Consider the game G in Fig. 1. As can be seen, b nicely weakly dominates m, and t weakly dominates b (but not nicely, since 1 is indifferent between (t, l) and (b, l), but 2 is not). The unique full reduction of G by NWD, R_1, is reached by the order of eliminations m, l, b. However, the order of eliminations by weak dominance b, r yields R_2. This is a full reduction by weak dominance, but clearly is not equivalent to a subset of R_1.

The error in the proof can be traced to Lemma B, which claims that if a subset T can be obtained from some subset W of the pure strategies by iteratively (i) removing a very weakly dominated strategy,[2] (ii) removing a redundant strategy, or (iii) removing a strategy and substituting one to which it is redundant on the set of strategies then remaining, then for every $s \in W$ there is some $s' \in T$ that very weakly dominates s on T.

Consider T in Fig. 1. T can be obtained from G by eliminating first b and then r by weak dominance and then substituting b for m (this is valid since b and m are redundant given l). So, by the claimed Lemma B, r must be very weakly dominated on T by a strategy in T. Since l is all that remains for player 2, this is clearly false. The difficulty is that, at the moment when b

[2] *Very weak dominance* is weak dominance without the requirement of the strict inequality.

was substituted for m, r was already gone. So, while b and m are redundant given l, they are not given r. Lemma B is correct with substitution removed as an operator.

Since Lemma B is needed to prove Lemma C, Lemma C is also false as stated, as shown by the example. Lemma C is correct if one restricts attention to *nice* dominances.

LEMMA C (Corrected). *Let W be a restriction of S, let $\hat{W}$ be a reduction of W by nice very weak dominance, and let V be a reduction of W by nice very weak dominance. Then there exists $\hat{V}$ equivalent to a subset of $\hat{W}$, where $\hat{V}$ is obtainable from V by the iterative removal of strategies that are either nicely weakly dominated or redundant.*

In Appendix A, we provide a proof of the corrected Lemma C that does not rely on the incorrect implication of Lemma B. The original Lemma C differs from the corrected version by placing the second and third appearances of the word "nice" in parentheses, allowing some of the dominances involved in getting to V from W in the antecedent (and hence to $\hat{V}$ from V in the consequent) not to be nice. Note also that with the removal of the third pair of parentheses, it is equivalent to saying that $\hat{V}$ is a reduction of V by nice very weak dominance (see Observation 2).

Given the corrected Lemma C, it is immediate that Proposition 1 without parentheses is correct. Theorem 1 and Corollary 1 are also correct since they only involve nice weak dominances.

Similar corrections apply to the mixed strategy results and are sketched in Appendix B. Proposition 2 holds once the parentheses are removed.[3] Hence, as in the case with pure strategies, Theorem 2 and Corollary 2 are correct.

Appendix A

We begin with a preliminary lemma establishing the corrected Lemma C for the case in which V differs from W by only one strategy.

LEMMA 1. *Let Q be a restriction of S, let $\hat{Q}$ be a reduction of Q by nice very weak dominance (NVWD), and let $Z \equiv Q \backslash y$ be a reduction of Q by the removal of a single NVWDed strategy. Then there exists $\hat{Z}$ equivalent to a subset of $\hat{Q}$, where $\hat{Z}$ is a reduction of Z by NVWD.*

[3]In fact, Proposition 2 now holds under the stronger notion of equivalence used in the pure strategy results (rather than only under equivalence*, which allows pure strategies to be mapped onto mixed strategies).

Proof of Lemma 1. By Lemma A, we can write $\hat{Q} = Q\backslash x^1, \ldots, x^m$, where for each $k \in \{1, \ldots, m\}$, $Q\backslash x^1, \ldots, x^k$ can be obtained from $Q\backslash x^1, \ldots, x^{k-1}$ by eliminating the single strategy x^k using NVWD. Proceed to remove the strategies $x^1, \ldots, x^m$, in order, from Z as long as they are NVWDed on the set remaining (if for some k, $x^k = y$, then skip this step). Let x^κ be the last valid such removal. If $\kappa = m$, we have arrived at either $\hat{Q}$ or $\hat{Q}\backslash y$ and so we are done.

Assume $\kappa < m$. Let $X \equiv Q\backslash x^1, \ldots, x^\kappa$ and let i be the player to whom $x^{\kappa+1}$ belongs. Assume y does not belong to i. Now, $x^{\kappa+1}$ is NVWDed on X by some $r \in X$. By the definition of NVWD it is clear that r also NVWDs $x^{\kappa+1}$ on $X\backslash y$. Since r belongs to i, and so $r \neq y$, $r \in X\backslash y$. Hence, $x^{\kappa+1}$ is NVWDed on $X\backslash y$. This is a contradiction.

So y belongs to i. Then, since $x^{\kappa+1}$ is not NVWDed on $X\backslash y$, but is NVWDed on X, it must be that $y \in X$ and that y NVWDs $x^{\kappa+1}$ on X. We claim that there is $z \in X\backslash y$ that NVWDs y on $X\backslash y$.[4] Since z NVWDs y on $X\backslash y$, it also NVWDs y on X, since the two sets are the same for players other than i. Thus, since z NVWDs y and y NVWDs $x^{\kappa+1}$ on X, it follows that z NVWDs $x^{\kappa+1}$ on X. Hence, if $z \neq x^{\kappa+1}$, then $x^{\kappa+1}$ is a valid removal, which is a contradiction. So $z = x^{\kappa+1}$. But then, since $z = x^{\kappa+1}$ NVWDs y on X while y NVWDs $x^{\kappa+1}$ on X, strategies $x^{\kappa+1}$ and y are redundant on X. Since both $x^{\kappa+1}$ and y are members of X, it follows that $X\backslash y$ and $X\backslash x^{\kappa+1}$ are equivalent.

Thus, one can continue from $X\backslash y$ in the same way as from $X\backslash x^{\kappa+1}$, up to the replacement of y by $x^{\kappa+1}$. To see this, if y appears among $x^{\kappa+2}, \ldots, x^m$, let τ be such that $x^\tau = y$. Otherwise, let $\tau = m + 1$. Consider the following sequence of removals from $X\backslash y$. For $\kappa + 2 \leq k \leq m$, $k \neq \tau$, remove x^k. If $\tau \leq m$, then for $k = \tau$, remove $x^{\kappa+1}$. As this sequence of removals results in either $\hat{Q}$ or $\hat{Q}\backslash y \cup x^{\kappa+1}$, we are done. ■

Proof of Lemma C (Corrected). By Lemma A, we can write V as $W\backslash y^1, \ldots, y^m$, where for each $k \in \{1, \ldots, m\}$, $W\backslash y^1, \ldots, y^k$ can be obtained from $W\backslash y^1, \ldots, y^{k-1}$ by eliminating the single strategy y^k using NVWD. Trivially, a set W^0 equivalent to a subset of $\hat{W}$ (namely $W^0 = \hat{W}$) can be obtained from W by the iterative removal of strategies by NVWD. We proceed by induction. For given $j \in \{0, \ldots, m - 1\}$, assume that a set $\hat{W}^j$ equivalent to a subset of $\hat{W}$ can be obtained from $W\backslash y^1, \ldots, y^j$ by the iterative removal of strategies by NVWD. By Lemma 1, a set $\hat{W}^{j+1}$

[4] To see this, note that since y was removed by NVWD from Q, there exists $z' \in Q\backslash y$ that NVWDs y on Q (and hence on $Q\backslash y, x^1, \ldots, x^\kappa$). If $z' \in Q\backslash y, x^1, \ldots, x^\kappa$, then $z = z'$ serves. If not, then at the point in the sequence from $Q\backslash y$ to $Q\backslash y, x^1, \ldots, x^\kappa$ at which z' was removed, z' was NVWDed on the set remaining by some $z'' \neq z'$ in the set. Since this set is at least as large as $Q\backslash y, x^1, \ldots, x^\kappa$, strategy z'' also NVWDs z' on $Q\backslash y, x^1, \ldots, x^\kappa$. The claim follows by induction.

equivalent to a subset of $\hat{W}^j$ can be obtained from $W\backslash y^1, \ldots, y^{j+1}$ by the iterative removal of strategies by NVWD. Since "equivalent to a subset" is a transitive relation, $\hat{W}^{j+1}$ is thus also equivalent to a subset of $\hat{W}$. But then, $\hat{V} \equiv \hat{W}^q$ meets our condition, and we are done. ■

Appendix B

LEMMA 2. *Let Q be a restriction of S, let $\hat{Q}$ be a reduction of Q by NVWD*, and let $Z \equiv Q\backslash y$ be a reduction of Q by the removal of a single NVWDed * strategy. Then there exists $\hat{Z}$ equivalent to a subset of $\hat{Q}$, where $\hat{Z}$ is a reduction of Z by NVWD*.*

Proof of Lemma 2. As before, by Lemma A*, write $\hat{Q} = Q\backslash x^1, \ldots, x^m$, and eliminate strategies in the same order from Z, where x^κ is the last valid such removal. If $\kappa = m$, we are done.

Assume $\kappa < m$, and define $X \equiv Q\backslash x^1, \ldots, x^\kappa$. Let i be the player to whom $x^{\kappa+1}$ belongs. Assume y does not belong to i. Let $\zeta \in \Delta(X)_i$ NVWD* $x^{\kappa+1}$ on X. As before, since y does not belong to i, $\zeta \in \Delta(X\backslash y)_i$, and so $x^{\kappa+1}$ is NVWDed* on $X\backslash y$, a contradiction.

So, y belongs to i. Then, since $x^{\kappa+1}$ is NVWDed* on X but not on $X\backslash y$, it must be that $y \in X$ and that there is a $\gamma \in \Delta(X)_i$ with $\gamma(y) > 0$, where γ NVWDs* $x^{\kappa+1}$ on X. We claim that there is $\sigma \in \Delta(X\backslash y)_i$ that NVWDs* y on $X\backslash y$ and hence X.[5] Thus, $\phi \equiv \gamma\{y \to \sigma\}$ places no weight on y and ϕ NVWDs* $x^{\kappa+1}$ on X. If $\phi(x^{\kappa+1}) < 1$, then define ϕ' to be ϕ with the weight on $x^{\kappa+1}$ removed and the other weights rescaled to sum to one. Then ϕ' NVWDs* $x^{\kappa+1}$ on X and $\phi' \in \Delta\left(X\backslash y, x^{\kappa+1}\right)_i$. Thus, $x^{\kappa+1}$ is NVWDed* on $X\backslash y$, which is a contradiction. So $\phi(x^{\kappa+1}) = 1$. Since $\phi \equiv \gamma\{y \to \sigma\}$ and $\gamma(y) > 0$, this implies that $\sigma(x^{\kappa+1}) = 1$ and that γ puts weight only on y and possibly $x^{\kappa+1}$. Since γ NVWDs* $x^{\kappa+1}$ on X, it follows that y NVWDs* $x^{\kappa+1}$ on X. Since σ NVWDs* y, and since $\sigma(x^{\kappa+1}) = 1$, it follows that $x^{\kappa+1}$ NVWDs* y on X. Thus, strategies $x^{\kappa+1}$ and y are redundant on X. Since both $x^{\kappa+1}$ and y are members of X, it follows that $X\backslash y$ and $X\backslash x^{\kappa+1}$ are equivalent. Much as before, this means that the removals from $X\backslash x^{\kappa+1}$ can be mimicked in $X\backslash y$. In this sequence, $x^{\kappa+1}$ plays the role of y, both replacing it in any mixed strategy which is doing the dominating and being removed at the step (if any) at which y is removed. ■

*Lemma C** (Corrected). Let W be a restriction of S, let $\hat{W}$ be a reduction of W by nice very weak dominance*, and let V be a reduction of W

[5] As before, start with σ' that NVWDs* y on Q. At the first point when a pure strategy x^j in the support of σ' is dominated by γ and removed, replace σ' by $\sigma'' = \sigma'\{x^j \to \gamma\}$. That is, redistribute the weight on x^j according to γ. Proceed iteratively.

by nice very weak dominance*. Then there exists $\hat{V}$ equivalent to a subset of $\hat{W}$, where $\hat{V}$ is a reduction of V by nice very weak dominance*.

The proof is analogous to that for the corrected Lemma C.

REFERENCE

Marx, L. M., and Swinkels, J. M. (1997). “Order Independence for Iterated Weak Dominance,” *Games Econ. Behavior* **18**, 219–245. [Article No. GA970525]