Pareto Optimal Solution Analysis of Convex Multi-Objective Programming Problem

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Abstract—The main method of solving multi-objective programming is changing multi-objective programming problem into single objective programming problem, and then get Pareto optimal solution. Conversely, whether all Pareto optimal solutions can be obtained through appropriate method, generally the answer is negative. In this paper, the methods of norm ideal point and membership function are used to solve the multi-objective programming problem. In norm ideal point method, norm and ideal point are given to structure the corresponding single objective programming problem. Then prove that for any Pareto optimal solution there exist weights such that Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. Membership function method, firstly construct membership function for every objective function, then establish the single objective programming problem, after then solve the single objective programming problem, finally prove that for any Pareto optimal solution there exist weights such that the Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. At last, two examples are given to illustrate that the two methods are effective in getting Pareto optimal solution.

Index Terms—convex multi-objective, Pareto optimal solution, M-Pareto optimal solution, weight

I. INTRODUCTION

In multi-objective programming problem, it is difficult to find an optimal solution to achieve the extreme value of every objective function, so that the decision maker is searching for the compromise solution. Based on this idea, the concepts of Pareto optimal solution and weakly Pareto optimal solution are introduced into multi-objective programming problem [1]. The main method of solving multi-objective programming problem is turning multi-objective programming problem to single objective programming problem, and we can get Pareto optimal solution or weakly Pareto optimal solution. Then, whether we can get all Pareto optimal solutions or weakly Pareto optimal solutions through appropriate method, generally the answer is negative. So it is a very important topic to discuss in what condition we can get all Pareto optimal solutions or weakly Pareto optimal solutions. In this paper, we only discuss this problem for convex multi-objective programming problem. Some scholars have done a lot of work about the Pareto optimal solution of multi-objective programming problem.

K. Suga, S. Kato, and K. Hiyama analyzed structure of Pareto optimal solution sets, presented the analysis process as well as the case study details, and showed the method they proposed is effective at finding an acceptable solution for multi-objective optimization problems [2]. Y. Shang and B. Yu presented a constraint shifting combined homotopy method for solving convex multi-objective programming problem with both equality and inequality constraints. This method does not need the starting point to be an interior point or a feasible point. The existence and convergence of a smooth path to an effective solution were proved [3]. A. Eusebio and J. R. Figueira presented a new algorithm for identifying all supported non-dominated vectors in the objective space, as well as the corresponding effective solutions in the decision space for multi-objective integer network flow problem. The proposed approach used the connectedness property of the set of supported non-dominated effective solution to find all integer solutions in maximal non-dominated efficient facets [4]. B.A. Ghaznavi-ghosoni and E. Khorram analyzed the relationships between \( \varepsilon \)-efficient points of multi-objective optimization problem and \( \varepsilon \)-optimal solutions of the related scalarized problem and obtained necessary and sufficient conditions for approximating efficient points of a general multi-objective optimization problem via approximate solutions of the scalarized problem [5]. M. D. Monfared, A. Mohades and J. Rezaei introduced a new method for ranking the solutions of an evolutionary algorithm’s population, and the proposed algorithm was very suitable for the convex multi-objective optimization problems [6].

Y. Gao introduced the notion of \( \varepsilon \)-quasi weakly saddle point and obtained the necessary and sufficient conditions for the existence of \( \varepsilon \)-quasi weakly efficient solution in multi-objective optimization problem in terms of Hadamard directional derivatives and limiting
subdifferentials [7]. Y. Liu, Z. Peng and Y. Tan analyzed the relations among absolutely optimal solutions, effective solutions and weakly effective solutions of multi-objective programming problem [8]. F. He and Y. Shang gave the dynamic constraint combination homotopy method for minimal weakly efficient solutions for convex multi-objective programming problems on unbounded sets and proved the existence and convergence of the homotopy path [9]. X. Niu, T. Zhan and L. Xu studied the large scale multi-objective programming with trapezoidal structure, which was decomposed into several subproblems. The relations of effective solutions between the large-scale multi-objective programming and its subproblems have been studied and the existence of effective solution has been investigated [10]. X. Zhou proved the connectedness of cone-efficient solution set for cone-quasiconvex multi-objective programming in topological vector spaces [11]. Y. Li and Q. Zhang obtained some Kuhn-Tucker type optimality conditions for a class of non-smooth multi-objective semi-infinite programming involving generalized convexity and some non-smooth non-convex functions. [12]. Y. An and H. Liu used Taylor's formal of the vector-valued function and Gordon Selection principle to prove several necessary and sufficient conditions of weakly Pareto optimal solution for convex multi-objective programming, and turned the problem how to determine the weakly efficient solution of convex multi-objective programming into the problem of judging whether an equation had non-negative non-zero solution [13].

There are a lot of methods of turning multi-objective programming problem to single objective programming problem. In norm ideal point method, for the given weights, the optimal solution of the corresponding single objective programming problem is Pareto optimal solution of multi-objective programming problem. In membership function method, for the given weights, the optimal solution of the corresponding single objective programming problem is M-Pareto optimal solution of multi-objective programming problem, which is similar to Pareto optimal solution [14]. Conversely, when taking all weights, whether all Pareto optimal solutions or M-Pareto optimal solutions can be got, this problem for the convex multi-objective programming problem will be discussed.

The rest of this paper is organized as follows. For clear, Section II gives the preliminary knowledge of convex multi-objective programming problem and a lemma used in the following. In Section III, three theorems are given to proof that all Pareto optimal solutions and M-Pareto optimal solutions can be got through norm ideal point method and membership function method for convex multi-objective programming problem, and Pareto optimal solution is equal to M-Pareto optimal solution. Section IV present examples to illustrate that there exist weights such that Pareto optimal solution or M-Pareto optimal solution of multi-objective programming problem is the optimal solution of the corresponding single objective programming problem. Finally, the conclusions are given in Section V.

II. PRELIMINARY KNOWLEDGE

A convex multi-objective optimization problem can be stated as follows:

\[
\text{minimize } f(x) = (f_1(x), f_2(x), \ldots, f_n(x))
\]

subject to

\[
g_j(x) \leq 0; j = 1, 2, \ldots, p
\]

where \(x\) is an \(n\)-dimensional vector of decision variables, \(f_1(x), f_2(x), \ldots, f_n(x)\) are convex functions defined on \(X\), and \(X = \{x \mid g_j(x) \leq 0; j = 1, 2, \ldots, p\}\) is convex set. Problem (1) is also called convex multi-objective programming problem.

Definition 1. \(x^* \in X\) is said to be a Pareto optimal solution of problem (1), if and only if there does not exist another \(x \in X\) such that \(f_i(x) \leq f_i(x^*), i = 1, 2, \ldots, m\), with strict inequality holding for at least one \(i\).

Definition 2. \(x^* \in X\) is said to be a weakly Pareto optimal solution of problem (1), if and only if there does not exist another \(x \in X\) such that \(f_i(x) < f_i(x^*), i = 1, 2, \ldots, m\).

We denote that \(R^n_* = \{w \in R^n \mid w \geq 0\}\), where \(w = (w_1, w_2, \ldots, w_n)^T, w \geq 0 \Leftrightarrow w_i \geq 0, i = 1, 2, \ldots, m\).

Lemma 1 [15]. Suppose that \(S\) is a nonempty convex set in normed linear space, and \(f_i(x), i = 1, 2, \ldots, m\) are convex functions defined on \(S\). Then convex function inequalities \(F(x) < 0, x \in S\) are incompatible if and only if there exists \(w \in R^n_* \setminus \{0\}\) such that \(w^T F(x) \geq 0, \forall x \in S\), where \(F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T\), \(F(x) < 0 \Leftrightarrow f_i(x) < 0, i = 1, 2, \ldots, m\).

III. THE DISCUSSION OF PARETO OPTIMAL SOLUTION

In order to get Pareto optimal solution or weakly Pareto optimal solution of problem (1), the main method is changing multi-objective programming problem into single objective programming problem. Next, two methods are used to discuss whether all Pareto optimal solutions can be obtained.

A. Norm Ideal Point Method

For the problem (1), firstly give ideal value \(\overline{f}_i\) for every objective function \(f_i(x)\), which satisfies \(\overline{f}_i \leq \min_{x \in X} f_i(x), i = 1, 2, \ldots, m\), \(\overline{f} = (\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n)\) is called ideal point, after then introduce the norm \(\|\cdot\|\), finally get the feasible solution which is having the nearest distance with the given ideal point \(\overline{f}\) in the norm.
Use absolute value norm to structure the corresponding single objective programming ($P_2$):

$$\min_{x \in X} \sum_{i=1}^{m} w_i |f_i(x) - \tilde{f}_i|$$

(2)

where $w = (w_1, w_2, \cdots, w_m)^T \in R^m \setminus \{0\}$.

Because $\tilde{f}_i \leq f_i(x), i = 1, 2, \cdots, m$, ($P_2$) can be simplified to

$$\min_{x \in X} \sum_{i=1}^{m} w_i (f_i(x) - \tilde{f}_i)$$

For the given ideal point $\tilde{T}$ and weights $w \in R^m \setminus \{0\}$, the optimal solution of ($P_2$) is weakly Pareto optimal solution of problem (1) [16]. Conversely, if $x^*$ is a weakly Pareto optimal solution of problem (1), whether there exists $w$ such that $x^*$ is the optimal solution of the corresponding single objective programming problem ($P_2$). The following theorem is given.

**Theorem 1.** If $x^*$ is weakly Pareto optimal solution of problem (1), then there exists $w \in R^m \setminus \{0\}$ such that $x^*$ is the optimal solution of the corresponding single objective programming problem ($P_2$).

**Proof.** Suppose that $x^*$ is weakly Pareto optimal solution of problem (1), then there does not exist $x \in X$ such that $f_i(x) < f_i(x^*), i = 1, 2, \cdots, m$.

So there does not exist $x \in X$ such that $f_i(x) - \tilde{f}_i < f_i(x^*) - \tilde{f}_i, i = 1, 2, \cdots, m$.

This implies that the following inequalities are incompatible:

$$f_i(x) - \tilde{f}_i - (f_i(x^*) - \tilde{f}_i) < 0, i = 1, 2, \cdots, m.$$  

Because $f_i(x), i = 1, 2, \cdots, m$ are convex functions defined on $X$, $f_i(x) - \tilde{f}_i - (f_i(x^*) - \tilde{f}_i)$ are also convex functions defined on $X$.

By lemma 1, there exists $w = (w_1, w_2, \cdots, w_m) \in R^m \setminus \{0\}$ such that

$$\sum_{i=1}^{m} w_i[f_i(x) - \tilde{f}_i - (f_i(x^*) - \tilde{f}_i)] \geq 0.$$  

This means that

$$\sum_{i=1}^{m} w_i[f_i(x) - \tilde{f}_i] \geq \sum_{i=1}^{m} w_i[f_i(x^*) - \tilde{f}_i].$$

So $x^*$ is optimal solution of problem ($P_2$), and the theorem is proved.

Attention should be paid that Pareto optimal solution must be weakly Pareto optimal solution, which implies that if all weakly Pareto optimal solutions can be obtained, then all Pareto optimal solutions can be obtained. This also shows that theoretically all Pareto optimal solutions can be obtained through changing weights.

**B. Membership Function Method**

Firstly structure membership function $\mu_i(f_i(x))$ for every objective function $f_i(x)$, then use $\mu_i(f_i(x))$ as the new objective functions to structure the new multi-objective programming problem, and then turn the new multi-objective programming problem to single objective programming problem through some appropriate methods, finally solve the single objective programming problem to get the optimal solution, which is also the M-Pareto optimal solution of the original multi-objective programming problem.

Now, structure the membership function for every objective function as follows:

$$\mu_i(f_i(x)) = \frac{f_i(x) - f_i^*}{f_i^* - \tilde{f}_i^*}, i = 1, 2, \cdots, m,$$

where $f_i^* = \min_{x \in X} f_i(x), f_i^* = \max_{x \in X} f_i(x), i = 1, 2, \cdots, m$.

Without loss of generality, suppose $f_i^* < f_i^*, i = 1, 2, \cdots, m$.

**Definition 3.** $x^* \in X$ is said to be a M-Pareto optimal solution of problem (1), if and only if there does not exist another $x \in X$ such that $\mu_i(f_i(x)) \geq \mu_i(f_i(x^*)), i = 1, 2, \cdots, m$, with strict inequality holding for at least one $i$.

**Definition 4.** $x^* \in X$ is said to be a weakly M-Pareto optimal solution of problem (1), if and only if there does not exist another $x \in X$ such that $\mu_i(f_i(x)) > \mu_i(f_i(x^*)), i = 1, 2, \cdots, m$.

It is obvious that M-Pareto optimal solution of problem (1) must be weakly M-Pareto optimal solution of problem (1).

Furthermore, the following conclusion is true.

**Theorem 2.** $x^*$ is M-Pareto optimal solution of problem (1), if and only if $x^*$ is Pareto optimal solution of problem (1).

**Proof.** "⇒"  

If $x^*$ is M-Pareto optimal solution of problem (1), then by definition 3, there does not exist another $x \in X$, with strict inequality holding for at least one $i$, such that $\mu_i(f_i(x)) \geq \mu_i(f_i(x^*)), i = 1, 2, \cdots, m$.

So there does not exist another $x \in X$, with strict inequality holding for at least one $i$, such that

$$f_i(x) - f_i^* \geq f_i(x^*) - f_i^*, i = 1, 2, \cdots, m.$$  

Therefore, there does not exist another $x \in X$ such that

$$f_i(x) \leq f_i(x^*), i = 1, 2, \cdots, m,$$

with strict inequality holding for at least one $i$, so $x^*$ is Pareto optimal solution of problem (1).

"⇐"  

If $x^*$ is Pareto optimal solution of problem (1), then there does not exist another $x \in X$ such that

$$f_i(x) \leq f_i(x^*), i = 1, 2, \cdots, m,$$

with strict inequality holding for at least one $i$ by definition 1.

So there does not exist another $x \in X$, with strict inequality holding for at least one $i$, such that

$$f_i(x) - f_i^* \leq f_i(x^*) - f_i^*, i = 1, 2, \cdots, m.$$  

This implies that there does not exist another $x \in X$, with strict inequality holding for at least one $i$, such that

$$f_i(x) - f_i^* \leq f_i(x^*) - f_i^*, i = 1, 2, \cdots, m.$$
\[ f_i(x) - f_i^* \geq f_i(x^*) - f_i^*, \quad i = 1, 2, \ldots, m. \]

Therefore, there does not exist another \( x \in X \) such that
\[ \mu_i(f_i(x)) > \mu_i(f_i(x^*)), \quad i = 1, 2, \ldots, m, \]
with strict inequality holding for at least one \( i \), so \( x^* \) is M-Pareto optimal solution of problem (1).

In order to turn multi-objective programming into single objective programming problem, consider the linear weighted model (\( P_w^* \)):
\[ \max_{x \in X} \sum_{i=1}^{m} w_i \mu_i(f_i(x)) \quad (3) \]
where \( w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n \setminus \{0\} \).

For the given \( w \in \mathbb{R}^n \setminus \{0\} \), the optimal solution of problem (\( P_w^* \)) is weakly M-Pareto optimal solution of problem (1) [16].

Conversely, if \( x^* \) is weakly M-Pareto optimal solution of problem (1), whether there exists \( w \in \mathbb{R}^n \setminus \{0\} \) such that \( x^* \) is the optimal solution of the corresponding single objective programming problem (\( P_w^* \)). We will discuss this problem below.

Because \( \mu_i(f_i(x)) \) is nonincreasing linear function about \( f_i(x) \) and \( f_i(x) \) is convex, \( \mu_i(f_i(x)) \) is concave function [17]. We have the following theorem.

**Theorem 3.** If \( x^* \) is weakly M-Pareto optimal solution of problem (1), then there exists \( w \in \mathbb{R}^n \setminus \{0\} \) such that \( x^* \) is the optimal solution of the corresponding problem (\( P_w^* \)).

**Proof.** Suppose that \( x^* \) is weakly M-Pareto optimal solution of problem (1), then there does not exist \( x \in X \) such that
\[ \mu_i(f_i(x)) > \mu_i(f_i(x^*)), \quad i = 1, 2, \ldots, m. \]
This implies that the following inequalities are incompatible:
\[ \mu_i(f_i(x^*)) - \mu_i(f_i(x)) < 0, \quad i = 1, 2, \ldots, m. \]

Because \( \mu_i(f_i(x)), i = 1, 2, \ldots, m \) are concave functions, \( \mu_i(f_i(x^*)) - \mu_i(f_i(x)) \) are convex functions defined on \( X \).

By lemma 1, there exists \( w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n \setminus \{0\} \) such that
\[ \sum_{i=1}^{m} w_i [\mu_i(f_i(x^*)) - \mu_i(f_i(x))] \geq 0. \]
Thus
\[ \sum_{i=1}^{m} w_i \mu_i(f_i(x^*)) \geq \sum_{i=1}^{m} w_i \mu_i(f_i(x)). \]

Therefore \( x^* \) is the optimal solution of problem (\( P_w^* \)), and the theorem is proved.

In the above, the membership function of objective function is set by using simple linear function. According to the properties of the composition of convex function, if the membership function of \( f_i(x) \) is nonincreasing and concave about \( f_i(x) \), then the above conclusion still holds.

**IV. AN ILLUSTRATIVE EXAMPLE**

**A. Norm Ideal Point Method**

Consider the following linear multi-objective programming problem:
\[
\begin{align*}
\min & \{ -x + 2y, 2x + y \} \\
\text{subject to } & -x - y + 2 \leq 0 \\
& 3x - 2y - 1 \leq 0 \\
& -2x + 3y - 6 \leq 0
\end{align*}
\] (4)

For convenience, the feasible region of the problem is denoted as \( X \).

The vertices of feasible region formed by constraints are \( A = (0, 2), B = (1,1), C = (3,4) \), and the function values of the two objective functions in vertices are given in the following table I:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-x + 2y)</td>
<td>(2x + y)</td>
</tr>
<tr>
<td>A</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
</tr>
</tbody>
</table>

**Figure 1.** The graph of feasible region.

Because the particularity of linear programming, it is easy to know \( \min_{x \in X} \{ -x + 2y \} = 1 \), \( \min_{x \in X} \{ 2x + y \} = 2 \), and all Pareto optimal solutions of problem (4) are \( A = (0, 2), B = (1,1) \).

Next, ideal point method is used to solve the multi-objective programming problem (4), and illustrate that there exists \( w \) such that each Pareto optimal solution of problem (4) is the optimal solution of the corresponding single objective programming problem. In the ideal point method, we take ideal point \( \bar{f} = (\bar{f}_1, \bar{f}_2) \), where
\[ \bar{f}_1 = \min_{x \in X} \{ -x + 2y \} = 1, \quad \bar{f}_2 = \min_{x \in X} \{ 2x + y \} = 2. \]

So the linear multi-objective programming problem (4) can be turned into the following single objective programming problem:
\[
\begin{align*}
\min w_1 & \left[(-x + 2y) - 1\right] + w_2 \left[(2x + y) - 2\right] \\
\text{subject to} & \quad -x - y + 2 \leq 0 \\
& \quad 3x - 2y - 1 \leq 0 \\
& \quad -2x + 3y - 6 \leq 0
\end{align*}
\] (5)

The objective function of problem (5) can be written as
\[
(-w_1 + 2w_2)x + (2w_1 + w_2)y - w_1 - 2w_2
\]

In order to take minimum in point \(A = (0,2)\), according to the characteristic of linear programming, the slope of objective function only need to satisfy
\[
-w_1 + 2w_2 > 0 \text{ and } \frac{w_1 - 2w_2}{2w_1 + w_2} < -1
\] (6)

or
\[
-w_1 + 2w_2 > 0 \text{ and } \frac{w_1 - 2w_2}{2w_1 + w_2} > \frac{2}{3}
\] (7)

Specially, take \(w_1 = \frac{1}{5}, w_2 = \frac{4}{5}\) and the multi-objective programming problem (4) can be turned into the following single objective programming problem:
\[
\begin{align*}
\min & \quad \frac{1}{5} \left[(-x + 2y) - 1\right] + \frac{4}{5} \left[(2x + y) - 2\right] \\
\text{subject to} & \quad -x - y + 2 \leq 0 \\
& \quad 3x - 2y - 1 \leq 0 \\
& \quad -2x + 3y - 6 \leq 0
\end{align*}
\] (8)

The optimal solution of single objective programming problem (8) is \(A = (0,2)\), which is also the Pareto optimal solution of problem (4). So there exists weights \(w_1 = \frac{2}{5}, w_2 = \frac{3}{5}\) such that \(B = (1,1)\) is the optimal solution of the corresponding single objective programming problem (11).

To sum up, for all Pareto optimal solutions of multi-objective programming problem (4), there exist weights such that the Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. And the weight \(w\) is not unique. From the above example, it is easy to see that if weight to satisfy inequality (6) or (7) the Pareto optimal solution \(A = (0,2)\) can be obtained. To get Pareto optimal solution \(B = (1,1)\) the weight only need to satisfy inequality (9) or (10). So it is easy to choose the weight \(w\) and get all Pareto optimal solutions of multi-objective programming problem.

**B. Membership Function Method**

In the following, membership function method is used to solve the multi-objective programming problem. Consider the following linear multi-objective programming problem:
\[
\begin{align*}
\min & \quad \{-3x - y, x - 2y, 4x + 3y\} \\
\text{subject to} & \quad -2x - y + 6 \leq 0 \\
& \quad 2x + 3y - 18 \leq 0 \\
& \quad 2x - 3y - 6 \leq 0
\end{align*}
\] (12)

For convenience, denote that \(f_1 = -3x - y\), \(f_2 = x - 2y, f_3 = 4x + 3y\). The feasible region of the problem is denoted as \(X\).
The vertices of feasible region formed by constraints are \( A=(0,6), \ B=(3,0), \ C=(6,2) \), and the function values of the three objective functions in vertices are given in the following table II:

**TABLE II.**

<table>
<thead>
<tr>
<th></th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>-6</td>
<td>-12</td>
<td>18</td>
</tr>
<tr>
<td>( B )</td>
<td>-9</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>( C )</td>
<td>-20</td>
<td>2</td>
<td>30</td>
</tr>
</tbody>
</table>

Because the particularity of linear programming, it is obvious that

\[
\begin{align*}
\min_{x \in X} f_1 & = -20, \quad \max_{x \in X} f_1 = -6, \\
\min_{x \in X} f_2 & = -12, \quad \max_{x \in X} f_2 = 3, \\
\min_{x \in X} f_3 & = 12, \quad \max_{x \in X} f_3 = 30.
\end{align*}
\]

Next, for every objective function, membership function can be structured as follows:

\[
\begin{align*}
\mu_1(f_1) & = -\frac{f_1 - 6}{14}, \\
\mu_2(f_2) & = -\frac{f_2 + 3}{15}, \\
\mu_3(f_3) & = -\frac{f_3 + 30}{18}.
\end{align*}
\]

The membership function values in vertices are given in the following table III:

**TABLE III.**

<table>
<thead>
<tr>
<th></th>
<th>( \mu_1(f_1) )</th>
<th>( \mu_2(f_2) )</th>
<th>( \mu_3(f_3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>0</td>
<td>1</td>
<td>2/3</td>
</tr>
<tr>
<td>( B )</td>
<td>3/14</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( C )</td>
<td>1</td>
<td>1/15</td>
<td>0</td>
</tr>
</tbody>
</table>

From the definition of M-Pareto optimal solution, the M-Pareto optimal solutions of problem (12) are
\( A=(0,6), \ B=(3,0), \ C=(6,2) \).

Next, membership function method is used to solve the multi-objective programming problem (12), and illustrate that there exists \( w \) such that the M-Pareto optimal solution of problem (12) is the optimal solution of the corresponding single objective programming problem. According to model \((P_w)\), the multi-objective programming problem (12) can be turned into the following single objective programming problem:

\[
\begin{align*}
\max & (w_1 \mu_1(f_1) + w_2 \mu_2(f_2) + w_3 \mu_3(f_3)) \\
\text{subject to} & -2x - y + 6 \leq 0, \\
& 2x + 3y - 18 \leq 0, \\
& 2x - 3y - 6 \leq 0.
\end{align*}
\]

Because of the way of structuring membership function of objective function \( f_1, f_2 \) and \( f_3 \), the above model can be written as

\[
\begin{align*}
\max & \left( \frac{-f_1 - 6}{14} + \frac{-f_2 + 3}{15} + \frac{-f_3 + 30}{18} \right) \\
\text{subject to} & -2x - y + 6 \leq 0, \\
& 2x + 3y - 18 \leq 0, \\
& 2x - 3y - 6 \leq 0.
\end{align*}
\]

The above model can also be described by the following formula (15):

\[
\begin{align*}
\max & \left( \frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 \right)x + \left( \frac{1}{14}w_1 + \frac{2}{15}w_2 - \frac{3}{18}w_3 \right)y \\
& + \left( \frac{-6}{14}w_1 + \frac{3}{15}w_2 - \frac{30}{18}w_3 \right)
\end{align*}
\]

subject to

\[
\begin{align*}
-2x - y + 6 & \leq 0, \\
2x + 3y - 18 & \leq 0, \\
2x - 3y - 6 & \leq 0.
\end{align*}
\]

In order to take maximum in point \( A=(0,6) \), the slope of objective function only need to satisfy

\[
\begin{align*}
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 > 0,
\end{align*}
\]

or satisfy simultaneously

\[
\begin{align*}
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 < 0
\end{align*}
\]

and

\[
\begin{align*}
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 < -2.
\end{align*}
\]

Here take

\[
\begin{align*}
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 = -4.
\end{align*}
\]

Specially, take \( w_1 = \frac{5}{7}, w_2 = \frac{14}{15}, w_3 = 279 \), and the multi-objective programming problem (12) can be turned into the following single objective programming problem:

\[
\begin{align*}
\max & \left( \frac{228}{1080}x - \frac{57}{1080}y + \frac{491}{280} \right) \\
\text{subject to} & -2x - y + 6 \leq 0, \\
& 2x + 3y - 18 \leq 0, \\
& 2x - 3y - 6 \leq 0.
\end{align*}
\]

The optimal solution of single objective programming problem (16) is \( A=(0,6) \), which is also the M-Pareto optimal solution of problem (12). So there exists weights \( w_1 = \frac{5}{7}, w_2 = \frac{14}{15}, w_3 = 279 \), such that \( A=(0,6) \) is the optimal solution of the corresponding single objective programming problem (16).

In order to take maximum in point \( B=(3,0) \), the slope of objective function only need to satisfy

\[
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 > 0,
\]

or satisfy simultaneously

\[
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 < 0
\]

and

\[
\begin{align*}
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 < -2.
\end{align*}
\]

Here take

\[
\begin{align*}
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 = -4.
\end{align*}
\]

Specially, take \( w_1 = \frac{5}{7}, w_2 = \frac{14}{15}, w_3 = 279 \), and the multi-objective programming problem (12) can be turned into the following single objective programming problem:

\[
\begin{align*}
\max & \left( \frac{228}{1080}x - \frac{57}{1080}y + \frac{491}{280} \right) \\
\text{subject to} & -2x - y + 6 \leq 0, \\
& 2x + 3y - 18 \leq 0, \\
& 2x - 3y - 6 \leq 0.
\end{align*}
\]

The optimal solution of single objective programming problem (16) is \( A=(0,6) \), which is also the M-Pareto optimal solution of problem (12). So there exists weights \( w_1 = \frac{1}{5}, w_2 = \frac{10}{14}, w_3 = 279 \), such that \( A=(0,6) \) is the optimal solution of the corresponding single objective programming problem (16).

In order to take maximum in point \( B=(3,0) \), the slope of objective function only need to satisfy

\[
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 > 0,
\]

or satisfy simultaneously

\[
\frac{3}{14}w_1 - \frac{1}{15}w_2 - \frac{4}{18}w_3 < 0
\]
\[ \frac{3}{14} w_1 - \frac{1}{15} w_2 - \frac{4}{18} w_3 < 0 \]
and
\[ -2 < -\frac{3}{14} w_1 - \frac{1}{15} w_2 - \frac{4}{18} w_3 < 0, \]

or satisfy simultaneously
\[ \frac{3}{14} w_1 - \frac{1}{15} w_2 - \frac{4}{18} w_3 > 0 \]
and
\[ 0 < -\frac{3}{14} w_1 - \frac{1}{15} w_2 - \frac{4}{18} w_3 < \frac{2}{3}. \]

Here take
\[ \frac{3}{14} w_1 - \frac{1}{15} w_2 - \frac{4}{18} w_3 = -1. \]
Specially, take
\[ w_1 = \frac{2}{3}, w_2 = \frac{1}{7}, w_3 = \frac{18}{35} \]
and the multi-objective programming problem (12) can be turned into the following single objective programming problem:
\[
\begin{align*}
\max & \left( -\frac{4}{105} x - \frac{4}{105} y + \frac{11}{14} \right) \\
\text{subject to} & -2x - y + 6 \leq 0, \\
& 2x + 3y - 18 \leq 0, \\
& 2x - 3y - 6 \leq 0.
\end{align*}
\]

The optimal solution of single objective programming problem (17) is \( B = (3, 0) \), which is also the M-Pareto optimal solution of problem (12). So there exists weights \( w_1 = \frac{2}{3}, w_2 = \frac{1}{7}, w_3 = \frac{18}{35} \), such that \( B = (3, 0) \) is the optimal solution of the corresponding single objective programming problem (18).

Therefore, for all M-Pareto optimal solutions of multi-objective programming problem (12), there exist weights such that M-Pareto optimal solution is the optimal solution of the corresponding single objective programming problem.

From the two examples above, it is easy to see that weight \( w \) is not unique. In fact, choose weight \( w \) in a large range, the corresponding Pareto optimal solution or M-Pareto optimal solution can be obtained. This also shows that the two methods discussed in this paper are very effective to obtain the Pareto optimal solution for convex multi-objective optimization problem.

V. CONCLUSION

This paper discusses the Pareto optimal solution and M-Pareto optimal solution for convex multi-objective programming problem. Generally, all Pareto optimal solutions and M-Pareto optimal solutions cannot be obtained through changing weights. But for convex multi-objective programming problem, all Pareto optimal solutions and M-Pareto optimal solutions can be obtained through taking out all the weights. At last, linear programming examples are given to illustrate that for any Pareto optimal solution there exist weights such that Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. And from the examples we can see that the weight is not unique. It is not difficult to choose weight such that the Pareto optimal solution is the optimal solution of the corresponding single objective programming problem. It shows that the two methods discussed in this paper not only have very important theoretic significant but also have widely practical value.

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