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Weakly Connected Independent and Weakly Connected Total Domination in a Product of Graphs

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Abstract

In this paper we characterize the weakly connected independent and weakly connected total dominating sets in the lexicographic product graphs. The weakly connected independent and weakly connected total domination numbers of this graph are also determined.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = X \cup N(X)$. A subset S of $V(G)$ is an *independent set* if for every $x, y \in S$, $xy \notin E(G)$. The *independence number* $\beta(G)$ of G is the largest cardinality of an independent set of G. A subset S of $V(G)$ is called *weakly connected* if the subgraph $\langle S \rangle_w = (N_G[S], E_W)$ weakly induced by S, is connected, where E_W is the set of all edges with at least one vertex in S.

A subset S of $V(G)$ is a (*total*) *dominating set* of G if for every $(v \in V(G))$ $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The (*total*) *domination number* $(\gamma_t(G))$ $\gamma(G)$ of G is the smallest cardinality of a (total) dominating set of G. A dominating set of G which is also independent is called an *independent dominating set*. The *independent domination number* $i(G)$ of G is the smallest cardinality of an independent dominating set of G. An independent dominating set of G which is weakly connected is called a *weakly connected independent dominating set*. The *weakly connected independent domination number* $i_w(G)$ of G is the smallest cardinality of a weakly connected independent dominating set of G. A total dominating set of G which is weakly connected is called a *weakly connected total dominating set*. The *weakly connected total domination number* $\gamma_{wt}(G)$ of G is the smallest cardinality of a weakly connected total dominating set of G.

A survey of selected recent results on independent domination in graphs is given in [2].

In [1], Dunbar et al. investigated the concept of weakly connected domination. They showed that every connected graph has a weakly connected independent dominating set. Thus, the $i_w(G)$ of a connected graph G always exists. Relation of this parameter with other domination parameters are also considered by the authors. The concept was also studied in [3] and [4].

2 Weakly Connected Independent Domination in the Lexicographic Product of Graphs

The *lexicographic product* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either

$uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Observe that any non-empty subset C of $V(G) \times V(H)$ (in fact, any set of ordered-pairs) can be written as $C = \cup_{x \in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforth, we shall use this form to denote any subset C of $V(G) \times V(H)$.

The first two results are found in [9].

Lemma 2.1 *Let G be a connected graph. A subset S of $V(G)$ is weakly disconnected (i.e. $\langle S \rangle_w$ is not connected) if and only if the following property is satisfied: (N) There exist $x, y \in S$ with $x \neq y$ such that $N_G[x] \cap N_G[y] = \emptyset$ and for every x-y path $P = [x, a_1, a_2, \cdots, a_k, y]$ in G, there exist $i \in \{1, 2, ..., k-1\}$ with $x_i, x_{i+1} \in V(P) \backslash S$.*

Lemma 2.2 *Let G and H be connected graphs and let $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$. Then C is weakly connected in $G[H]$ if and only if S is weakly connected in G.*

Theorem 2.3 *Let G be a nontrivial connected graph and H any graph. A subset $C = \bigcup_{\in S} (\{x\} \times T_x)$ of $V(G[H])$ is a weakly connected independent dominating set of $G[H]$ if and only if S is a weakly connected independent dominating set of G and T_x is an independent dominating set of H for every $x \in S$.*

Proof: Suppose that $C = \bigcup_{x \in S} (\{x\} \times T_x)$,where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a weakly connected independent dominating set of $G[H]$. Let $v \in V(G) \backslash S$. Then there exists $b \in V(H)$ such that $(v, b) \in V(G[H]) \backslash C$. Since C is a dominating set of $G[H]$, there exists $(u, a) \in C$ such that $(u, a)(v, b) \in E(G[H])$. This means that $u \in S$ and $uv \in E(G)$. Thus, S is a dominating set of G. Let $x, y \in S$ such that $x \neq y$. Pick $c \in T_x$ and $d \in T_y$. Then $(x, c), (y, d) \in C$ and $(x, c) \neq (y, d)$. Since C is an independent set, $(x, c)(y, d) \notin E(G[H])$. This implies that $xy \notin E(G)$. Hence, S is an independent set of G. Moreover, by Lemma 2.2, S is weakly connected.

Next, let $y \in S$ and let $b \in V(H) \backslash T_y$. Then $(y, b) \notin C$ and since C is a dominating set, there exists $(x, a) \in C$ such that $(x, a)(y, b) \in E(G[H])$. Since S is an independent set of G, $xy \notin E(G)$. By the definition of $G[H]$, $x = y$ and $ab \in E(H)$, where $a \in T_x = T_y$. Thus, T_x is a dominating set of H. Let $x \in S$ and let $c, d \in T_x$ such that $c \neq d$. Then $(x, c), (x, d) \in C$ with $(x, c) \neq (x, d)$. Since C is independent, $(x, c)(x, d) \notin E(G[H])$. This means that $cd \notin E(H)$. Hence, T_x is an independent set of H.

Conversely, suppose S is a weakly connected independent dominating set of G and T_x is an independent dominating set of H for every $x \in S$. From the definition of $G[H]$, it is evident that C is independent and dominating. To show that it is weakly connected, let $(x, a), (y, b) \in C$ with $(x, a) \neq (y, b)$

and $(x,a)(y,b) \notin E(G[H])$. Then $x, y \in S$. If $x = y$, then we are done. Suppose $x \neq y$. Since $\langle S \rangle_w$ is connected, by Proposition 1.3, there exists an x-y path $P = [x, x_1, x_2, ..., x_k, y]$ in G with $V(P) \subseteq V(\langle S \rangle_w)$. Since S is weakly connected and independent, $x_i \in S$ if i is even and $x_i \in N_G(S)$ if i is odd. For each $x_i \in S$, pick $b_i \in T_{x_i}$ and if $x_i \in N_G(S)$, let $b_i = a \in V(H)$. Then $(x_i, b_i) \in C$ if i is even and $(x_i, b_i) \in N_{G[H]}(C)$ if i is odd. Hence, $P' = [(x,a), (x_1,b_1), (x_2,b_2), ..., (x_k,b_k), (y,b)]$ is an (x,a)-(y,b) path in $G[H]$ with $V(P') \subseteq V(\langle C \rangle_w)$. □

Corollary 2.4 *Let G be a nontrivial connected graph and H any graph. Then $i_w(G[H]) = i_w(G)i(H)$.*

Proof: Let $C = \bigcup_{x \in S}(\{x\} \times T_x)$ be a minimum weakly connected independent dominating set of $G[H]$. By Theorem 2.1, S is a weakly connected independent dominating set of G and T_x is an independent dominating set of H for every $x \in S$. Hence,

$$i_w(G[H]) = |C| = |\sum_{x \in S}(\{x\} \times T_x)| \geq i_w(G)i(H).$$

Next, let S be a minimum weakly connected independent dominating set of G and D a minimum independent dominating set of H. For each $x \in S$, let $T_x = D$. By Theorem 2.1 $C = \bigcup_{x \in S}(\{x\} \times T_x)$ is a weakly connected independent dominating set of $G[H]$. Thus,

$$i_w(G[H]) \leq |C| = |\sum_{x \in S}(\{x\} \times T_x)| = i_w(G)i(H).$$

Therefore, $i_w(G[H]) = i_w(G)i(H)$. □

The following results immediately follow from Corollary 2.2 and the fact that $i(K_n) = 1$ and $i(\overline{K_n}) = n$.

Corollary 2.5 *Let G be a connected graph and K_n the complete graph of order $n \geq 1$. Then*
(i) $i_w(G[K_n]) = i_w(G)$, and
(ii) $i_w(G[\overline{K_n}]) = [i_w(G)]n$.

3 Weakly Connected Total Domination in the Lexicographic Product of Graphs

The next result is found in [4].

Lemma 3.1 *Let G be a connected non-trivial graph and let S be a weakly connected dominating set of G. Then $\gamma_{wt}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. In particular, $\gamma_{wt}(G) \leq 2\gamma_w(G)$.*

Theorem 3.2 *Let G and H be connected non-trivial graphs. Then a subset $C = \cup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a weakly connected total dominating set in $G[H]$ if and only if and either*

(i) S is a weakly connected total dominating set in G or

(ii) S is a weakly connected dominating set of G and T_x is a total dominating set of H for every $x \in S \setminus N_G(S)$.

Proof: Suppose C is a weakly connected total dominating set of $G[H]$. By Lemma 2.2, S is weakly connected. Let $u \in V(G)\backslash S$ and pick $b \in V(H)$. Since C is a dominating set of $G[H]$, there exists $(y,c) \in C$ such that $(y,c)(u,b) \in E(G[H])$. This implies that $y \in S$ and $u \in N_G(y)$. This shows that S is a dominating set in G. If S is a total dominating set of G, then we are done. So suppose S is not a total dominating set of G. Then $S \setminus N_G(S) \neq \emptyset$. Let $x \in S \setminus N_G(S)$. Suppose there exists $y \in V(H) \setminus N_H(T_x)$. Then $yz \notin E(H)$ for all $z \in T_x$. This implies that $(x,y) \notin N_{G[H]}(C)$, contrary to our assumption that C is a total dominating set in $G[H]$. Therefore, $N_H(T_x) = V(H)$, i.e., T_x is a total dominating set in H.

For the converse, suppose that $C = \cup_{x\in S}(\{x\} \times T_x)$, where S is weakly connected in G, and let $(u,t) \in V(G[H])$. By Lemma 2.2, C is weakly connected. Assume first that S is a total dominating set of G. Then there exists $x \in S \setminus \{u\}$ such that $u \in N_G(x)$. Choose $d \in T_x$. Then $(x,d) \in C$ and $(u,t)(x,d) \in E(G[H])$. Hence, $(u,t) \in N_{G[H]}(C)$. This shows that C is a weakly connected total dominating set in $G[H]$.

Suppose now that (ii) holds. If $u \notin S$, then because S is a dominating set of G, there exists $y \in S$ such that $u \in N_G(y)$. Pick $a \in T_y$. Then $(y,a) \in C$ and $(u,t)(y,a) \in E(G[H])$. Suppose $u \in S$. If $u \in N_G(z)$ for some $z \in S \setminus \{u\}$, then there exists $(z,b) \in C$ such that $(u,t)(z,b) \in E(G[H])$. If $u \notin N_G(z)$ for all $z \in S \setminus \{u\}$, then by assumption, T_u is a total dominating set in H. This implies that there exists $s \in T_u$ such that $ts \in E(H)$. It follows that $(u,s) \in C$ and $(u,t)(u,s) \in E(G[H])$. Thus, $(u,t) \in N_{G[H]}(C)$. Therefore C is a weakly connected total dominating set in $G[H]$. □

Corollary 3.3 *Let G and H be connected non-trivial graphs with $\gamma_t(H) = 2$. Then a subset $C = \cup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum weakly connected total dominating set in $G[H]$ if and only if either*

(i) S is a minimum weakly connected total dominating set of G and $|T_x| = 1$ for all $x \in S$ or

(ii) S is a weakly connected dominating set of G such that $|S \cap N_G(S)| + 2|S \setminus N_G(S)| = \gamma_{wt}(G)$, $|T_x| = 1$ for each $x \in S \cap N_G(S)$, and T_x is a minimum total dominating set in H (hence, $|T_x| = 2$) for every $x \in S \setminus N_G(S)$.

Proof: Suppose $C = \cup_{x\in S}(\{x\} \times T_x)$ is a minimum weakly connected total dominating set of $G[H]$. By Theorem 3.2, S is a weakly connected total dominating set of G or S is a weakly connected dominating set of G and T_x is a total dominating set in H for every $x \in S \setminus N_G(S)$. Suppose first that S is weakly connected total dominating set. Suppose further that that $|T_z| \geq 2$ for some $z \in S$. Let $a \in T_z$ and define $T_z^* = \{a\}$. Then $C^* = [\cup_{x\in S\setminus\{z\}}(\{x\} \times T_x)] \cup (\{z\} \times T_z^*)$ is a weakly connected total dominating set by Theorem 3.2(i). This, however, is impossible because $|C^*| < |C|$. Thus, $|T_x| = 1$ for all $x \in S$ and (i) holds.

Suppose now that S is not a weakly connected total dominating set in G. Suppose first that $\gamma_{wt}(G) < |S \cap N_G(S)| + 2|S \setminus N_G(S)| = |C|$. Choose a minimum weakly connected total dominating set R of G and set $S_x = \{v\}$ for every $x \in R$, where $v \in V(H)$. Then $Y = [\cup_{x\in R}(\{x\} \times S_x)]$ is a weakly connected total dominating set by Theorem 3.2(i). It follows that $\gamma_{wt}(G) = |R| = |Y| < |C|$, contrary to our assumption of C. Thus, $\gamma_{wt}(G) = |S \cap N_G(S)| + 2|S \setminus N_G(S)|$.

Next, suppose that there exists $z \in S \cap N_G(S)$ with $|T_z| \geq 2$. Let $a \in T_z$ and define $T_z^* = \{a\}$. Then $C^* = [\cup_{x\in S\setminus\{z\}}(\{x\} \times T_x)] \cup (\{z\} \times T_z^*)$ is a weakly connected total dominating set by Theorem 3.2(i). This is not possible because $|C^*| < |C|$. Therefore, $|T_x| = 1$ for all $x \in S \cap N_G(S)$. Finally, suppose there exists $w \in S \setminus N_G(S)$ such that T_w is not a minimum total dominating set in H. By Theorem 3.2, $|T_w| > 2$. Let $L_w = \{a, b\}$ be a minimum total dominating set in H. Then $C_1 = [\cup_{x\in S\setminus\{w\}}(\{x\} \times T_x)] \cup (\{w\} \times L_w)$ is a total dominating set by Theorem 3.2(i). Again, this is not possible because $|C_1| < |C|$. Therefore, T_x is a minimum total dominating set in H for every $x \in S \setminus N_G(S)$.

The converse is also easy. □

Corollary 3.4 *Let G and H be nontrivial connected graphs with $\gamma_t(H) \neq 2$. Then a subset $C = \cup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum weakly connected total dominating set of $G[H]$ if and only if S is a minimum weakly connected total dominating set of G and $|T_x| = 1$ for all $x \in S$.*

Proof: Suppose $C = \cup_{x\in S}(\{x\} \times T_x)$ is a minimum weakly connected total dominating set of $G[H]$. Then S is weakly connected. Suppose S is not a total dominating set. Then S is a dominating set of G and T_x is a total dominating set in H for every $x \in S \setminus N_G(S)$, by Theorem 3.2. Since $\gamma_t(H) \neq 2$, it follows that $|T_x| > 2$ for every $x \in S \setminus N_G(S)$. Now, by Lemma 3.1, $\gamma_{wt}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. Since $|C| = \sum_{x\in S\cap N_G(S)} |T_x| + \sum_{x\in S\setminus N_G(S)} |T_x|$, it follows that $\gamma_{wt}(G) < |C|$. Let S_1 be a minimum weakly connected total dominating set of G and set $Q_x = \{a\}$ for every $x \in S$, where $a \in V(H)$. Put $Q = \cup_{x\in S_1}(\{x\} \times Q_x)$. Then Q is a weakly connected total dominating set of $G[H]$ by Theorem 3.2. Moreover, $|Q| = |S_1| = \gamma_{wt}(G)$. Thus, $|Q| < |C|$, contrary to our assumption of C. Therefore, S is a weakly connected total dominating set of G. Using a similar argument, it can be shown that S minimum weakly connected total dominating set of G and $|T_x| = 1$ for all $x \in S$.

For the converse, suppose that $C = \cup_{x \in S}(\{x\} \times T_x)$ and S is a minimum weakly connected total dominating set of G with $|T_x| = 1$ for all $x \in S$. By Theorem 3.2, C is a weakly connected total dominating set of $G[H]$. If $C_1 = \cup_{x \in S_1}(\{x\} \times L_x)$ is another total dominating set of $G[H]$, then, by Theorem 3.2, S_1 is a weakly connected dominating set of G. Let $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. By Theorem 3.2,

$$|D_1| + 2|D_2| \leq \sum_{x \in D_1} |L_x| + \sum_{x \in D_2} |L_x| = |C_1|.$$

Thus, by Lemma 3.1, $\gamma_{wt}(G) = |C| \leq |C_1|$. This implies that C is a minimum weakly connected total dominating set of $G[H]$. □

The next result gives the weakly connected total domination number of the lexicographic product of two connected non-trivial graphs.

Corollary 3.5 *Let G and H be nontrivial connected graphs. Then $\gamma_{wt}(G[H]) = \gamma_{wt}(G)$.*

Proof: Let S be a minimum weakly connected total dominating set of G. Pick $a \in V(H)$ and set $T_x = \{a\}$ and $C = \cup_{x \in S}(\{x\} \times T_x)$. By Corollary 3.3 and Corollary 3.4, C is a minimum total dominating set of $G[H]$. Thus, $\gamma_{wt}(G[H]) = |C| = |S| = \gamma_{wt}(G)$. □

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