Static Output Feedback Passivation of a SISO System Characterized by State Matrices

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Abstract—Necessary and sufficient conditions for the existence of a static feedback controller for a LTI single-input single-output system, which renders the system passive, are derived. The novelty is that these conditions are expressed in terms of the state space matrices of the open loop system. The derived conditions pertain to all possible cases: proper transfer functions, strictly proper transfer functions, state matrix A invertible or singular.

I. INTRODUCTION

It is well known that a physical system which is passive has properties which makes the behavior of the system “friendly”. For example, it is well known that a negative feedback connection of two passive systems is always asymptotically stable, i.e. the composite system consisting of the minimal realization of the two systems is asymptotically stable.

A passive linear time-invariant system is a system whose transfer function is positive real. The notion of a positive real (PR) function and of a strict positive real (SPR) function is cardinal in many areas of engineering systems, in particular in control theory and in circuit theory. A partial list of such areas would be design of passive filters, absolute stability theory, output feedback stabilization of uncertain systems, adaptive control systems, switching systems, variable structure systems, etc. There are numerous articles on these subjects since positive real functions were first introduced by O. Brune [1]. A brief review of this literature is given in [10].

Hence, a problem of equal importance would be the following: given a rational transfer function $G(s)$ which is not SPR, what are the conditions such that using an output feedback, the closed loop transfer function would become SPR or, in other words, passive. It is claimed in [2, Theorem 2] that if no static output feedback such that the transfer function of the closed loop system is SPR exists, then there does not exist an output dynamic feedback such that the transfer function of the closed loop system is SPR, as well. Thus, we only have to consider the conditions on $G(s)$ such that there exists a static output feedback rendering a closed loop SPR transfer function.

Some recent work has been done with regard to this latter problem. In [2], necessary and sufficient conditions for the existence of a static output feedback rendering a closed loop SPR transfer function are given, but these conditions depend on the existence of a positive definite matrix complying with a certain matrix inequality. In [3] and [4], necessary and sufficient conditions for the existence of a static output feedback rendering a closed loop SPR transfer function are given. These are expressed in terms of the transfer function of the open loop. Based on these conditions, an algorithm using linear matrix inequalities (LMI) is proposed in [5] to find a constant tandem matrix and a constant output feedback matrix which render a closed loop SPR transfer function. A different approach from the above, namely to use a feedforward action in order to obtain an SPR transfer function, is proposed in [6].

In this paper we consider single-input single-output systems and derive necessary and sufficient conditions for the existence of a static output feedback rendering a closed loop SPR transfer function, but the novelty is that our conditions are expressed in terms of the state matrices of the system. Our results are completely general, in the sense that we treat the following three cases: $d \neq 0$ (the transfer function is proper but not strictly proper), $d = 0$ and the matrix $A$ is invertible (the transfer function is strictly proper), and $d = 0$ and the matrix $A$ is singular.

For the sake of completeness, we give the definitions of positive real (PR) rational function and of strictly positive real (SPR) rational function. Note, as usual, the square matrix $A$ is said to be stable if all its eigenvalues are in the open left half of the complex plane.

Definition 1: A rational function $F(s)$ is PR if:
1) $F(s)$ is real for real values of $s$.
2) $\text{Re}[F(s)] \geq 0$ for $\text{Re}[s] \geq 0$.

Equivalently,

Definition 2: A rational function $F(s)$ is PR if:
1) $F(s)$ is real for real values of $s$.
2) $F(s)$ has no poles in the open right half of the complex plane.
3) If $F(s)$ has purely imaginary poles (including $s = 0$ and $s = \infty$), they are simple and have real positive residues.
4) $\text{Re}[F(j\omega)] \geq 0$ for $-\infty \leq \omega \leq \infty$. 
Definition 3: [7] A rational function $F(s)$ is SPR if $F(s - \varepsilon)$ is PR for some $\varepsilon > 0$.

Equivalently,

Definition 4: A rational function $F(s)$ is SPR if:
1) $F(s)$ is real for real values of $s$.
2) $F(s)$ has no poles in the closed right half of the complex plane.
3) $\text{Re}\left[F(j\omega)\right] > 0$ for $-\infty < \omega < \infty$.
4) $[7] \lim_{\omega \to \infty} \left\{\omega^2 \text{Re}\left[F(j\omega)\right]\right\} > 0$.

II. STATEMENT OF THE PROBLEM
Consider the linear time-invariant single-input single-output system:
\[ \Sigma : \dot{x} = Ax + bu \]
\[ y = c^T \dot{x} + du \]
where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{n \times 1}$, $d \in \mathbb{R}$.

As mentioned in the Introduction, our purpose is to derive necessary and sufficient conditions on the state matrices $A, b, c, d$ of a single-input, single-output, linear, time-invariant system $\Sigma$, such that a static output feedback can passivate the closed-loop system. These necessary and sufficient conditions are derived for the following three cases:

Case No. 1: $d \neq 0$
Case No. 2: $d = 0$, $A$ is invertible
Case No. 3: $d = 0$, $A$ is singular, but the system is controllable and observable.

Before dwelling into the theorems and proofs, we should note the following, on which our results are based. The transfer function of the system $\Sigma$ in (1) is given by
\[ G(s) = d + c^T (sI - A)^{-1} b = \frac{P(s)}{Q(s)} = \frac{\sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i} . \]  
(2)

It can be deduced from [3] that the necessary and sufficient conditions for $G(s)$ to be passivated by static output feedback are the following three points:

A. No zeros of $G(s)$ are contained in the closed right half complex plane.
B. $n \leq m + 1$
C. In the case when $n = m + 1$ the leading coefficient of $G(s)$ is positive
\[ b_m \frac{a_0}{a_n} > 0; \quad b_m \neq 0, \quad a_0 \neq 0 . \]

Note that for a state space characterization of the system, as in (2), the transfer function must have $m \leq n$. Thus, in conjunction with point B above, we must have $m \leq n \leq m + 1$, which leaves only the two possibilities $n = m$ or $n = m + 1$.

III. THE CASE $d \neq 0$

Theorem 1: The necessary and sufficient condition for the system $\Sigma$ in (1) to be passivated by a static output feedback is that the eigenvalues of the matrix
\[ A - \frac{1}{d} bc^T \]  
(3)
are in the open left half complex plane (“stable”), except possibly eigenvalues in the closed right half complex plane (“unstable”) which are identical to the “unstable” eigenvalues of the matrix $A$, with the same multiplicity.

PROOF:

Sufficiency: Consider point A above, which yields
\[ d + c^T (sI - A)^{-1} b \neq 0 \quad \text{in} \quad \text{Re}[s] \geq 0 . \]  
(4)
Since any two vectors $U$ and $V$ comply with the identity
\[ 1 + V^T U = \text{det}\left[I + UV^T\right] \]  
(5)
and we define
\[ V^T = \frac{1}{d} c^T ; \quad U = (sI - A)^{-1} b , \]  
(6)
we have instead of (4),
\[ d \left\{ \text{det}\left[I + (sI - A)^{-1} bc^T \frac{1}{d}\right]\right\} \neq 0 \quad \text{in} \quad \text{Re}[s] \geq 0 . \]  
(7)
If $A$ is stable, we have
\[ \text{det}\left[sI - A\right] \neq 0 \quad \text{in} \quad \text{Re}[s] \geq 0 . \]  
(8)
Therefore, multiplying (7) by (8) and recalling that $d \neq 0$, the condition (4) becomes
\[ \text{det}\left[sI - A + bc^T \frac{1}{d}\right] \neq 0 \quad \text{in} \quad \text{Re}[s] \geq 0 . \]  
(9)
In this case, the condition (9), and hence point A above, are satisfied if, and only if, the matrix $A - \frac{1}{d} bc^T$ is stable.

If $A$ is not stable, then there are values of $s$, say $s_i$, with $\text{Re}[s_i] \geq 0$, such that
\[ Q(s_i) = \text{det}(sI - A) = 0 . \]  
(10)
For these values of $s = s_i$, the closed loop transfer function becomes:
\[ H(s_i) = \frac{G(s_i)}{1 + KG(s_i)} = \frac{P(s_i)}{Q(s_i) + KP(s_i)} = \frac{1}{K} , \]  
(11)
where $K$ is the output feedback
\[ u = Ky . \]  
(12)
Therefore the closed loop transfer function is strictly positive real (passive) at these points $s = s_i$, for any positive $K$ without...
additional conditions.

Consider now point B above, \( n \leq m + 1 \). Clearly, for \( d \neq 0 \), \( m = n \), which complies with point B without any additional constraints.

Point C above is irrelevant, since \( n \neq m + 1 \) in this case. This concludes the sufficiency proof of Theorem 1.

**Necessity:** If \( A \) is stable, then the sufficiency proof above obviously proves necessity as well. If \( A \) is not stable, suppose the matrix \( [A + \frac{1}{d} bcT] \) has an eigenvalue at \( s = s_i \), \( \text{Re}[s_i] \geq 0 \), which is not identical to an eigenvalue of \( A \), in contradiction to Theorem 1. This implies that for \( s = s_i \), (9) is violated but (8) is satisfied:

\[
\det \left[ sI - A + bcT \frac{1}{d} \right] = 0; \quad \text{Re}[s_i] \geq 0 ,
\]

(13)

\[
\det \left[ sI - A \right] \neq 0 .
\]

(14)

Since (9) is the product of (7) and (8), this implies that (7) is violated for \( s = s_i \),

\[
\det \left( I + (sI - A)^{-1} bcT \frac{1}{d} \right) = 0
\]

and so is (4), which is point A above.

This concludes the necessity proof of Theorem 1.

**Comment 1:** It is obvious from the proof of Theorem 1 that a sufficient condition for the system \( \Sigma \) in (1) to be passivated by a static output feedback is that the matrix \( [A + \frac{1}{d} bcT] \) is stable, whether the matrix \( A \) is stable or not.

**Corollary 1:** If the matrix \( A \) is stable, then the necessary and sufficient condition for the system \( \Sigma \) in (1) to be passivated by a static output feedback is that the matrix \( [A + \frac{1}{d} bcT] \) be stable.

The proof of Corollary 1 is evident from the proof of Theorem 1 above.

IV. THE CASE \( d = 0, A \) IS INVERTIBLE

In this case, \( G(s) \) is strictly proper having a simple zero at \( s = \infty \).

**Theorem 2:** The necessary and sufficient conditions for the system \( \Sigma \) in (1) to be passivated by a static output feedback are:

1) The eigenvalues of the matrix

\[
\left[ A^{-1} - \frac{1}{c' A^{-1} b} A^{-1} bcT A^{-1} \right]
\]

are in the open left half complex plane (“stable”), except for an eigenvalue of multiplicity one at the origin and except possibly eigenvalues in the closed right half complex plane (“unstable”) which are identical to the “unstable” eigenvalues of the matrix \( A^{-1} \), with the same multiplicity.

2) \( c^T b > 0 \).

This condition appears also in [2].

**Proof:**

The closed loop transfer function

\[
H(s) = \frac{G(s)}{1 + KG(s)}
\]

(18)

is SPR if, and only if, there exists \( \varepsilon > 0 \) such that

\[
H(s - \varepsilon) = \frac{G(s - \varepsilon)}{1 + KG(s - \varepsilon)}
\]

is positive real. and \( H(s^{-1} - \varepsilon) \) is positive real if, and only if,

\[
H \left( \frac{1}{s} - \varepsilon \right) = \frac{G \left( \frac{1}{s} - \varepsilon \right)}{1 + KG \left( \frac{1}{s} - \varepsilon \right)}
\]

(19)

(20)

is positive real. However, (20) is positive real if, and only if,

\[
H \left( \frac{1}{s} - \varepsilon \right) = \frac{1}{G \left( \frac{1}{s} - \varepsilon \right)} + K \triangleq F(s)
\]

(21)

is positive real.

Now, for \( F(s) \) to be positive real, the following requirements should be satisfied:

1) No poles of \( F(s) \), which are the poles of \( 1 / \left[ G \left( \frac{1}{s} - \varepsilon \right) \right] \), are in the open right half complex plane. In other words, all poles of \( 1 / \left[ G \left( \frac{1}{s} \right) \right] \) are in the open left half complex plane. Hence, this requirement is formulated as “all the zeros of \( G \left( \frac{1}{s} \right) \) should be in the open left half complex plane, except for a zero of multiplicity one at the origin \( s = 0 \)” (which refers to \( G(\omega) \)).

2) If there are purely imaginary poles of \( F(s) \), they must be simple and have positive real residues. However, the previous requirement ensures that there are no such purely imaginary poles of \( F(s) \), except the pole at the origin. Hence, requirement number 2 becomes that the residue of the pole of \( F(s) \) at the origin, which is the residue of the pole of \( 1 / G(s) \) at infinity \( (s = \infty) \), be positive. Thus, requirement number 2 becomes

\[
a_n / b_m > 0 \quad \text{where} \quad m = n - 1 .
\]

(22)

3) The last requirement is that

\[
\text{Re} \left[ F(j\omega) \right] = \text{Re} \left[ \frac{1}{G \left( \frac{1}{j\omega} - \varepsilon \right)} \right] + K \geq 0
\]

(23)

for \(-\infty \leq \omega \leq \infty \).

Let \( \Omega = -1/\omega \). Then this requirement becomes:

\[
\text{Re} \left[ \frac{1}{G(j\Omega - \varepsilon)} \right] + K \geq 0 \quad \text{for} \quad -\infty \leq \Omega \leq \infty .
\]

(24)
Since the zeros of $G(j\Omega - \varepsilon)$ at finite values of $\Omega$ are *discrete* points, and since we can always replace $\varepsilon$ with some $\delta$, $0 < \delta < \varepsilon$, then we can always make sure that $\text{Re} \left[ \frac{1}{G(j\Omega - \varepsilon)} \right]$ is finite for some value of $\varepsilon > 0$ and finite $\Omega$. Therefore, since $K > 0$ can be chosen as large as desired, the requirement (24) can be satisfied for finite $\Omega$, with no additional conditions. The only point which remains to be checked is what happens to

$$\text{Re} \left[ \frac{1}{G(j\Omega - \varepsilon)} \right] \quad \text{as} \quad \Omega \to \infty.$$ (25)

Recalling (2), it can readily be shown by direct substitution and calculation that

$$\lim_{\Omega \to \infty} \left\{ \text{Re} \left[ \frac{1}{G(j\Omega - \varepsilon)} \right] \right\} = a_{n-1}b_{n-1} - a_nb_{n-2}. \quad (26)$$

Since the expression in (26) is finite, requirement number 3 can always be satisfied with $K > 0$ as large as desired, with no additional conditions.

We conclude that $G(s)$ in this case can be passivated by a static output feedback if, and only if, all the zeros of $G(1/s)$ are in the open left half complex plane, except for a zero of multiplicity one at the origin (requirement number 1), and complying with Eq. (22) (requirement number 2).

Using [8] we have

$$G \left( \frac{1}{s} \right) = \tilde{c}^T(sI - \tilde{A})^{-1}\tilde{b} + \tilde{d}, \quad (27)$$

where

$$\tilde{A} = A^{-1}, \quad (28a)$$

$$\tilde{b} = -A^{-1}b \quad (28b)$$

$$\tilde{c}^T = c^TA^{-1} \quad (28c)$$

$$\tilde{d} = -c^TA^{-1}b. \quad (28d)$$

The definitions in Eqs. (28) are meaningful, since we assume that $A$ is invertible. Therefore, according to the previous arguments in the proof of Theorem 1, requirement number 1 is that the eigenvalues of

$$\tilde{A} - \frac{1}{\tilde{d}}\tilde{b}\tilde{c}^T, \quad (29)$$

or,

$$A^{-1} - \frac{1}{c^TA^{-1}b}A^{-1}bc^TA^{-1} \quad (30)$$

are “stable”, except for an eigenvalue of multiplicity one at the origin and except possibly “unstable” eigenvalues which are identical to the “unstable” eigenvalues of the matrix $\tilde{A} = A^{-1}$, with the same multiplicity, which is the first condition of Theorem 2.

For the case $d = 0$, it is easily verified that

$$b_n/a_n = c^Tb = b^Tc \quad (31)$$

so that requirement number 2 becomes the second condition of Theorem 2 (17). Q.E.D.

*Comment 2:* If $c^Tb = 0$ and $d = 0$, then $n > m + 1$. That is, the open loop transfer function $G(s)$ cannot be passivated by static output feedback.

V. THE CASE $d = 0$, $A$ IS SINGULAR, $\Sigma$ IS CONTROLLABLE AND OBSERVABLE

If the system is controllable and observable, this case can be readily transformed to Case No. 2, where $A$ is not singular, by applying a pre-conditioning static output feedback. It is proved in [9] that if an output feedback

$$u = k^*y + u^*$$

is applied to a controllable and observable system as in (1), with $d = 0$ and $A$ singular, then $(A + bk^*c^T)$ in the resulting system

$$\dot{x} = (A + bk^*c^T)x + bu^*$$

$$y = c^Tx$$

is non-singular for almost all $k^*$. Here “almost all” refers to the fact that $(A + bk^*c^T)$ is non-singular for generic gain $k$, and so one can just choose a random $k^*$ and verify that $(A + bk^*c^T)$ is non-singular.

Thus, Theorem 2 includes case No. 3 where $A$ is singular as well, by replacing $A$ with $(A + bk^*c^T)$.

VI. EXAMPLE

Consider the minimum phase LTI system which is not passive:

$$\begin{align*}
\dot{x} &= Ax + bu \\
y &= c^Tx + du
\end{align*}$$

with

$$A = \begin{bmatrix}
-4 & -5 & -1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, b = \begin{bmatrix}
-2 \\
0 \\
0
\end{bmatrix}, c = \begin{bmatrix}
-2 \\
-2 \\
0
\end{bmatrix}, \quad (32)$$

and $d = -2$. In this case it can be confirmed that the conditions of Theorem 1 apply, so that there exists a static output feedback which results in the closed loop system becoming passive. In particular, the closed loop system is passive with the controller:

$$u = ky + \bar{u}$$

where $-\infty < k < -1.23$.

VII. CONCLUSIONS

Necessary and sufficient conditions for the existence of a static output feedback rendering a passive closed loop single-input single-output system are derived. The novelty is that these conditions are expressed in terms of the state space matrices of the open loop system. Future work will report results for MIMO and Descriptor systems.
REFERENCES


