Constraint-Tightening and Stability in Stochastic Model Predictive Control

Matthias Lorenzen¹, Fabrizio Dabbene², Roberto Tempo², and Frank Allgöwer¹

Abstract—Constraint tightening to non-conservatively guarantee recursive feasibility and stability in Stochastic Model Predictive Control is addressed. Stability and feasibility requirements are considered separately, highlighting the difference between existence of a solution and feasibility of a suitable, a priori known candidate solution. Subsequently, a Stochastic Model Predictive Control algorithm which unifies previous results is derived, leaving the designer the option to balance an increased feasible region against guaranteed bounds on the asymptotic average performance and convergence time. Besides typical performance bounds, under mild assumptions, we prove asymptotic stability in probability of the minimally robust positive invariant set obtained by the unconstrained LQ-optimal controller. A numerical example, demonstrating the efficacy of the proposed approach in comparison with classical, recursively feasible Stochastic MPC and Robust MPC is provided.

Index Terms—Stochastic model predictive control, constrained control, chance constraints, discrete-time stochastic systems, receding horizon control, linear systems

I. INTRODUCTION

It is well-known that a moving horizon scheme like Model Predictive Control (MPC) might incur significant performance degradation in the presence of uncertainty and disturbances. This fact was already recognized in early publications on dynamic programming, see for instance [1] Chapter 9.5. To cope with this disadvantage, in recent years Robust MPC has received a great deal of attention for linear systems [2], [3] as well as for nonlinear systems [4], [5]. In many cases, a stochastic model can be formulated to represent the uncertainty and disturbance, as for instance in the case of inflow material quality and purity in a chemical process or wind speed and turbulence in aircraft or wind turbine control. This fact, and the inherent conservativeness of robust formulations, has led to an increasing interest in Stochastic Model Predictive Control (SMPC). A probabilistic description of the disturbance or uncertainty allows to optimize the average performance or appropriate risk measures. Furthermore, allowing a (small) probability of constraint violation, by introducing so-called chance constraints, seems more appropriate in some applications, e.g. meeting the demand in a warehouse or bounds on the temperature or concentrations in a chemical reactor. Besides, chance constraints lead to an increased region of attraction without changing the prediction horizon. Still, hard constraints, e.g. due to physical limitations, can be considered in the same setup.

The first problem in Stochastic MPC is the derivation of computationally tractable methods to propagate the uncertainty for evaluating the cost function and the chance constraints. Both are multivariate integrals, whose evaluation requires the development of suitable techniques. A second problem in SMPC is related to the difficulty of establishing recursive feasibility, which is essential for stability. Indeed, in classical MPC, recursive feasibility is usually guaranteed through showing that the planned input trajectory remains feasible in the next optimization step. This idea is extended in Robust MPC by requiring that the input trajectory remains feasible for all possible disturbances.

In Stochastic MPC, a certain probability of future constraint violation is in general allowed, which leads to significantly less conservative constraint tightening for the predicted input and state, because worst-case scenarios become very unlikely. However, in this setup, the probability distribution of the state prediction at some future time depends on both the current state and the time to go. Hence, even under the same control law, the violation probability changes from time k to time k + 1 and this might render the optimization problem infeasible.

The first problem, uncertainty propagation and tractable reformulation of chance constraints, has gained significant attention and different methods to exactly evaluate, approximate or bound the desired quantities have been proposed in the MPC literature. An exact evaluation is in general only possible in a linear setup with Gaussian noise or finitely supported uncertainties [6]. Approximate solutions include particle approaches [7] or polynomial chaos expansion [8].

Bounding methods with guaranteed confidence include [9], [10], where the authors use the so-called scenario approach to cope with the chance constraint and determine at each iteration an optimal feedback gain ([9]) or feed-forward input ([10]), respectively. While this approach allows for nearly arbitrary uncertainty in the system, the online optimization effort increases dramatically and recursive feasibility cannot be guaranteed. In [11], [12] the authors use an online sampling approach as well, but show how the number of samples can be decreased significantly. For linear systems with parametric uncertainty, [13] proposes to decompose the uncertainty tube
into a stochastic part computed offline and a robust part which is computed online. The paper [14] computes online a stochastic tube of fixed complexity using a sampling technique, but a mixed integer problem needs to be solved online. In [13] layered sets for the predicted states are defined and a Markov Chain models the transition from one layer to another.

For linear systems with additive stochastic disturbance, the system is usually decomposed into a deterministic, nominal part and an autonomous system involving only the uncertain part. The approaches can then be divided into (i) computing a confidence region for the uncertain part and using this for constraint tightening, see [16] for an ellipsoidal confidence region, and (ii) directly tightening the constraints given the evolution of the uncertain part, e.g. [17] and [18]. A slightly different approach is taken in [19], where the authors first determine a confidence region for the disturbance sequence, as well, but then employ robust optimization techniques. Using the same setup, in [20] the focus is put on guaranteeing bounded variance of the state under hard input constraints.

The second problem, recursive feasibility, has seemingly attracted far less attention. The issue has been highlighted in [21] and a rigorous solution has been provided in [16], [17], where “recursively feasible probabilistic tubes” for constraint tightening are proposed. Instead of considering the probability distribution $\ell$ steps ahead given the current state, the probability distribution $\ell$ steps ahead given any realization in the first $\ell - 1$ steps is considered. This essentially leads to a constraint tightening with $\ell - 1$ worst-case and one stochastic prediction for each prediction time $\ell$. In [18] the authors propose to compute a control invariant region and to restrict the next state to be inside this region. This procedure leads to a feasible region which is less restrictive, given the affine feedback structure in the MPC control law, but stability issues are not discussed.

The main contribution of this paper is to propose a non-conservative Stochastic MPC scheme that is computationally tractable, while guaranteeing recursive feasibility. This is achieved by introducing a novel approach which unifies in an original way the previous results, combining the asymptotic performance bound of [17] with the advantages of the least restrictive approach in [18]. Unlike previous works, we explicitly study the case when the optimized input sequence does not remain feasible in the next time instance – but only up to a desired probability $\varepsilon_f \in [0,1]$. Recursive feasibility is guaranteed through an additional constraint on the first step. With $\varepsilon_f = 1$ a scheme similar to [18] and with $\varepsilon_f = 0$, SMPC with recursively feasible probabilistic tubes is recovered. We introduce a constraint tightening, which allows the parameter $\varepsilon_f$ to be used as a tuning parameter to balance the guarantees on convergence speed and performance against the size of the feasible region. Under mild assumptions, we prove stability in probability of the minimally robust positive invariant region obtained by the unconstrained LQ-optimal controller. The proof reveals the relation of $\varepsilon_f$ to bounds on the expected time until convergence. As suggested in [22] the online algorithm is kept simple and the main computational effort is offline. The resulting offline chance constrained programs are briefly discussed and an efficient solution strategy using a sampling approach is provided.

The remainder of this paper is organized as follows. Section II introduces the receding horizon problem to be solved. In Section III the proposed finite horizon optimal control problem is derived, starting with a suitable constraint reformulation, followed by recursive feasibility considerations of the optimization problem and the candidate solution. The section concludes with a summary of the algorithm. The theoretical properties are summarized in Section IV where a performance bound and stability result are derived. A discussion on constraint tightening concludes the section and demonstrates the advantages of the approach. The computation of the offline constraint tightening is discussed in Section V followed by numerical examples that underline the advantages of the proposed scheme. Finally, Section VI provides some conclusions and directions for future work.

Preliminary results have been presented in [23]. Building on these results, methods to bound the probability that the candidate solution remains feasible are introduced and the implication on system theoretic properties stability and performance are analyzed thoroughly. A discussion on how to deal with joint chance constraints is presented and the numerical example has been updated to support the theory. Related results for systems with parametric uncertainty have been presented in [24], where constraint tightening via offline uncertainty sampling is addressed.

**Notation:** The notation employed is standard. Uppercase letters are used for matrices and lower case for vectors. $[A]_j$ and $[a]_j$ denote the $j$-th row and entry of the matrix $A$ and vector $a$, respectively. Positive (semi)definite matrices $A$ are denoted $A \succ 0$ ($A \succeq 0$) and $\| x \|_A^2 = x^T Ax$. The set $\mathbb{N}_{\geq 0}$ denotes the positive integers and $\mathbb{N}_{\geq 0} = \{ 0 \} \cup \mathbb{N}_{>0}$, similarly $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$. We use $x_k$ for the (real, measured) state at time $k$ and $x_{j/k}$ for the state predicted $l$ steps ahead at time $k$. The set of finite control action sequences $v_{0/jk}, \ldots, v_{r-1/jk}$ of length $T$ will be denoted $v_{T/k}$. $A \oplus B = \{ a+b | a \in A, b \in B \}$, $A \ominus B = \{ a \in A | a+b \in A \ominus b \in B \}$ denotes the Minkowski sum and Pontryagin set difference, respectively. To simplify notation, we use the convention $\sum_{k=a}^b c_k = 0$ for $a > b$.

**II. Problem Setup**

In this section, we first describe the system to be controlled and afterwards introduce the basic Model Predictive Control algorithm.

**A. System Dynamics, Constraints, and Objective**

Consider the following linear, time-invariant system with state $x_k \in \mathbb{R}^n$, control input $u_k \in \mathbb{R}^m$ and additive disturbance $w_k \in \mathbb{R}^{m_w}$

$$x_{k+1} = Ax_k + Bu_k + B_w w_k.$$  \hspace{1cm} (1)

The disturbance sequence $(w_k)_{k \in \mathbb{N}_{\geq 0}}$ is assumed to be a realization of a stochastic process $(W_k)_{k \in \mathbb{N}_{\geq 0}}$ on which we take the following assumption.

**Assumption 1.** $W_k$ for $k = 0, 1, 2, \ldots$ are independent and identically distributed $\mathbb{W}$-valued, zero mean random variables. $\mathbb{W}$ is bounded and convex.
The system is subject to probabilistic constraints on the state and hard constraints on the input

\[ \mathbb{P}_k\{[H]_j x_{k+l} \leq [h]_j \} \geq 1 - [\varepsilon]_j, \quad j \in [1, p], \quad l \in \mathbb{N}_{>0} \quad (2a) \]

\[ Gu_{k+l} \leq g \quad k \in \mathbb{N}_{\geq 0} \quad (2b) \]

with \( H \in \mathbb{R}^{p \times m}, \ G \in \mathbb{R}^{q \times m}, \ h \in \mathbb{R}^p, \ g \in \mathbb{R}^q, \ \varepsilon \in [0, 1]^p \) and the assumption that \( u_{k+l} \) is a measurable function in \( x_{k+l} \). The notation \( \mathbb{P}_k\{\mathcal{A}\} = \mathbb{P}\{\mathcal{A}|x_k\} \) denotes the probability of an event \( \mathcal{A} \) given the realization of \( x_k \). Equation (2a) restricts to \( [\varepsilon]_j \) the probability of violating the linear state constraint \( j \) at the future time \( k+l \), given the realization of the current state \( x_k \).

The control objective is to (approximately) minimize \( J_{\infty} \), the expected value of an infinite horizon quadratic cost

\[ J_{\infty} = \lim_{T \to \infty} \mathbb{E}_{\mathcal{F}} \left\{ \frac{1}{T} \sum_{i=0}^{T} x_i^T Q x_i + u_i^T R u_i \right\} \quad (3) \]

with \( Q \in \mathbb{R}^{m \times m}, \ Q > 0, \ R \in \mathbb{R}^{p \times p}, \ R > 0. \)

**B. Receding Horizon Optimization**

To solve the control problem, a Model Predictive Control algorithm is considered. The approach consists of repeatedly solving an optimal control problem with horizon \( T \), but implementing only the first control action. The main goal is to suitably design the cost and constraints of the finite horizon optimal control problem, such that in closed-loop operation the constraints (2) are satisfied, the online optimization remains feasible and the system is stabilized.

In order to decrease the uncertainty in the state predictions, the simulated future state of the system \( x_{l+1|k} = z_{l|k} + e_{l|k} \) is split into a deterministic, nominal part \( z_{l|k} \) and a stochastic error part \( e_{l|k} \). Let \( K \in \mathbb{R}^{n} \) be a stabilizing feedback gain such that \( A + BK \) is Schur, i.e., its eigenvalues are within the open unit circle. A prestabilizing error feedback \( u_e = Ke \) is employed, which leads to the predicted input \( u_{l|k} = K e_{l|k} + v_{l|k} \) with \( v_{l|k} \) being the free MPC optimization variables. Hence, the dynamics of the predicted nominal state and error are given by

\[ z_{l+1|k} = Az_{l|k} + B v_{l|k} \quad z_{0|k} = x_k \quad (4a) \]

\[ e_{l+1|k} = A e_{l|k} + B w_{l|k} + e_{0|k} = 0. \quad (4b) \]

The finite horizon cost \( J_T(x_k, u_{T|k}) \) to be minimized at time \( k \) is defined as

\[ J_T(x_k, u_{T|k}) = \mathbb{E}_k \left\{ \sum_{j=0}^{T-1} (z_{l|k}^T Q z_{l|k} + u_{l|k}^T R u_{l|k}) + z_{T|k}^T P z_{T|k} \right\} \quad (5) \]

where \( P \) is the solution to the discrete-time Lyapunov equation \( A_{cl}^T P A_{cl} + Q + K^T R K = P \). It should be remarked that the expected value can be solved explicitly, which gives a quadratic, finite horizon cost function in the deterministic variables \( z_{l|k} \) and \( v_{l|k} \)

\[ J_T(z_{0|k}, v_{T|k}) = \sum_{j=0}^{T-1} (z_{l|k}^T Q z_{l|k} + v_{l|k}^T R v_{l|k}) + z_{T|k}^T P z_{T|k} + c \quad (6) \]

where \( c = \mathbb{E}_k \left\{ \sum_{j=0}^{T-1} e_{ik}(Q + K^T R K)e_{ik} + e_{ik}^T Pe_{ik} \right\} \) is a constant term and can therefore be neglected in the optimization.

The full finite horizon optimization problem to be solved online is given in the following definition. The constraint sets \( Z_i \) and \( V_i \) will be derived from the chance constraints (2a) and some suitable terminal constraint as described in the next section.

**Definition 1** (Finite Horizon Optimization Problem). Given the system dynamics (4), cost (6) and nominal constraint sets \( Z_i \), \( V_i \) and \( Z_f \), the finite horizon optimization problem to be solved in each time step is

\[ \min_{v_{T|k}} J_T(z_{0|k}, v_{T|k}) \]

\[ \text{s.t.} \quad z_{l+1|k} = A z_{l|k} + B v_{l|k}, \quad z_{0|k} = x_k \]

\[ z_{l|k} \in Z_i, \quad l \in [1, T] \]

\[ v_{l|k} \in V_i, \quad l \in [0, T - 1] \]

\[ z_{T|k} \in Z_f. \]

**III. CONSTRAINT TIGHTENING AND STOCHASTIC MPC ALGORITHM**

This section addresses the Stochastic MPC synthesis part. First, the deterministic, nonconservative constraint sets \( Z_i \) and \( V_i \) are derived such that the constraints (2a) hold in closed-loop operation. Thereafter, the constraints are further modified in order to provide stochastic stability guarantees and recursive feasibility under all admissible disturbance sequences.

**A. Constraint Tightening**

Given the evolution of the disturbance (4b), similar to (17), (25), we directly compute tightened constraints offline. However, we neither aim at the computation of recursively feasible probabilistic tubes nor at robust constraint tightening for the input.

**State Constraints**: The probabilistic state constraints (2a) can non-conservatively be rewritten in terms of convex, linear constraint sets \( Z_i \) on the predicted nominal state \( z_{l|k} \), as stated in the following proposition.

**Proposition 1.** The real system (1) satisfies the chance constraints (2a) for \( k = 1, \ldots, T \) and \( j = 1, \ldots, p \) if and only if the nominal system (4a) satisfies the constraints

\[ Hz_{l|k} \leq \eta \quad l \in [1, T] \quad (8) \]

where \( v_i \) can be computed as

\[ [\eta]_j = \max_{\eta} \quad \eta \in \mathbb{R} \quad (9) \]

\[ \mathbb{P}_k \{ \eta \leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j. \]

**Proof**: The constraint (2a) can be rewritten in terms of \( z_{l|k} \) and \( e_{l|k} \) as

\[ \mathbb{P}_k \{ [H]_j z_{l|k} \leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j \]

with \( e_{l|k} \) being the solution to (4b). Equation (10) is equal to \( \mathbb{P}_k \{ \eta \leq [h]_j - [H]_j e_{l|k} \} \geq 1 - [\varepsilon]_j \). This is equal to \( [H]_j z_{l|k} \leq \eta \), with \( \eta = \max_{\eta} \quad \eta \in \mathbb{R} \) s.t.
\( \mathbb{P}_k \{ \bar{\eta} \leq [h] - [H]e_{jk} \} \geq 1 - [\varepsilon]_j \). The maximum value exists as (9) can equivalently be written as

\[
-\lceil \eta \rceil_j = \min \eta \\
\text{s.t. } \mathbb{P}_k([H]e_{jk} - [h] \leq \eta) \geq 1 - [\varepsilon]_j.
\]

By the assumptions on the disturbance, the cumulative density function \( F_{He-h} \) for the random variable \([H]e_{jk} - [h] \) exists and is right-continuous. Using \( F_{He-h} \), the constraint can be written as \([F_{He-h}(\eta)] \geq 1 - [\varepsilon]_j \) which concludes the proof. ■

Proposition 1 leads to \( TP \) independent, one dimensional, linear chance constrained optimization problems (9) that need to be solved offline. Efficient computational methods will be presented in Section 7.

**Input Constraints**: Instead of a robust constraint tightening for the hard constraints on the input \( u \), we propose a stochastic constraint tightening. In other words, we take advantage of the probabilistic nature of the disturbance and require that the combination of MPC feedforward input sequence and static error feedback remains feasible for most, but not for all possible disturbance sequences. This is in line with the fact that at a later time the optimal input is recomputed and adapted to the disturbance realization.

Let \( \eta_a \in [0, 1) \) be a small probability. Similarly to the state constraint tightening, we replace the original constraint (2a) with

\[
Gv_{jk} \leq \mu_l \quad l \in [0, T-1]
\]

where \( \mu_l = [\mu_{1l} \ldots \mu_{ql}] \) are the solutions to \( gT \) one dimensional, linear chance constrained optimization problems (11)

\[
\mu_l = \max \mu \\
\text{s.t. } \mathbb{P}_k \{ \mu \leq [g]_j - [G]_j Ke_{jk} \} \geq 1 - \varepsilon_a.
\]

**Terminal Constraint**: We first construct a recursively feasible admissible set under the local control law and then employ a suitable tightening to determine the terminal constraint \( Z_f \) for the nominal system.

**Proposition 2**. For the system (1) with input \( u = Kx \) let \( \mathcal{X}_f = \{ x \mid H_j x \leq h_j \} \) be a (maximal) robust positive invariant polytope inside the set

\[
\mathcal{X}_f = \{ x \mid HA_{jk}x \leq \eta_1, G_{jk}x \leq g \}
\]

with \( \eta_1 = [\eta_{11}, \eta_{12}, \ldots, \eta_{1p}] \) according to (9). For any initial condition in \( \mathcal{X}_f \) the constraints (2a) are satisfied in closed-loop operation with the local control law \( u = Kx \) for all \( k \geq 0 \).

**Proof**: By definition, the set \( \mathcal{X}_f \) is forward invariant for all disturbances and constraint (2a) holds for all \( x_k \in \mathcal{X}_f \). Furthermore

\[
\mathbb{P}_k \{[H]x_k \leq [h] \mid x_{k-1} \} \geq 1 - [\varepsilon]_j \quad \forall j \in [1, p]
\]

is satisfied for all states \( x_{k-1} \in \mathcal{X}_f \), which is sufficient for (2a).

1 For an in-depth theoretical discussion, practical computation and polytopic approximations of \( \mathcal{X}_f \) see [26] and [27] for an overview or [27] for details.

To define the terminal constraint \( Z_f \) for the nominal system, a constraint tightening approach similar to (9) is needed. Let \( \varepsilon_f \in [0, 1) \) be a small probability, we define the terminal region

\[
Z_f = \{ z \mid H_f z \leq \eta_f \}
\]

with

\[
\eta_f = \max \eta \\
\text{s.t. } \mathbb{P}_k \{ \eta \leq [h]_j - [H]_j e_{jk} \} \geq 1 - \varepsilon_f.
\]

**B. Recursive Feasibility**

As it has been pointed out in previous works, e.g. [17], [21], the probability of constraint violation \( \ell \) steps ahead at time \( k \) is not the same as \( \ell - 1 \) steps ahead at time \( k + 1 \) given the realization of state \( x_{k+1} \). Hence, the tightened constraints (8), (11) and (12) do not guarantee recursive feasibility.

A commonly used approach to recover recursive feasibility is to use a mixed worst-case/stochastic prediction for constraint tightening [17], [18]. In [18] the authors point out that this approach is rather restrictive and leads to higher average costs if the optimal solution is “near” a chance constraint. Alternatively, the authors propose to use a constraint only on the first input to obtain a recursively feasible algorithm which is, given the affine feedback structure in the MPC, least restrictive.

In the following, we propose a hybrid strategy: We impose a first step constraint to guarantee recursive feasibility and the previously introduced stochastic tube tightening with terminal constraint and cost to guarantee stability. At the cost of further offline reachability and controllability set computation, the proposed approach has the advantage of being less conservative compared to recursively feasible stochastic tubes, but yet guaranteed to stabilize the system at the minimal positive invariant region.

Let

\[
C_T = \left\{ \begin{bmatrix} z_{0jk} \\ v_{0jk} \end{bmatrix} \in \mathbb{R}^{n+m} \mid \begin{array}{c}
\exists v_{1jk}, \ldots, v_{T-1jk} \in \mathbb{R}^{m} \\
\forall j, l \in [1,T] \\
\forall j \end{array} \right\}
\]

be the \( T \)-step set and allowed first step input for the nominal system under the tightened constraints. The set can be computed via standard recursion e.g. [26]. \( C_T \) defines the feasible states and first inputs of the receding horizon optimization.

Since \( C_{T,x} = \text{Proj}_x(C_T) \), the projection onto the first \( n \) coordinates, is not necessarily robustly positive invariant, it is important to further compute a (maximal) robust control invariant polytope \( C_{T,x}^\ast \), with the constraint \( (x,u) \in C_T \). This can be computed recursively, see [26], [27], and references therein for algorithms and their finite termination.

**Remark 1**. The computation of the sets \( C_T \) and \( C_{T,x} \) may be involved for high dimensions and limits the proposed approach. Nevertheless, this is a long-standing, standard problem in (linear) controller design and efficient algorithms to exactly calculate or to approximate those sets exist, e.g. [27].
C. Feasibility of the Candidate Solution

To prove asymptotic stability, not only existence of a feasible solution at each time \( k \) is of interest, but also feasibility of a suitable candidate solution at time \( k+1 \).

Let \( \epsilon_f \) be the probability of infeasibility of the candidate solution. As shown in the next section, \( \epsilon_f \) appears explicitly in the average cost and implicitly influences the transient phase until the state is inside the terminal region. In this subsection, we introduce and define a candidate solution for time \( k+1 \), given a feasible solution at time \( k \). We further propose an extension to the constraint tightening introduced in Subsection III-A in order to explicitly bound this probability a priori. This extension essentially closes the gap between “recursively feasible probabilistic tubes” [17] and the “least restrictive” scheme presented in [18].

**Proposition 3.** Let the state, input and terminal constraints in (7) be given by

\[
Z_l = \{z \mid Hz \leq \tilde{\eta}_l\} \\
V_l = \{v \mid GV \leq \bar{\mu}_l\} \\
Z_f = \{z \mid H_f z \leq \tilde{\eta}_f\}
\]

and assume \( Z_f \subseteq Z_T \).

If \( \bar{v}_{T|k} \) is a feasible solution at time \( k \), the candidate solution \( \bar{v}_{k+1} \) at time \( k+1 \) is feasible for the state, input and terminal constraints with probability no smaller than \( 1 - \epsilon_f \).

**Proof:** Let \( \bar{z}_{k+1} \) be the state prediction derived from the candidate solution. With probability \( 1 - \epsilon_f \) it holds \( w_k \in W_f \), hence it suffices to show recursive feasibility for \( w_k \in W_f \).

Assume \( w_k \in W_f \), recursive feasibility of the terminal constraint follows from robust recursive feasibility of the terminal region and robust reachability for all \( w_k \in W_f \). Furthermore \( \tilde{z}_{T|k+1} \in Z_T \) is implied by the assumption \( Z_f \subseteq Z_T \).

Constraint satisfaction for the state constraints for \( l < T \) follows inductively

\[
[H\tilde{z}_{l+1}|k]_j = [H\tilde{z}_{l+1}|k]_j + HA_{cl}^j B_ww_k]_j \leq [\tilde{\eta}_{l+1} + HA_{cl}^j B_ww_k]_j
\]

\[
= \min_{i=0,\ldots,l} \left\{ [\tilde{\eta}_{l+1}|i]_j + [H]_j A_{cl}^l B_ww_k \right\}
\]

\[
= \min_{i=0,\ldots,l} \left\{ [\tilde{\eta}_{l+1}|i]_j - \max_{w_k \in W_f} \left[ [H]_j \sum_{k=1}^{l+1-k} A_{cl}^{l+1-k} B_ww_k \right] + [H]_j A_{cl}^l B_ww_k \right\}
\]

\[
\leq \min_{i=0,\ldots,l} \left\{ [\tilde{\eta}_{l+1}|i]_j - \max_{w_k \in W_f} \left[ [H]_j \sum_{k=2}^{l+1} A_{cl}^{l+1-k} B_ww_k \right] + [H]_j A_{cl}^l B_ww_k \right\}
\]

\[
= \min_{i=0,\ldots,l} \left\{ [\tilde{\eta}_{l+1}|i]_j - \max_{w_k \in W_f} \left[ [H]_j \sum_{k=1}^{l+1} A_{cl}^{l+1-k} B_ww_k \right] \right\} = \tilde{\eta}_l
\]

for all \( k, l \) and \( j \). Similar for the input, replacing \( H \) and \( \tilde{\eta}_l \) by \( GK \) and \( \mu_l \).

**D. Resulting Stochastic MPC Algorithm**

The final MPC algorithm can be divided into two parts: (i) an offline computation of the involved sets and (ii) the repeated online optimization. In the following, we present the algorithm and state its control theoretic properties.

**Offline:** Solve (9), (12) and (14) to determine \( \eta_l, \mu_l \) and \( \eta_f \). Determine the first step constraint \( C_{T,x}^\infty \) according to the previous section.

**Online:** For each time step \( k = 1, 2, \ldots \)

1) Measure the current state \( x_k \).
2) Solve the linearly constrained quadratic program (7) subject to state and input constraints (8), (11), first step constraint \( C_{T,x}^\infty \) and terminal constraint (15), i.e.

\[
\bar{v}_{T|k} = \arg\min_{\bar{v}_{T|k}} J_T(x_k, v_{T|k})
\]

s.t. \( z_{l+1|k} = Az_{l|k} + Bv_{T|k} \) \( z_{0|k} = x_k \)

\[
H_{T} z_{T|k} \leq \tilde{\eta}_f, \quad l \in [1, T] \\
G v_{T|k} \leq \bar{\mu}_l, \quad l \in [0, T - 1] \\
H_f z_{T|k} \leq \tilde{\eta}_f
\]

3) Apply \( u_k = v_{0|k} \)

**IV. PROPERTIES OF THE PROPOSED SMPC SCHEME**

In this section, we formally derive the control theoretic properties of the proposed SMPC scheme. In particular the influence of \( \epsilon_f \), the probability of the candidate solution being infeasible, is computed. We first derive a bound on the asymptotic average state cost, which highlights the connection to [17]. This is followed by a proof of asymptotic stability in probability of the minimally robust invariant set, which is novel in Stochastic MPC and shows the connection to tube based Robust MPC approaches like [3], [25] and proves a conjecture made in [29]. The section concludes with a discussion on offline relaxation of chance constraints in MPC.
A. Asymptotic Average Performance

Prior to a stability analysis, it is necessary to prove recursive feasibility of the MPC algorithm, which is provided by the following proposition.

**Proposition 4.** The MPC optimization remains feasible if the initial state is inside $C^\infty_{T,x}$.

Proof: Since $C^\infty_{T,x}$ is a robust control invariant subset of the feasible set $C_{T,x}$, a solution exists for $x_k \in C^\infty_{T,x}$. The state $x_{k+1} \in C^\infty_{T,x}$ because $x_{k+1} = z^*_l | x_k + B_w w_k \in C^\infty_{T,x}$ for all $w_k \in \mathcal{W}$ which is guaranteed by last constraint in (15). 

Due to the persistent excitation, it is clear that the system does not converge asymptotically to the origin, but “oscillates” with bounded variance around it. The following theorem summarizes the constraint satisfaction and provides a bound on the asymptotic average stage cost.

**Theorem 1.** If $x_0 \in C^\infty_{T,x}$, then the closed-loop system under the proposed MPC control law satisfies the hard and probabilistic constraints in (3) for all future times and

$$
\lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \mathbb{E}\left\{ \|x_k\|_Q^2 \right\} \leq (1 - \epsilon_f) \mathbb{E}\left\{ \|B_w w\|_T^2 \right\} + \epsilon_f C
$$

with $\epsilon_f$ the maximum probability that the previously planned trajectory does not remain feasible, $C = L \max_{w \in \mathcal{W}} \|B_w w\|$ and $L$ the Lipschitz constant of the optimal value function $J_f(\cdot, v^*_l | k)$ of (15).

Proof: By Proposition 4, the SMPC algorithm is recursively feasible. Then, chance constraint satisfaction follows from Proposition 4 and hard input constraint satisfaction from $e_{0|k} = 0$ and hence $\mu_0 = g$.

To prove the second part, we use the optimal value of (15) as a stochastic Lyapunov function. Let $V(x_k) = J(x_k, v^*_l | k)$ be the optimal value of (15) at time $k$. The optimal value function is known to be continuous, convex and piecewise quadratic in $x_k$ [30]. Hence, a Lipschitz constant $L$ on $C^\infty_{T,x}$ exists. The old input trajectory does not remain feasible with at most probability $\epsilon_f$, but we can bound the cost increase of $V(x_{k+1}) - V(x_k)$ in that case by $L \max_{w \in \mathcal{W}} \|B_w w\|$.

Let $\mathbb{E}\{V(x_{k+1}) | x_k, \tilde{v}_{T+1|k+1} \text{ feasible}\}$ be the expected optimal value at time $k + 1$, conditioning on the state at time $k$ and feasibility of the candidate solution $\tilde{v}_{T+1|k} = v^*_{T+1|k} + K A^T_{t+1} B_w w_k$.

$$
\mathbb{E}\{V(x_{k+1}) | x_k, \tilde{v}_{T+1|k+1} \text{ feasible}\} - V(x_k) \\
\leq \sum_{l=1}^{T} \left\{ \|z^*_l | k\|_Q^2 + \|v^*_l | k\|_R^2 \right\} + \|z^*_{T+1|k} | (Q + K^T RK) \|_P^2 + \|z^*_{T+1|k} \|_P^2 \right\} \\
+ \mathbb{E}\left\{ \sum_{l=0}^{T-1} \|A^T_{l+1} B_w w_k\|_Q | (Q + K^T RK) + \|A^T_l B_w w_k\|_R \right\} \\
- \left( \sum_{l=0}^{T-1} \left\{ \|\epsilon_l\|_Q^2 + \|v^*_l | k\|_R^2 \right\} + \|z^*_l | k\|_P^2 \right\} \\
= \|z^*_l | k\|_Q^2 | (Q + K^T RK) + \|z^*_{T+1|k} \|_P - \|z^*_0 | k\|_Q - \|v^*_0 | k\|_R - \|z^*_T | k\|_P \right\} + \mathbb{E}\left\{ \|B_w w_k\|_R^2 \right\} \\
\leq -\|z^*_0 | k\|_Q^2 + \mathbb{E}\left\{ \|B_w w_k\|_R^2 \right\} = -\|x_k\|_Q^2 + \mathbb{E}\left\{ \|B_w w_k\|_P^2 \right\}
$$

where $v^*_{l|k}$, $z^*_{l+1|k}$, $l = 0, \ldots, T - 1$ denote the optimal solution of (15), respectively predicted state at time $k$ and $z^*_{T+1|k} = (A + BK)^* z^*_T | k$. Note that the expected value of all $w$-$z$ cross-terms equals zero because of the zero-mean and independence assumption. Furthermore, since we defined the terminal cost as the solution to the discrete-time Lyapunov equation it holds that $A_j^* P A_j + Q + K^T RK = P$.

Combining both cases we obtain by the law of total expectation

$$
\mathbb{E}\{V(x_{k+1}) | x_k\} - V(x_k) \\
\leq (1 - \epsilon_f) \left( \mathbb{E}\{V(x_{k+1}) | x_k, \tilde{v}_{T+1|k+1} \text{ feasible}\} - V(x_k) \right) + \epsilon_f \left( -\|x_k\|_G^2 + L \max_{w \in \mathcal{W}} \|B_w w\| \right) \\
\leq -\|x_k\|_G^2 + (1 - \epsilon_f) \mathbb{E}\{\|B_w w_k\|_P^2\} + \epsilon_f C.
$$

The final statement follows by taking iterated expectations. 

Remark 2. A terminal region, which is forward invariant with probability $\epsilon_f$, can be used instead of a robust forward invariant terminal region. In this case, Theorem 1 still holds.

B. Asymptotic Stability

In the following, we assume that the state feedback gain $K$ is chosen to be the unconstrained LQ-optimal solution and we let $\mathcal{X}_m$ be the minimally robust positive invariant set for the system (1) with input $u = Kx$.

In this section, we prove that $\mathcal{X}_m$ is asymptotically stable in probability for the closed-loop system under the proposed Stochastic MPC algorithm. In particular, the proposed MPC control law stabilizes the system to the same set as the robust MPC proposed in [25] or Stochastic MPC proposed in [17]. The different constraint tightening leads to a possibly different transient phase. The price for a larger feasible region is a possibly longer convergence time before the terminal set is reached.

**Definition 3** (Asymptotic Stability in Probability). A compact set $\mathcal{S}$ is said to be asymptotically stable in probability for system (1) with control law $u = k(x)$, if for each $\epsilon \in [0, 1]$ and $\rho \in (0, 1]$ such that $\|x_0\|_S \leq \delta \Rightarrow \mathbb{P}\{\sup_{k \geq 0} \|x_k\|_S \leq \epsilon \} \leq \rho$

and for a neighborhood $\mathcal{N}_2$ of $\mathcal{S}$, for all $\epsilon_2 \in [0, 1]^+$

$$
x_0 \in \mathcal{N}_2 \Rightarrow \lim_{K \to \infty} \mathbb{P}\{\sup_{k \geq K} \|x_k\|_S < \epsilon_2 \} = 1.
$$

$\mathcal{N}_2$ is called region of attraction.

Next, we make the following nonrestrictive assumption that the minimally robust positive invariant set is a proper subset of the terminal region.

**Assumption 2.** Let $\mathcal{B}$ be an open unit ball in $\mathbb{R}^n$. It exists $\lambda \in \mathbb{R}_{>0}$ such that $\mathcal{X}_m + \lambda \mathcal{B} \subseteq \mathcal{X}_f$.

**Theorem 2** (Asymptotic Stability). Under Assumption 2 the set $\mathcal{X}_m$ is asymptotically stable in probability with region of attraction $C^\infty_{T,x}$ for the system (1) with the proposed SMPC controller.
We prove the theorem by first providing the statement under the condition that the candidate solution remains feasible at each time step. Then, we prove that it exists a set $\mathcal{S}$ where this assumption holds and that for every probability $\rho \in (0,1)$ and state $x_0$ in $C_T^x$, it exists a time $N \in \mathbb{N}_{\geq 0}$ such that $P\{x_N \in \mathcal{S}\} \geq 1 - \rho$.

The proof differs from standard proofs using a stochastic Lyapunov function because of the nonzero probability that the candidate solution does not remain feasible during a transient phase.

The following lemma is inspired by Theorem 8 in [25], where robust MPC is considered.

**Lemma 1.** Given the system (1) with $x_0 \in C_T^x$ and the proposed SMPC controller. If Assumption 2 holds and the candidate solution $\tilde{v}_{T|k}$ remains feasible for all $k > 0$, then the state $x_k = \tilde{z}_k + \xi_k$ can be separated into a part $\tilde{z}_k$ and a part $\xi_k$, such that the origin is asymptotically stable for $\tilde{z}_k$ and $\xi_k \in \mathbb{R}^n, \forall k \geq 0$.

Proof: Let

\[
\begin{align*}
\tilde{z}_{k+1} &= A_{cl} \tilde{z}_k + B(u_k - K(\tilde{z}_k + \xi_k)) & \tilde{z}_0 &= x_0 \quad (16a) \\
\xi_{k+1} &= A_{cl} \xi_k + B_{u\xi_w} u_k & \xi_0 &= 0. \quad (16b)
\end{align*}
\]

Given recursive feasibility, in [25] it has been shown that $c_k = u_k - K(\tilde{z}_k + \xi_k)$ is bounded and $c_k \to 0$ for $k \to \infty$. Since $A_{cl}$ is Schur stable, the system (16a) is input to state stable (ISS) with respect to the input $c_k$ and hence $\xi_k$ converges to the origin. Furthermore, for $x_k \in \mathcal{X}_f$ it holds that $c_k = 0$, which together with Assumption 2 implies asymptotic stability of the origin for system (16a).

**Corollary 1.** If Assumption 2 holds and the candidate solution $\tilde{v}_{T|k}$ remains feasible for all $k > 0$, then exists $N_k$ such that $\|\tilde{x}_k\| < \varepsilon$ for all $k \geq N_k$. In particular, there exists $N_f \in \mathbb{N}_{\geq 0}$ such that $x_{k} \in \mathcal{X}_f$ for all $k \geq N_f$ and $x_0 \in C_T^x$.

Proof: From asymptotic stability, it follows that the origin is a uniform attractor for $x_{k} \in \mathcal{X}_f$ and hence $\exists N_f$ such that $\tilde{x}_k \in \lambda \mathcal{B} \\forall k > N_f$ and all $x_0 \in C_T^x$. Using $\tilde{x}_k \in \mathcal{X}_f$, by Assumption 2 this implies $x_k \in \mathcal{X}_f \\forall k > N_f$ and $x_0 \in C_T^x$.

It can be shown that the candidate solution remains feasible for all $k > 0$ if $x_0$ is inside the terminal region. In this case, Lemma 1 holds and we only need to consider $x_k \notin \mathcal{X}_f$.

**Lemma 2.** The terminal region $\mathcal{X}_f$ is robust forward invariant for the closed-loop system under the proposed SMPC algorithm and the candidate solution remains feasible for all $k \geq k_0$ if $x_{k_0} \in \mathcal{X}_f$.

Proof: The unconstrained optimal solution to (15) equals the control inputs generated by the LQR, $v_{ik|k}^* = K_{ik}^* z_{ik|k}^*$. For $z_{ik|k} \in \mathcal{X}_f$ robust forward invariance of the terminal region implies constraint satisfaction of the unconstrained optimal solution because

\[
HA_{cl}(z_{ik|k} + e_{ik|k}) \leq n_1 \quad \forall e_{ik|k}
\]

\[
H_{ik|k+1} \leq n_1 - A_{cl} e_{ik|k} \quad \forall e_{ik|k}
\]

\[
H_{ik|k} \leq n_1 + 1
\]

and similarly for the input and terminal constraints. Hence, in the terminal region, the proposed SMPC controller equals the unconstrained LQR. Since $\mathcal{X}_f$ is robust forward invariant under the unconstrained LQ optimal controller the statement follows.

Under Assumption 2, Lemma 1 and 2 suffice for stability of the proposed algorithm. Before proving attractivity, we need another lemma.

**Lemma 3.** Let $I_k = [k_0, k_0 + N_f - 1]$ denote some interval of length $N_f$ and $A_{k_0}$ the event that the candidate solution $\tilde{v}_{T|k}$ is feasible $\forall k \in I_k$. For each $\rho \in (0,1)$ there exists $N_\rho \in \mathbb{N}_{\geq 0}$

\[
P\{\{N_\rho > k_0 \neq A_{k_0}\} \geq 1 - \rho. \quad (17)
\]

Lemma 2 states that, for each probability $1 - \rho$, we can find a long enough horizon such that, at some point within this horizon, the candidate solution stays feasible $N_f$ consecutive times and hence, by Corollary 1, enters the terminal region.

Proof: Let $1 - \varepsilon I_f$ denote the probability that the candidate solution stays feasible in the next sampling instant. The left hand side of (17) can be crudely over-approximated by the probability of staying feasible during one of the time periods $I_i = [(i+1)N_f - 1]$ for $i \in [0, \frac{N_f}{\varepsilon}].$ For each $I_i$ we have

\[
P\{A_{N_f} \geq (1 - \varepsilon)N_f = 1 - \beta_f. \quad \text{And hence } P\{\{N_\rho > k_0 \neq A_{k_0}\} \geq 1 - (\beta_f)^{\frac{N_f}{\varepsilon}}, \beta_f \in [0,1), \text{the right hand side of the inequality is increasing with } N_\rho \text{ and converges to 1.}
\]

**Lemma 4 (Attractivity).** For all $\varepsilon_2 \in \mathbb{R}_{>0}$

\[
x_0 \in C_T^x \Rightarrow \lim_{K \to \infty} \limsup_{k \to K} \mathbb{E}\{\|x_k\|_x < \varepsilon_2\} = 1.
\]

By Corollary 1 and Lemma 2, the closed-loop converges as soon as the candidate solution remains feasible for $N_f$ consecutive time-steps. By Lemma 3 for any given probability $\rho$, there exists a long enough horizon $N_\rho$ such that this happens with probability $\rho$. We use the Borel-Cantelli Lemma and Fatou’s Lemma to show that in fact the probability grows fast enough.

Proof: Let $N = \max\{N_{\varepsilon_2}, N_f\}$ with $N_{\varepsilon_2}, N_f$ according to Corollary 1 and define the event $B_n = \{\sup_{k>n}\|x_k|_x < \varepsilon_2\}$. By Corollary 1 Lemma 2 it holds $\mathbb{P}\{B_n\} \leq 1 - \mathbb{P}\{\cup_{k_0 = 0}^{N_\rho} A_{k_0}\}. \quad \text{Inserting the explicit bound derived in the proof of Lemma 3 leads to}$

\[
\sum_{n=0}^{\infty} \mathbb{P}\{B_n\} \leq \sum_{n=0}^{\infty} \beta_f^n < \infty.
\]

By the Borel-Cantelli Lemma we have that $\mathbb{E}\{\lim_{n \to \infty} \sup B_n\} = 0$ and hence by Fatou’s Lemma $\lim_{n \to \infty} \sup \mathbb{E}\{B_n\} = 0$ which concludes the proof.

**Proof (Theorem 2):** Stability follows from the robust case together with robust feasible invariance in the terminal region (Lemma 2). Attractivity follows from Lemma 4.

A direct corollary of Theorem 2 is a tighter bound on the asymptotic average performance.
Corollary 2. Under Assumption \([2] \) it holds
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \mathbb{E} \left\{ \|x_k\|_Q^2 \right\} \leq \mathbb{E} \left\{ \|B_n w\|_{\mathbf{P}}^2 \right\}.
\]

C. Discussion: Offline Relaxation of Chance Constraints

In this section, we briefly discuss joint vs single chance constraints of the form
\[
\mathbb{P}\{Hx \leq h\} \geq 1 - \epsilon,
\]
\[
\mathbb{P}\{|H|x \leq |h|\}_{j} \geq 1 - |\epsilon|\}_{j},
\]
respectively, and contrast the approaches taken in [19] and [16] with our approach.

It is a known result in Robust Tube MPC, that the tightened constraint \(Z\) involves at most as many linear inequalities as the original constraint \(X\). For a joint chance constraint, an analogue does not hold. A tightened constraint on \(z\) can in general not be expressed by a finite number of linear constraints and in the cases in which this can be done (e.g., polytopic \(W\) with uniform distribution), the tightened constraint involves in general more linear inequalities than the original one, see Fig. [1] for an example.

Using single chance constraints to approximate joint chance constraints does not lead to an increased number of tightened constraints, but is in general either more conservative or increases the probability of constraint violation at certain points in state space.

We remark that the often proposed option of determining a confidence region for the uncertainty \(w_k\) or the uncertain state \(e_l\) and then requiring the constraints to hold for all realizations within these sets, leads to the same result. This does not approximate the true joint chance constraint but tighter versions of the original constraints, as well. Furthermore, choosing a parametrization for these sets increases the conservatism unless it fits the underlying distribution perfectly, e.g., spheroids for normal distributed uncertainties. In contrast, the approach taken here is tight for arbitrary distributions and constraints given by [2].

With the following illustrative example, we show the advantage of tightening each constraint according to a predefined probability of violating it, instead of jointly optimizing the tightening of all constraints with a given overall violation probability or similarly jointly determining the parameters for the confidence region of the uncertainty. In particular, we show that the latter approach might lead to undesired, conservative results.

Example 1. Consider the one dimensional case \(x = z + e\) with \(e\) having a non symmetric pdf, e.g., \(e + 0.2 \sim \text{Gamma}(2, 0.1)\) and constraint \(\mathbb{P}\{|x| \leq 1\} \geq 1 - \epsilon\). Optimizing the constraint tightening jointly, i.e.
\[
\max_{\eta} \eta_1 + \eta_2
\]
\[
s.t. \ \mathbb{P}\{|\eta_1| \leq 1 - \epsilon, \ \eta_2 \leq 1 + \epsilon\} \geq 1 - \epsilon
\]
to derive the nominal constraints \(-\eta_2 \leq z \leq \eta_1\) or optimizing the bounds of a confidence interval \([\gamma_1, \gamma_2]\) for \(e\)
\[
\max_{\gamma} \gamma_2 - \gamma_1
\]
\[
s.t. \ \mathbb{P}\{|\gamma_1| \leq \epsilon \leq \gamma_2\} \geq 1 - \epsilon
\]
to derive the constraint \(-1 \leq \gamma_1 \leq z \leq 1 - \gamma_2\) leads to a biased outcome. As illustrated in Fig. [2] with \(\epsilon = 0.05\), the result will be \(\eta_1 \approx 0.73\), \(\eta_2 = 0.8\), respectively \(\gamma_1 = -0.2\), \(\gamma_2 \approx 0.27\). The constraint \(x \geq 0\) holds with zero probability of violation and \(x \leq 0\) with \(\epsilon\) probability of violation. If we want to maximize \(x\), the result of the deterministic problem with tightened constraints will be equal to the original chance constrained problem. If we want to minimize \(x\), the result will be equal to the robust problem.

![Figure 1: Constraint \(x_{1,2} \geq 0\) and tightened constraints for \(z\) with \(x = z + e\) and \(e_{1,2} \sim \mathcal{N}(0, 0.2^2)\) or uniform \(e_1 \sim \mathcal{U}(-0.5, 0.5)\) distributed and violation probability \(\epsilon = 0.1\). Dashed lines show tightened constraints for single chance constraints with \(e_j = \epsilon/2\).

![Figure 2: Constraints \(|x| \leq 1\) (solid line), resulting tightened constraints for the nominal state \(z\) (dashed line) and probability density functions for \(x\) with \(z = -0.8\) and \(z = 0.73\), respectively (blue). A joint chance constraint evaluation as described in Example [1] leads to a biased outcome: The lower bound is satisfied with probability 1, whereas the upper bound is satisfied with probability \(1 - \epsilon\). The example shows that jointly optimizing multiple parameters to determine a minimal size confidence region might lead to overly conservative results equal to the deterministic, robust program. Finally, consider the case where the probability density function of \(e\) is single valued over some region, e.g. uniform disturbance model. In this case both optimizations in Example [1] might not have a unique optimizer, a standard problem in determining a confidence region. Hence the conservativeness of the resulting deterministic problem depends on the chosen optimizer and start value.

We conclude this section with emphasizing that direct tightening of single chance constraints gives the best worst-case value in terms of conservativeness of approximating a joint chance constraint by means of offline probability calculation.

Remark 3. If the constraint tightening can be given as a function of the violation probability \(\epsilon_j\), then by using Boole’s Inequality and dynamic risk allocation [33] we can significantly reduce the conservatism at the cost of higher online computations.
V. IMPLEMENTATION AND NUMERICAL EXAMPLE

In this section, we briefly review practical considerations for solving the single chance constrained programs to determine the proposed constraint tightening. Thereafter, the non-conservativeness of the approach with respect to the allowed probability of constraint violation and the increased feasible region is demonstrated in a numerical example.

A. Solving the Single Chance Constrained Programs

There is a vast literature on how to (approximately) solve optimization programs involving single chance constraints. In the following, we briefly state the deterministic solution and then show how to efficiently solve the offline problems \((9), (12)\) and \((14)\) using a sampling approach.

1) Deterministic: Chance constraints are constraints on multivariate integrals. In particular, if the random variable \(W\) has a known probability density function \(f_W(w)\), we can write \((9)\) as

\[
\eta_j = \max \eta \text{s.t. } \int_{\Omega} \mathbf{1}_{\{\eta_j - (u_j) \in \mathcal{F}(\omega_j)\}} \prod_{i=0}^{l-1} f_W(w_i) \, dw_0 \cdots \, dw_{l-1} \geq 1 - [\varepsilon]_j
\]

with \(\varepsilon_{ijk} = \sum_{i=1}^{l} A_{ij} b_i w_i\) and \(\mathbf{1}_{\{\cdot\}}\) being the indicator function. The multivariate integral can be further simplified if the convolution of the distributions of \(B_i w_i\) is known, e.g. if \(W\) is Gaussian. If the inverse cumulative distribution function \(H_i\) of \(W_i\) is known or can be approximated, then

\[
[\eta_j]_i = [h_j]_i - Q(1 - [\varepsilon]_j).
\]

For further discussions on convexity and explicit numerical solutions, see e.g. [34] and references therein.

2) Sampling: Recently, sampling techniques to solve robust and chance constrained problems have gained increased interest [35], [36]. They are independent of the underlying distribution, easy to implement and specific guarantees about their solution can be given. Furthermore, they allow to directly use complicated simulations or measurements of the error, instead of determining a probability density function.

The chance constrained problems \((9), (12), (14)\) can be efficiently solved to the desired accuracy by drawing a sufficiently large number \(N_s\) of samples \(w^{(i)}\) from \(W\) and require the constraint to hold for all, but a fixed number \(r\) of samples. In [37] the authors give the explicit conditions

\[
\begin{align*}
 r & \leq \varepsilon u N_s - \sqrt{2 \varepsilon u N_s} \ln \left(\frac{\varepsilon u N_s}{\beta}\right), \\
 r & \geq \varepsilon u N_s - 1 + \sqrt{3 \varepsilon u N_s} \ln \left(\frac{2}{\beta}\right)
\end{align*}
\]

(18)

to select \(r\) and \(N_s\) such that with confidence \(\beta\) the solution to the sampled program is equal to a chance constrained programs \((9), (12)\) and \((14)\) with \(\varepsilon \in [\varepsilon_l, \varepsilon_u]\).

In general, one has to solve a mixed integer problem or use heuristics to discard samples in a (sub)optimal way. Here, due to the simple structure, a sort algorithm is used to solve the sampled approximation of \((9)\) exactly.

**Proposition 5.** Let \(N_s\) and \(r\) be chosen according to \((18)\). Let \(q_{1-r/N_s}\) be the \((1 - r/N_s)\)-quantile of the set \(\left\{H_i\right\}_{i=1,\ldots,N_s}\) with \(\varepsilon_{ijk} = \sum_{i=1}^{l} A_{ij} b_i w_i^{(i)}\) independently chosen samples from \(W^l\). Then with confidence \(\beta\)

\[
[\eta_j]_i = [h_j]_i - q_{1-r/N_s}
\]

solves \((9)\) with \(\varepsilon \in [\varepsilon_l, \varepsilon_u]\).

B. Numerical Example

In the following, the performance and enlarged region of attraction of the proposed Stochastic MPC scheme is demonstrated. To this end, the DC-DC converter example, previously considered in [16], is implemented. The linearized system is of the form \((1)\) with

\[
A = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix}, \quad B = \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix}, \quad B_w = I_2.
\]

The MPC cost weights are \(Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}\), \(R = 1\) and the prediction horizon is \(T = 8\). For disturbance attenuation in the predictions \((45)\) and terminal region \(X_f\), the unconstrained LQR is chosen. The disturbance distribution is assumed to be a truncated Gaussian with the covariance matrix \(\Sigma = \frac{1}{2\pi} I_2\) truncated at \(||w||^2 \leq 0.02\).

For the robust set calculations we chose a polytopic outer approximation with 8 hyperplanes. For the stochastic constraint tightening we used the described sampling approach with \(\varepsilon_l = 0.95\varepsilon\), \(\varepsilon_u = 1.05\varepsilon\) and confidence \(\beta = 10^{-4}\). The constrained tightening was performed without explicitly bounding \(\varepsilon_f\), the probability of the candidate solution not being feasible, as described in Section III-C.

The time for computing the MPC input using quadprog with the standard interior point algorithm in Matlab R2014b was approximately 4ms for each scheme on an Intel Core i7 with 3.4GHz.

**Constraint Violation:** First, consider the single chance constraint

\[
P_k \{x_1 \leq 2\} \geq 0.8 \quad (19)
\]

for the system above with initial state \(x_0 = [2.5, 2.8]^\top\).

In [16] it has been shown that Stochastic MPC achieves lower closed-loop cost compared to Robust MPC. The approach presented in [16], using a confidence region, yields 14.4% constraint violation in the first 6 steps.

In contrast, the approach taken here, i.e. a direct constraint tightening, achieves a closed-loop operation tight at the constraint. A Monte Carlo simulation with \(10^4\) realizations showed an average constraint violation in the first 6 steps of 20% and an even lower closed-loop cost. Simulation results of the closed-loop system for 500 random disturbances are shown in Figure [3]. The left plot shows the complete trajectories for a simulation time of 15 steps. The right plot shows the constraint violation in more detail; \((19)\) is satisfied non-conservatively hence leaving more control authority for optimizing the performance.

For comparison, we remark that Robust MPC achieves 0% constraint violation and that the LQ optimal solution violates the constraint 100% in the first 3 steps.
Feasible Region: The main advantage of the proposed SMPC scheme is the increased feasible region. To illustrate this feature, we assume the same setup as before, but with additional chance constraints on the state and hard input constraints

\[
\begin{align*}
\mathbb{P}_k \{ |x_1| \leq 2 \} &\geq 0.8 \\
\mathbb{P}_k \{ |x_2| \leq 3 \} &\geq 0.8 \\
|u| &\leq 0.2.
\end{align*}
\]

According to the described setup, we allowed 5\% constraint violation in the predictions for the input and a probability of 0.05 of not reaching the terminal region. In closed-loop operation the input was treated as hard constraint.

Figure 4 shows the different feasible regions of Robust MPC, Stochastic MPC with constraint tightening using recursively feasible probabilistic tubes and the proposed method using probabilistic tubes and a first step constraint. The feasible region of the proposed Stochastic MPC has 1.7 times the size of the feasible region of standard SMPC and 3.4 times the size of the feasible region of Robust MPC. The Robust MPC scheme has been taken from [3] and is only included here for a more complete comparison, it is of course significantly smaller than having stochastic constraints.

In Figure 5 the decrease in the size of the feasible region is plotted, when a constraint tightening as described in Section II-C is employed. We note that even for moderate values of $\varepsilon_f$ a significant increase in the feasible region can be gained.

VI. CONCLUSIONS AND FURTHER WORK

The proposed stabilizing Stochastic MPC algorithm provides a significantly increased feasible region through separating the requirements of recursive feasibility and stability. The algorithm unifies the results obtained in [17] and [18] allowing to balance convergence speed and performance guarantees against the size of the feasible region. Absolute bounds of the disturbance are used to provide a first step constraint to guarantee recursive feasibility. The stochastic information about the disturbance is used to prove an asymptotic bound on the closed-loop performance, which naturally resembles the bound obtained by the unconstrained LQ-optimal controller. Furthermore, under mild assumptions, asymptotic stability with probability one of the set $X_\infty$ has been proven, which is novel in the Stochastic MPC literature.

The online computational effort is equal to that of nominal MPC. An efficient, broadly applicable solution strategy based on randomized algorithms is presented to solve, to the desired accuracy, the offline chance constrained problems for determining the constraint tightening.

Future work will be focused on improving the performance through an online evaluation of the expected cost taking into account possible infeasibility of the optimized input trajectory. Similarly, the idea to incorporate a first step constraint to
guarantee recursive feasibility could be further exploited. In the future this could be applied in a broader context, e.g. it could be nicely combined with ideas of (incomplete) decision trees which show very good results in practice [38], but have no recursive feasibility or stability guarantees. Finally, for a broader applicability, it is necessary to relax the assumption of identically and independently distributed disturbance and allow for parametric uncertainty.

REFERENCES