Control of chaos in nonlinear systems with time-periodic coefficients

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In this study, some techniques for the control of chaotic nonlinear systems with periodic coefficients are presented. First, chaos is eliminated from a given range of the system parameters by driving the system to a desired periodic orbit or to a fixed point using a full-state feedback. One has to deal with the same mathematical problem in the event when an autonomous system exhibiting chaos is desired to be driven to a periodic orbit. This is achieved by employing either a linear or a nonlinear control technique. In the linear method, a linear full-state feedback controller is designed by symbolic computation. The nonlinear technique is based on the idea of feedback linearization. A set of coordinate transformation is introduced, which leads to an equivalent linear system that can be controlled by known methods. Our second idea is to delay the onset of chaos beyond a given parameter range by a purely nonlinear control strategy that employs local bifurcation analysis of time-periodic systems. In this method, nonlinear properties of post-bifurcation dynamics, such as stability or rate of growth of a limit set, are modified by a nonlinear state feedback control. The control strategies are illustrated through examples. All methods are general in the sense that they can be applied to systems with no restrictions on the size of the periodic terms.

Keywords: nonlinear control systems; time-periodic systems; chaos; feedback control; bifurcation control

1. Introduction

The phenomenon of chaos in nonlinear systems has been investigated extensively in the last two decades. Recently, there has been a significant interest in controlling chaotic systems. Among the control strategies used to suppress chaos, perhaps the most well known is the Ott–Grebogi–Yorke method (Ott et al. 1990). This method relies on the facts that chaotic systems are very sensitive to initial conditions and that there is typically an infinite number of unstable periodic orbits embedded in the chaotic attractor. One of these unstable orbits is stabilized by controlling perturbations. The method uses the Poincaré map of the system. Some of its limitations are that the parameter changes can only be discrete and only certain embedded orbits can be stabilized. Another control

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method has been suggested by Pyragas (1992). In his approach, the stabilization is achieved either by periodic external perturbation or by feedback control with time delay. A detailed review of chaos control can be found in the work by Fradkov & Evans (2002).

Most of the work, however, has dealt with autonomous systems. In the physics literature, a non-autonomous system is often converted into an autonomous, discrete system using a Poincaré map in which the fixed points of the Poincaré map correspond to periodic orbits of the system. In this case, since it is not easy to obtain an analytical expression for the Poincaré map, one often needs to obtain an approximate expression based on numerical methods or experimental data. Local stabilization of periodic orbits in chaotic systems by feedback control has been dealt with in previous works by several authors (Cheng & Dong 1993a, b; Chen 1996). However, in all these papers, the non-autonomous nature of the problems was ignored and the Routh–Hurwitz criterion was applied as if the systems were autonomous. Since the problem of stabilization of a periodic orbit leads to a non-autonomous system with time-periodic coefficients, Floquet theory must be used to guarantee stability of the system. Sinha et al. (2000) have proposed a general approach in the design of active controllers for nonlinear systems exhibiting chaos. In this method, it is shown that a system exhibiting chaos can be driven to a desired periodic motion by combining a feed-forward and a feedback controller. The feedback controller is designed using a method proposed by Sinha & Joseph (1994), where a time-invariant auxiliary system is constructed and stabilized with pole placement method. However, this method uses a least-square approximation approach and in certain parameter ranges, the stability of the periodic orbits may not be guaranteed.

In this paper, three approaches are suggested for local chaos control in nonlinear systems with time-periodic coefficients. The first one is based on the symbolic computation of the Floquet transition matrix (FTM) associated with the linear part of the system. By using the technique introduced by Sinha & Butcher (1997), the state transition matrix (STM) of the resulting system is calculated symbolically. The symbolic expression of the STM contains the unknown control gains and these can be assigned by placing the Floquet multipliers at the desired locations inside the unit circle of the complex plane.

The second method is based on the idea of feedback linearization of nonlinear periodic systems. Feedback linearization is an important branch of nonlinear control. In the last two decades, feedback linearization of affine nonlinear systems has been investigated extensively (Khalil 2002). However, these works deal with autonomous systems only. There are no results for nonlinear time-periodic systems. Our technique employs time-periodic normal form reduction (Arnold 1998) and the introduction of a nonlinear periodic state feedback. The two ideas, when combined, can transform the nonlinear periodic closed-loop system into an equivalent linear time-periodic system. Then, the symbolic approach can be used to design the controller and transform it back to the original coordinates, where it takes the form of a nonlinear time-periodic state feedback.

The third approach employs the concept of nonlinear bifurcation control. The goal in this case is to modify the nonlinear characteristics of a bifurcation (such as stability, size or rate of growth of a limit cycle) along the route to chaos, such that the onset of chaos is delayed. Bifurcation control is an emerging area of nonlinear control, and already has an extensive literature for autonomous
systems. An excellent review can be found in Chen et al. (2000). Recent results obtained by Dávid & Sinha (2000, 2003) show that the idea of nonlinear bifurcation control can be extended to systems with time-periodic coefficients. The method is based on the construction of dynamically equivalent time-invariant normal forms. It involves periodic centre-manifold reduction, time-dependent normal form theory and construction of versal deformations. A purely nonlinear feedback controller is designed in the transformed domain for the equivalent autonomous system, and then transformed back to the original coordinates.

2. Statement of the problem

Consider the nonlinear control system with time-periodic coefficients given by

\[ \dot{x} = F(x, u(x), \alpha, t), \]  

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R} \), \( \alpha \in \mathbb{R}^m \) and \( F(x, u, \alpha, t+T) = F(x, u, \alpha, t) \). The vector \( \alpha \in \mathbb{R}^m \) contains the parameters of the system. We assume that \( \{x=0, u=0\} \) is an equilibrium point. Let us note that this can be assumed without any loss of generality. If our purpose is to study an autonomous or periodic system around a known periodic solution, a simple coordinate transformation leads to the same type of system. Also, the same type of equation is obtained by attempting to drive an autonomous or periodic system to a desired periodic orbit. Let us assume that there exists a range of parameters \( \{\alpha_l, \alpha_h\} \) for which the system exhibits chaotic behaviour. Our goal is to eliminate chaos from the given range of the parameters. This is achieved by three different methods: employing a linear controller using symbolic computation, designing a nonlinear controller designed by feedback linearization and implementing a purely nonlinear controller obtained by local bifurcation control.

3. Linear full-state feedback control using symbolic computation

In this case, we seek the control input \( u(x) \) to be a linear function of \( x \) and we assume that equation (2.1) has the form

\[ \dot{x} = F(x, \alpha, t) + L_u(\alpha, t)u. \]  

We seek a control input \( u(x) \in \mathbb{R} \) such that for any \( \alpha \in \{\alpha_l, \alpha_h\} \) it drives the system to a desired periodic orbit \( x_t(t) = x_t(t+ iT) \), where \( i=1, 2, 3, ... \) or \( i=1/2, 1/3, 1/4, ... \). It is important to assume that the periodicity of this orbit and that of the system are integer multiples of each other. Otherwise, we would obtain a closed-loop system with quasi-periodic coefficients. Consider a control law consisting of two parts: a feed-forward \( u_t \) and a feedback \( u_t \) given by

\[ L_u(t)u = L_u(u_t + u_t) = \dot{x}_t - F(x_t, \alpha, t) - L_u k(x - x_t), \quad k = \{k_1, k_2, ..., k_n\}. \]  

Defining \( e = x - x_t \), as the error between the actual and desired trajectories, equation (3.1) in the error variables becomes

\[ \dot{e} = G(e, \alpha, t) - L_u(\alpha, t)ke. \]  

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After linearizing about the zero equilibrium, we obtain

\[ \dot{e} = (L_e(a, t) - L_u(a, t)k)e := L_k(a, t)e, \]  

(3.4)

where \( L_k \) is a time-periodic matrix with period \( T \). Now, the objective is to design \( k \), the control gain vector, such that \( \lim_{t \to \infty} e = 0 \). The stability of equation (3.4) is determined by its FTM, \( \Phi_k(a, k) \). According to the Floquet theory, if all the eigenvalues of the FTM are within the unit circle of the complex plane, the system is asymptotically stable. The FTM is obtained by evaluating the STM of equation (3.4) at the end of the principal period \( T \). Sinha & Butcher (1997) have developed a symbolic computation technique for obtaining the STM of a linear periodic system explicitly as a function of the system parameters. The method employs Picard iterations and expansion into shifted Chebyshev polynomials, and has been shown to be very efficient. The equivalent integral form of equation (3.4) can be written as

\[ e(t) = e(0) + \int_0^t L_k(a, \tau)e(\tau)d\tau. \]  

(3.5)

Applying the method of Picard iteration, we find the \( (k+1) \)th approximation

\[ e^{(k+1)}(t) = e(0) + \int_0^t L_k(a, \tau_k)e^{(k)}(\tau_k)d\tau_k \]

\[ = \left[ I + \int_0^t L_k(a, \tau_k)d\tau_k + \int_0^t L_k(a, \tau_k) \int_0^\tau L_k(a, \tau_{k-1})d\tau_{k-1}d\tau_k + \cdots \right] e(0), \]  

(3.6)

where \( \tau_0, \ldots, \tau_k \) are dummy variables. The expression in the square brackets is an approximation to the fundamental solution matrix \( \Phi(t, a) \) since it is truncated after a finite number of terms (iterations). After the period is normalized by the transformation \( t = T\tau \), the one-periodic system matrix \( \bar{L}_k(a, \tau) \) is expanded in \( m \)-shifted Chebyshev polynomials of the first kind. By substituting this and utilizing the integral and product operational matrices associated with Chebyshev polynomials, equation (3.6) yields a polynomial expression for the STM in terms of the system parameters and the control gains. Once the STM is found in such a form, the FTM, \( \Phi_k(a, k) \), can also be obtained as an explicit function of system parameters \( a \) and the unknown control gains \( k \). Then, the eigenvalues of \( \Phi_k(a, k) \) can be placed at the desired locations by choosing appropriate values of \( k \) for any \( a \in \{a_h, a_h\} \). However, since \( \Phi_k \) contains high-degree polynomial expressions of \( k \) and \( a \), computing the eigenvalues may not be easy. Instead, \( k \) may be selected by applying a stability criterion for maps, such as the Shur–Cohn criterion, for example. Alternatively, we may map the unit circle of the complex plane into the left half plane by a simple transformation and apply the Routh–Hurwitz criterion. Since the control gains \( k \) can be selected as functions of system parameters, by guaranteeing asymptotic stability of equation (3.4), chaos can be eliminated for the original system (3.1) from its entire parameter range \( \{a_h, a_h\} \).

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(a) An illustrative example

Let us consider the non-autonomous parametrically excited Lorenz equation:
\[ \dot{x} = -2x + 2y, \quad \dot{y} = [26.5 + 5 \cos(2\pi t)]x - y - xz, \quad \dot{z} = xy - 0.6z. \] (3.7)

We would like to drive the system to the desired periodic orbit
\[ x_t = \sin(2\pi t), \quad y_t = \sin(2\pi t), \quad z_t = 20 + \cos(2\pi t). \] (3.8)

We assume the control input vector in the form \( L_u = \{0 \ 1 \ 0\}^T \). By applying the feed-forward and feedback control as described earlier and linearizing the equation for the error variables, we obtain

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3
\end{bmatrix} =
\begin{bmatrix}
-2 & 2 & 0 \\
6.5 + 4 \cos(2\pi t) & -1 & -\sin 2\pi t \\
\sin 2\pi t & \sin 2\pi t & -0.6
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} u_t. \tag{3.9}
\]

Let \( u_t = -k\{e_1 \ e_2 \ e_3\}^T \) with \( k = \{k_1 \ k_2 \ k_3\} \). For simplicity, let us choose \( k_2 = k_3 = 0 \). Then, by applying the symbolic computation technique with 18 Chebyshev polynomials and 22 Picard iterations (cf. Sinha & Butcher 1997), we obtain that if \( k_1 \geq 5.18 \), it guarantees asymptotic stability. More details of controller design for such systems by symbolic computation may be found in the paper by Sinha et al. (2005). The uncontrolled as well as controlled trajectories are shown in figure 1. With two control gains \( k_1 \) and \( k_2 \) \((k_3=0)\), we can compute conditions for \( k_1 \) and \( k_2 \) to yield asymptotic stability. Figure 2 shows stable and unstable areas as a function of the control gains.

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Figure 2. The area of asymptotic stability for the Lorenz equation as a function of the control gains.

4. Local nonlinear control using feedback linearization

In this section, we aim to find a set of coordinate transformations that convert the original nonlinear time-periodic control problem to a dynamically equivalent linear time-periodic system. Assuming that it is possible, a linear controller may be designed via the symbolic approach or by other existing linear methods. When the controller is transformed back into the original state, it becomes nonlinear. One of the advantages of this type of control over linear techniques is that full-state feedback is not required. If a system is uncontrollable linearly or not all the states are available for feedback, a nonlinear method may still work. Also, nonlinear control may provide an increased region of stability when compared with linear methods.

Consider the nonlinear periodic system (3.1) at some fixed value of the system parameters \( \alpha \in \{ \alpha_1, \alpha_n \} \). At this point, the equation becomes parameter independent as

\[
\dot{x} = F(x, t) + L_u(t)u. \tag{4.1}
\]

We would like to drive this system to the periodic orbit \( x_0(t) \) (or to a fixed point), as before. We apply a feed-forward and a feedback control in the form

\[
L_u(t)u = L_u(t)(u_d + u_n) = \dot{x}_0 - F(x_0, t) + L_u(t)u_n. \tag{4.2}
\]

The closed-loop system in the error variable is obtained by substituting equation (4.2) into equation (4.1). After expanding the closed-loop system into Taylor series in the state variables, we have

\[
\dot{e} = L_c(t)e + Q_c(e, t) + C_c(e, t) + \cdots + F_r(e, t) + L_u(t)u_n, \tag{4.3}
\]

where \( Q_c, C_c \) and \( F_r \) are symmetric quadratic, cubic and \( r \)th-order forms of \( e \) with \( T \)-periodic coefficients, respectively. At this point, the real Lyapunov–Floquet (L–F) transformation \( e = Q(t)\tilde{e} \) is applied to transform the linear part into a time-invariant form and then a modal transformation converts the constant linear matrix into a Jordan canonical form as

\[
\tilde{\dot{e}} = J\tilde{e} + \tilde{Q}_c(\tilde{e}, t) + \tilde{C}_c(\tilde{e}, t) + \cdots + \tilde{F}_r(\tilde{e}, t) + L_u(t)u_n. \tag{4.4}
\]
Note that since $Q(t)$, in general, is $2T$-periodic, so is equation (4.4). Let us denote the eigenvalues of $J$ as $\lambda_j, j=1, \ldots, n$. We assume that the eigenvalues of $J$ are not critical and the nonlinear terms of this equation may be eliminated by applying the time-dependent normal form theory in the form of a sequence of near-identity transformations. We start with the quadratic terms and then proceed to eliminate the cubic terms and so on. Then

$$\ddot{\varepsilon} = y + \varphi_2(y, t) + \varphi_3(y, t) + \cdots + \varphi_r(y, t),$$

where $\varphi_j(y, t)$ are unknown functions with $2T$-periodic coefficients, containing only $j$th-order terms of $y$. We will demonstrate the approach only for the $r$th-order terms. We also introduce a transformation for the control input as

$$u_n = \alpha_2(y, t) + \alpha_3(y, t) + \cdots + \alpha_r(y, t) + \beta(y, t) v,$$

where $\alpha_i(y, t)$ and $\beta(y, t)$ are unknown functions. The function $\alpha_i(y, t)$ contains only $i$th-order terms in $y$, while $\beta(y, t)$ consists of terms only up to the $(r-1)$th orders in $y$. After applying these transformations, we obtain

$$\dot{y} = Jy + \dot{Q}_y + C_y + \cdots + \dot{L}_n(\alpha_2 + \alpha_3 + \cdots)$$

$$+ \left[ \ddot{F}_r + J \varphi_r - \frac{\partial \varphi_r}{\partial y} Jy - \frac{\partial \varphi_r}{\partial t} \dot{L}_n \alpha_r \right] + \left( I - \frac{\partial \varphi_r}{\partial y} \right) \ddot{L}_n \beta v,$$

assuming that there are no resonances in the terms lower than $r$th order (for the definition of resonance, see equation (4.10) and the following paragraph). The square bracket contains all the $r$th-order terms, and the transformation does not affect the lower orders. Therefore, if the homological equation

$$\ddot{F}_r + J \varphi_r - \frac{\partial \varphi_r}{\partial y} Jy - \frac{\partial \varphi_r}{\partial t} \dot{L}_n \alpha_r = 0,$$

is satisfied, $\ddot{F}_r$ can be removed. We seek a solution by expanding the known and unknown functions into finite Taylor–Fourier series. Let

$$\ddot{F}_r(y, t) = \sum_{j=1}^{n} \sum_{m_j=r} \sum_{k=-q}^{q} f_{j,k}(m_1, \ldots, m_n) e^{ik \frac{\pi}{T} t} y_1^{m_1} \cdots y_n^{m_n} e_j, $$

$$\varphi_r(y, t) = \sum_{j=1}^{n} \sum_{m_j=r} \sum_{k=-q}^{q} \varphi_{r,j,k}(m_1, \ldots, m_n) e^{ik \frac{\pi}{T} t} y_1^{m_1} \cdots y_n^{m_n} e_j,$$

$$\alpha_r(y, t) = \sum_{m_j=r} \sum_{k=-q}^{q} \alpha_{r,j,k}(m_1, \ldots, m_n) e^{ik \frac{\pi}{T} t} y_1^{m_1} \cdots y_n^{m_n},$$

where $e_j$ is the $j$th member of the natural basis. If we substitute these expressions into equation (4.8), we obtain a set of independent algebraic equations in the form

$$\left( ik \frac{\pi}{T} + \sum_{p=1}^{n} m_p \lambda_p - \lambda_j \right) \varphi_{r,j,k}(m_1, \ldots, m_n) = f_{r,j,k}(m_1, \ldots, m_n) + l_{nj} \alpha_{r,k}(m_1, \ldots, m_n).$$

Since it is assumed that the eigenvalues of $J$ are not critical, the term in the parentheses on the left-hand side is not zero and we can express $\varphi$ in terms of $\alpha$ and the nonlinear term corresponding to $f_{r,j,k}(m_1, \ldots, m_n)$ can be removed. If the term in
the parentheses is zero (called the resonance case), then, in general, it is not possible to eliminate the corresponding nonlinear term. Thus, by repeating this procedure for all the nonlinear terms, equation (4.4) can be reduced to

$$
\dot{y} = Jy + \left( I - \frac{\partial \varphi_1(\alpha)}{\partial y} \right) \cdots \left( I - \frac{\partial \varphi_n(\alpha)}{\partial y} \right) \tilde{L}_u \beta v.
$$

(4.11)

This equation is still nonlinear in the control term, but it may be reduced further by utilizing the unknown $\beta$ functions. Assume that

$$
\beta(y, t) = \beta_0(t) + \beta_1(y, t) + \beta_2(y, t) + \cdots + \beta_{n-1}(y, t),
$$

(4.12)

where $\beta_i(y, t)$ are $i$th-order polynomials of $y$ and are $2T$-periodic in time. We need to determine the $\beta$ and $\alpha$ functions such that

$$
\left( I - \frac{\partial \varphi_1(\alpha)}{\partial y} \right) \cdots \left( I - \frac{\partial \varphi_n(\alpha)}{\partial y} \right) \tilde{L}_u \beta = B(t),
$$

(4.13)

where $B(t)$ is a time-periodic vector. By using Fourier–Taylor expansions of $\beta_i(y, t)$ similar to those in equation (4.9), and collecting the powers of $y$, this becomes a set of algebraic equations for the $\beta$ and $\alpha$ coefficients. Generally speaking, we obtain a set of underdetermined equations that may not have a solution or have an infinite number of solutions. Of course, the number of equations depends on the number of Fourier terms in the $L$–$F$ transformation $Q(t)$ and the control influence matrix $L_u$. Here, we freely choose a periodic function, $\beta_0(t)$ and from equation (4.13) set

$$
\tilde{L}_u \beta_0(t) = B(t).
$$

(4.14)

Then, we have

$$
\left( I - \frac{\partial \varphi_1(\alpha)}{\partial y} \right) \cdots \left( I - \frac{\partial \varphi_n(\alpha)}{\partial y} \right) \tilde{L}_u \beta(y, t) - B(t) = 0.
$$

(4.15)

A detailed discussion on the solvability condition for the general case of multiple inputs and outputs will be presented in a future paper. Assuming that equation (4.15) is solvable, the final form of equation (4.4) is

$$
\dot{y} = Jy + B(t)v,
$$

(4.16)

which is linear, time periodic and dynamically equivalent to the original system. The controller can now be designed by linear methods developed for periodic systems, such as the symbolic computation described in §3a. Once a controller is obtained, the final step is to transform it back to the original domain where it becomes a nonlinear, periodic control.

(a) Illustrative example

Consider the system

$$
\begin{align*}
\dot{x}_1 &= (-1 + \alpha \cos^2 t)x_1 + (1 + \alpha \sin t \cos t)x_2 + \sin^2 t x_1^2 + x_1 x_2 + \cos^2 t u, \\
\dot{x}_2 &= (-1 + \alpha \sin t \cos t)x_1 + (1 + \alpha \sin^2 t)x_2 + \cos^2 t x_1 x_2 + \sin^2 t u.
\end{align*}
$$

(4.17)
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For this particular equation, since its linear part is commutative, the L–F transformation can be obtained in a closed form (this equation is chosen for its computational simplicity for illustrative purposes and it is not a representation of any physical system; however, the method is not restricted to this type of problem). We would like to drive this system to zero. This implies that the feedforward control $u_f$ is identically zero and equation (4.17) represents equation (4.3). For the parameter value $\alpha=1.1$, this system exhibits chaotic behaviour, as shown in figures 3 and 4. We seek our transformations in the form

$$\begin{cases}
  x_1 = Q(t)z_1, \\
  x_2 = Q(t)z_2
\end{cases},
$$

$$\begin{cases}
  z_1 = y_1 + \varphi_1(2,0)(t)y_1^2 + \varphi_1(1,1)(t)y_1y_2 + \varphi_1(0,2)(t)y_2^2 \\
  z_2 = y_2 + \varphi_2(2,0)(t)y_1^2 + \varphi_2(1,1)(t)y_1y_2 + \varphi_2(0,2)(t)y_2^2
\end{cases}$$

$$u = \alpha_{(2,0)}(t)y_1^2 + \alpha_{(1,1)}(t)y_1y_2 + \alpha_{(0,2)}(t)y_2^2 + (\beta_0(t) + \beta_{11}(t)y_1 + \beta_{12}(t)y_2)v.$$  \hfill (4.18)

If we set $\beta_{\alpha}(t)=1+\cos^2t+\cos^2t\sin t$, then there are still five unknown coefficient functions to find. Therefore, from equation (4.15), we have four equations and five unknowns. If we let $\alpha_{11}(t) = \sum_{k=-\infty}^{\infty} e^{i\omega t}$ and apply the transformations (4.18), we can solve the resulting algebraic equations for the rest of the $\alpha$ and $\beta$ functions. After applying the transformations we obtain the following reduced system:

$$\begin{cases}
  \dot{y}_1 = 0.1y_1 + \frac{1}{32}(2 + 44\cos t + 28\sin t - \cos 2t - 5\sin 2t)v, \\
  \dot{y}_2 = -y_2 + \frac{1}{32}(2 - 12\cos t + 12\sin t - \cos 2t - 3\sin 2t)v
\end{cases}$$ \hfill (4.19)

Seeking a full-state feedback controller of the form $v = -(k_1y_1 + k_2y_2)$, the closed-loop system from equation (4.19) is a linear periodic system with unknown gains $k_1$ and $k_2$. We use the symbolic computation method described in §3 to find the appropriate values for these control gains. For $k_1=1$ and $k_2=0.45$, the Floquet multipliers of equation (4.19) are $\mu_1 = -0.673$ and $\mu_2 = -0.003$, respectively, and the system is asymptotically stable. The final step is to transform the controller back into the original state variables. Figures 5 and 6 show uncontrolled and controlled motions in time, in the original $x_1$ variable.
5. Local bifurcation control

It is well known that, in most cases, a dynamic system undergoes a sequence of bifurcations as its behaviour changes from stable to chaotic. Therefore, one can expect that a chaotic behaviour can be delayed to a higher parameter range or even completely eliminated by controlling the bifurcations along the path that led to it. Since in bifurcation control we try to modify only the nonlinear

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characteristics by means of employing purely nonlinear controllers (without changing the linear part of the system), it might be more efficient than the linear methods in terms of control effort.

Let us reconsider our original system given by equation (2.1). We assume that there exists a critical value of the system parameters $\alpha_c \not\in \{\alpha_l, \alpha_h\}$ for which the linear system matrix $L_x$ has $n_1$ Floquet multipliers on the unit circle of the complex plane and $n_2$ multipliers with magnitude less than one, where $n_1 + n_2 = n$. This implies that the system is undergoing a bifurcation at that point. We are concerned with the dynamics in the neighbourhood of the critical point, more precisely, with the post-bifurcation limit sets. The goal is to find a control input $u$, such that $u$ is a purely nonlinear function of $x$ and

(i) the stability of the post-bifurcation limit set can be adjusted as desired for $\alpha \neq \alpha_c$,

(ii) the size of the post-bifurcation limit set can be modified to some desired value, and

(iii) the rate of growth (as a function of the bifurcation parameter) of the size of the post-bifurcation limit set can be chosen to be some desired value.

For the purpose of local control, we expand equation (2.1) into a Taylor series. After substituting $\alpha_c$, we obtain a parameter-independent system

$$
\dot{x} = L_x(t)x + L_u(t)u + Q_x(x, t) + L_{2x}(t)xu + L_{2u}(t)u^2 + C_x(x, t)
+ Q_{2x}(x, t)u + L_{3x}(t)xu^2 + L_{3u}(t)u^3 + \text{h.o.t},
$$

(5.1)

where $L_x = \frac{\partial F}{\partial x} |_{x=x_c, u=0}$ and $L_u = \frac{\partial F}{\partial u} |_{x=x_c, u=0}$ are the linear coefficient matrix and control input vector, respectively. $Q_x, Q_{2x}$ and $C_x$ are symmetric quadratic and cubic forms. It is well known that in the case of a codimension-one bifurcation, it is sufficient to keep terms only up to the cubic order because it provides adequate qualitative and quantitative approximation of the bifurcation phenomenon. Therefore, in this discussion, we ignore the terms higher than cubic. We note, however, that the method is not restricted to cubic approximations; one could consider higher-order terms as well. The control input is assumed in the form

$$
u(x) = x^T G(t)x + H(x, t),$$

(5.2)

where $G(t)$ is a matrix of the unknown gains of the quadratic terms; $H(x, t)$ an unknown cubic form of the states $x$ containing the cubic control gains; and both $G(t)$ and $H(x, t)$ are time-periodic with period $T$. With this, the closed-loop system becomes

$$
\dot{x} = L_x(t)x + Q^*(x, t) + C^*(x, t),
$$

(5.3)

where $Q^*$ and $C^*$ are quadratic and cubic forms of the states $x$, respectively, and they include unknown control gains. According to the Floquet theory, the linear part of this system can be brought into a time-invariant form by applying the L-F transformation, $x = Q(t)y$. This time-invariant linear system matrix can then be converted into a Jordan canonical form by the modal transformation $y = Mz$. After these transformations, equation (5.3) becomes

$$
\begin{pmatrix}
\dot{z}_c \\
\dot{z}_s
\end{pmatrix} =
\begin{pmatrix}
J_c & 0 \\
0 & J_s
\end{pmatrix}
\begin{pmatrix}
z_c \\
z_s
\end{pmatrix} +
\begin{pmatrix}
\hat{Q}_c^*(z_c, z_s, t) + \hat{C}_c^*(z_c, z_s, t) \\
\hat{Q}_s^*(z_c, z_s, t) + \hat{C}_s^*(z_c, z_s, t)
\end{pmatrix},
$$

(5.4)

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where \( \tilde{Q}^* = M^{-1}Q^1Q^* \), \( \tilde{C}^* = M^{-1}Q^1C^* \) and the subscripts \( c \) and \( s \) denote the critical and stable states, respectively. For systems of this type, the centre-manifold theory (Pandiyan & Sinha 1995) states that there exists a nonlinear transformation with time-periodic coefficients of the form

\[
z_t = h_{2c}(z_c, t) + h_{3c}(z_c, t),
\]

which, upon substitution into equation (5.4), decouples the critical states from the stable ones in the nonlinear terms. In equation (5.5), the indices 2 and 3 denote the order of the nonlinearities.

Up to this point, the procedure is valid for the general case, regardless of the type of the bifurcation. It should be observed that the nonlinear terms of equation (5.4) contain unknown control gains; therefore, we cannot begin with the computation of the centre-manifold relations, in general. However, we can derive formal expressions of the normal forms for each bifurcation without actually computing the centre-manifold relations. In §5a, the procedure is shown in some detail for the case of flip bifurcation. It has been worked out for the transcritical, symmetry breaking and the secondary Hopf bifurcations; however, we omit these cases for brevity.

(a) Flip (period doubling) bifurcation

Let us assume that for some critical value, \( \alpha_c \), of the system parameters, the FTM associated with the linear part of equation (5.3) has an eigenvalue (Floquet multiplier) that equals \(-1\) and all the other eigenvalues have magnitudes less than one. Under this assumption, equation (5.4) takes the form

\[
\left( \begin{array}{c}
\dot{z}_1 \\
\dot{z}_s
\end{array} \right) = \left( \begin{array}{cc}
0 & 0 \\
0 & J_s
\end{array} \right) \left( \begin{array}{c}
z_1 \\
z_s
\end{array} \right) + \left( \begin{array}{c}
\tilde{Q}_1'(z_1, z_s, t) + \tilde{C}_1'(z_1, z_s, t) \\
\tilde{Q}_s'(z_1, z_s, t) + \tilde{C}_s'(z_1, z_s, t)
\end{array} \right).
\]

In this case, because all the Floquet multipliers lie in the left half of the complex plane, the real L–F transformation and its inverse are \( 2T \)-periodic and have the symmetry property \( Q(t + T) = -Q(t) \). Therefore, the nonlinear part of equation (5.6) is also \( 2T \)-periodic and owing to the symmetry property, the coefficients of the quadratic terms have zero averages in time, over the period \( 2T \).

If the centre-manifold relations are assumed as

\[
z_i = h_{2,i}(t)z_1^2 + h_{3,i}(t)z_1^3, \quad i = 2, 3, \ldots, n,
\]

then the reduced equation on the one-dimensional centre manifold becomes

\[
\dot{z}_1 = \tilde{Q}_1'(z_1, z_2) + \left( \sum_{i=2}^{n} \tilde{Q}_1',(1,1,0,\ldots,0)(t)h_{2,i}(t) + \tilde{C}_1',(1,1,0,\ldots,0)(t) \right)z_1^3,
\]

where the subscripts in parentheses indicate the power of the nonlinear terms (for example, \( (0,0,1,\ldots,0) \) corresponding to \( z_1^0z_2^0z_3^1\ldots z_n^0 \)). To further simplify equation (5.6), we employ the time-dependent normal form theory. By using a near-identity transformation, it has been shown that most of the nonlinear terms can be removed and the only terms that cannot be removed are the time averages of the periodic coefficient functions. Therefore, the normal form becomes

\[
\dot{v} = \left( \sum_{i=2}^{n} \tilde{Q}_1',(1,1,0,\ldots,0)(t)h_{2,i}(t) + \tilde{C}_1',(1,1,0,\ldots,0)(t) \right)v^3 := av^3.
\]
where the bar denotes the average of the quantity over the period and, therefore, $a$ is a constant. We cannot compute the value of $a$ owing to the unknown control gains, but this form provides the necessary information about the stability in order to design the controller. Now, to study the dynamics in the neighbourhood of the bifurcation point, versal deformation of the normal form is constructed as (Dávid & Sinha 2000)

$$
\dot{v} = \mu(\alpha) v + av^3,
$$

(5.10)

where the versal deformation parameter $\mu$ is a function of the bifurcation parameter of the original system. This equation, together with the L–F transformation, describes a period-doubling bifurcation. A stable $2T$-periodic limit cycle exists for $\mu \geq 0$ if and only if $a < 0$. We observe that $\tilde{Q}^*$ contains only quadratic terms of the control input, while $\tilde{C}^*$ has unknowns from both quadratic and cubic control terms. Since our normal form does not have any quadratic terms, it is reasonable to choose the control input to be a purely cubic function. Then, $\tilde{Q}^*$ will not contain unknowns at all and the computation of the quadratic centre-manifold relations, $h_{2,i}(t)$, becomes possible. Also, the only effective term of the control input is the cubic term that is a function of the critical state, $z_1$, only. Therefore, in the transformed domain (after the application of L–F and modal transformations), we choose the control input as

$$
G(t) := 0, \quad \tilde{u}_c := \beta z_1^3,
$$

(5.11)

then the normal form becomes

$$
\dot{v} = \mu(\alpha) v + \left( \sum Q_{h_{2,i}} + \tilde{C} + \beta \right) v^3 = \mu(\alpha) v + av^3.
$$

(5.12)

It is important to note that $\tilde{Q}$ and $\tilde{C}$ here are the transformed versions of $Q_x$ and $C_x$ from the original equation (5.1), not the same as $\tilde{C}^*$ and $\tilde{C}^*$ of the transformed closed-loop system. Therefore, equation (5.12) contains no other unknowns than $\beta$. Now, $\beta$ can be chosen such that $a < 0$. This equation can be solved in a closed form. From the solution, it is easy to see how $\beta$ affects the size and the rate of growth of the limit cycle, and it can be chosen to adjust these characteristics to any desired value.

(b) Illustrative example

Consider a parametrically excited simple pendulum shown in figure 7. The control torque $U$ is applied at the suspension point. The equation of motion in the state space form, expanded into Taylor series about the bottom equilibrium point up to the fifth-order terms, is given as

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-(a + b \sin \omega t) & -d
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
(a + b \sin \omega t) \left( \frac{x_1^3}{3!} - \frac{x_2^5}{5!} \right) + u
\end{bmatrix},
$$

(5.13)

where $a = 4g/(\omega^2L)$, $b = A/L$, $d = 4c/(ML^2\omega^2)$ and $u = 4U/(ML^2\omega^2)$. Let $b$ be the bifurcation parameter, and set $a = 0.1$, $d = 0.31623$ and $\omega = 2$. For the critical value $b_c = 1.753802$, one of the Floquet multipliers is +1 and the system undergoes a supercritical symmetry-breaking bifurcation. As the parameter
is increased further, the bifurcated limit cycle undergoes a flip bifurcation at $b=2.19042$ and from there, a period-doubling cascade leads to chaos very quickly. Figure 8 illustrates the behaviour of the uncontrolled system at $b=2.5$. We can see that it is chaotic with large vibrations. Following the procedure described in §5, a nonlinear feedback controller is designed to change the rate of growth of the limit cycle after the first symmetry-breaking bifurcation. The cubic gain $a_3$ is chosen to be 50, which results in a very small amplitude and delays the onset of chaos beyond $b=3$. Figure 9 shows the bifurcation diagram for the uncontrolled and controlled systems.

6. Concluding remarks

Some possible techniques for the control of chaos in nonlinear time-periodic systems have been presented. The first suggestion is a linear control method that uses symbolic computation, while in the second, a nonlinear controller design...
based on feedback linearization is proposed. The last approach is a nonlinear technique, which employs bifurcation analysis. All the three methods are very general in the sense that they can be applied to the entire range of the system parameters and the parametric excitation terms need not be small. They are also computationally simple, although the symbolic computation may require more computer time and memory depending on the number of unknown control parameters. The linear controller design, based on symbolic computation and the bifurcation control methods, is more general. However, the feedback linearization method does have limitations in the sense that systems must be non-resonant and satisfy certain solvability conditions.

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References


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