A Class of Frequency Hop Codes with Nearly Ideal Characteristics for Use in Multiple-Access Spread-Spectrum Communications and Radar and Sonar Systems

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Abstract—In the paper we address the problem of constructing frequency hop codes for use in multiuser communication systems such as multiple-access spread-spectrum communications and multiuser radar and sonar systems. Previous frequency hopping techniques are reviewed. The construction of a new family of frequency hop codes called hyperbolic frequency hop codes is given. Exactly \( p-1 \) codes of length \( p-1 \) exist for every prime, \( p \). Additionally the fullness of the codes is proven. We review the concepts of multiple-access spread-spectrum communication systems andmultiuser radar and sonar systems and it is shown that the hyperbolic frequency hop codes possess nearly ideal characteristics for use in both types of system. Specifically, in multiple-access communications the codes achieve minimum error probability, while in radar and sonar systems the codes have at most two hits in their auto- and cross-ambiguity function. Examples of address assignment for multiple-access communications systems and radar and sonar auto- and cross-ambiguity functions are also given.

I. INTRODUCTION

The design of families of frequency hop codes suitable for use in multiple-access spread-spectrum communication systems and multiuser radar and sonar systems still remains of great interest. Although the nature and physical goals of the spread-spectrum communication systems and multiuser radar and sonar systems are different the requirements imposed on the coding of the signals are almost identical. In spread-spectrum communications the address assignment must be achieved in such a way that 1a) there is no ambiguity about the sender and the information it transmits and 2a) the received signal must interfere as little as possible with the reception of signals from other users. In multiuser radar and sonar systems the signals must 1b) possess high range and Doppler resolution and 2b) allow for as little as possible crosstalk between different users. While conditions 2a) and 2b) are clearly similar, the similarity of conditions 1a) and 1b) is less obvious. However, in the sequel it is seen that both 1a) and 1b) in some sense correspond to maximizing the auto-correlation function of the frequency hop signal. The similarity between the conditions arises from the fact that in all multiuser systems the employed frequency hop signals occupy the same frequency bandwidth. Nevertheless, construction of frequency hop codes for use in multiple-access communication systems or radar and sonar systems has always been treated separately. For instance in [1] it is shown that codes based on Reed–Solomon codes [2] can be successfully used in multiple-access communication systems, whereas in [3] and [4] Costas arrays that can again be constructed via Reed–Solomon codes are introduced as a family of frequency hop codes having nearly ideal auto-ambiguity functions.

In this paper we provide a construction of a family of frequency hop codes that when used in both of the above mentioned systems achieves all the necessary requirements for successful operation. The construction of codes is based upon the number theoretical concept of congruence equations, and the new family of codes are called hyperbolic frequency hop codes.

The paper is organized in the following manner. In Section II we give the definitions necessary for the sequel and the construction of our new family of hyperbolic frequency hop codes. In Sections III and IV we provide the descriptions of the spread-spectrum communication systems and radar and sonar systems utilizing the hyperbolic congruence codes and prove that these systems achieve the best possible performances. We conclude with a brief summary of results.

II. FREQUENCY HOP SIGNALS AND CODES

Frequency hop signals that are used in all systems of interest are defined as follows.

Definition 1: A frequency hop signal of length \( T \) seconds is a train of \( N \) equal-length pulses with the \( k \)th pulse being frequency modulated with frequency \( f_k \) about the carrier frequency \( f_0 \). The frequencies to be placed in various time slots are determined via a sequence of ordered integers \( y(k) \)

\[
y(k), \quad k = 1, \ldots, N
\]

which is called the placement operator, or a frequency hop code. The expression for the frequency \( f_k \) is

\[
f_k = y(k) \frac{B}{N}, \quad k = 1, \ldots, N
\]
where $B$ is the approximate signal bandwidth. The frequency hop signal is shown in Fig. 1 and can be written as:

$$u(t) = \sum_{k=1}^{N} p(t - kT) \cos[(f_0 + f_k)t + \theta_k] \quad (3)$$

where

$$p(t) = \begin{cases} 1, & t \leq T \\ 0, & \text{elsewhere} \end{cases} \quad (4)$$

and $f_0$ is the carrier frequency.

A convenient way of representing a frequency hop signal is through a $N \times N$ matrix where $N$ rows correspond to $N$ frequency channels and $N$ columns to $N$ time slots. The element of the matrix, $a_{k,l}$, is equal to one if and only if the frequency $f_k$ is transmitted in the interval $t$. Otherwise, $a_{k,l}$ is equal to zero [4].

The usual constraint imposed on frequency hop codes and signals is that each frequency channel and each time channel must be utilized once and only once [3].

This allows for the maximum usage of the available mediums bandwidth. And in the case of system reverberation prevents masking of the components in the same frequency channel. We refer to a frequency hop code that satisfies this requirement as a full frequency hop code.

Definition 2: Let $J(p)$ denote a finite field over a prime, $p$. Denote by $J_p$ the set containing the elements of the field. A full frequency hop code is a frequency hop code whose placement operator is a permutation of the set $J_p$ or $J_p^{\sim}$, where $J_p = J_p^{\sim}$ $\setminus \{0\}$. For example, Welch–Costas arrays [4], obtained through the placement operator,

$$y(k) = r^k \quad (\text{mod } p), \quad k = 1, \ldots, p - 1 \quad (5)$$

where $r$ is a primitive root of $p$ [5]. $J_p$ are known to be full frequency hop codes over the set $J_p^{\sim}$. Thus, according to definition 1, Welch–Costas arrays specify a frequency hop signal with $N = p - 1$ pulses.

We now give the definition of hyperbolic frequency hop codes.

Definition 3: A hyperbolic frequency hop code for a finite field, $J(p)$, is defined through the following placement operator

$$y(k) = \frac{a}{k} \quad (\text{mod } p), \quad k = 1, 2, \ldots, p - 1 \quad \text{and some } a \in J_p^{\sim} \quad (6)$$

Since the operation in (6) is performed over a finite field every element in the field has a unique inverse and it is immediately seen that the hyperbolic frequency hop codes are full codes with $N = p - 1$ pulses. Also, for each prime there are $p - 1$ different hyperbolic codes. An example of two hyperbolic frequency hop codes for $p = 11$, and $a = 1$ and $a = 3$ is shown in Fig. 2.

III. ADDRESS ASSIGNMENT FOR MULTIPLE-ACCESS

SPREAD-SPECTRUM COMMUNICATION SYSTEM

In a typical multiple-access spread-spectrum communication system a large number of transmitters needs to simultaneously communicate with one receiver. For instance such a system can be a satellite communication network in which a large number of earth stations needs to simultaneously communicate with one satellite receiver. Here, we consider a system proposed by Viterbi [6] in which each user has the entire bandwidth available and in order to be recognized by the receiver is assigned an address represented by a placement operator—frequency hop pattern. The addresses are transmitted via frequency hop signals which carry the address of the transmitter as well as the transmitted message. After the frequency hop signal is received the decoder determines which user has sent the message and decodes it as well. To wit, each of the $Q$ users, $u_1, u_2, \ldots, u_i, \ldots, u_Q$, is assigned an address from the set $A$ of frequency hop operators,

$$A = \{y_1(k), y_2(k), \ldots, y_i(k), \ldots, y_Q(k)\}, \quad (7)$$

in such a way that the user $u_i$ is assigned the address $y_i$. The users also transmit a message $m$ from the set $M$:

$$M = \{0, 1, \ldots, R\}. \quad (8)$$

We propose the assignment of addresses—frequency hop patterns—for the user $u_i$ through hyperbolic frequency hop codes. Namely,

$$y_i(k) \equiv \frac{i}{k} + m \quad (\text{mod } p), \quad k = 1, \ldots, p - 1, \quad i = 1, 2, \ldots, Q, \quad m \in M. \quad (9)$$

From the last equation we see that the address and message assignment allows for simultaneous operation of $Q = p - 1$ users with $M = p - 1$ different messages. An example of the address assignment, with $p = 7$, for user $u_7$ transmitting message $m = 1$ is shown in Fig. 3. Note that the actual transmitted pattern is a matrix of size $N = 1 + N$ due to the addition operation in (9).

The interference effects in this type of system (aside from the noise) come from the simultaneous use of the same frequency channel. This event is referred to as a coincidence.
or a hit. The receiver is not able to distinguish between the hit and the single use of the frequency channel, hence, the greater the number of hits the greater the probability of error.

We next show that in the synchronous case (signals from different users all aligned in time) the above address and message assignment achieves the minimal probability of error, that is maximum one hit between any two placement operators. Suppose, without loss of generality, that two frequency hop operators for users $u_1$ and $u_2$ are

$$y_1(1), \ldots, y_1(j), \ldots, y_1(N = p - 1)$$

and

$$y_2(1), \ldots, y_2(j), \ldots, y_2(N = p - 1)$$

The hit occurs when

$$y_1(q) = y_2(q) \quad \text{for any } q \in \mathbb{Z}_p .$$

Hence, two hits occur when equation (12) is satisfied simultaneously for two different values of $q$:

$$y_1(j) - y_2(j) = y_1(r) - y_2(r).$$

Inserting (9) in (13) we have

$$\frac{1}{j} = \frac{1}{r} \pmod{p} ,$$

and since the operations are done on the field the above equation can never be satisfied, hence two frequency hop signals can form only one hit.

The receiver possesses (stored in memory or generated) all the possible placement operators—addresses. Once the frequency hop pattern is received the decoder subtracts from it (in modulo $p$ arithmetic) all the prespecified addresses. Only the correct address, $y_i$, will completely fill the row $m$, indicating that the user $u_i$ was active transmitting the message $m$. Table I shows a system of four users with placement operators generated through (9), with prime, $p = 7$. The assigned messages and transmitted frequency hop patterns are also given.

In Fig. 4 we show the receiver matrix for the above example with the frequency hop pattern of the $i$th user denoted by $i$. As proved above two different patterns form at most one hit.

We now discuss the case in which frequency hop signals from different users are not synchronized in time; the effect of this being to randomly shift the frequency hop operators. The hit in this case occurs when

$$y_i(k + t) = y_j(k)$$

where $t$ is a random time shift. Inserting (9) into (15) gives

$$\frac{i}{k + t} + m_i - \frac{j}{k} - m_j \equiv 0 \pmod{p} ,$$

or combining the terms

$$\frac{ki + k(k + t)(m_i - m_j) - (k + t)j}{k(k + t)} \equiv 0 \pmod{p} .$$

It is important to note that since we are dealing with acyclic time shifts, $k + t < p$, therefore the denominator of the above congruence equation can never be zero. Rearranging the terms we have

$$k^2(m_i - m_j) + k(i - j + (m_i - m_j)t) - jt \equiv 0 \pmod{p} .$$

This is a second-order equation in $k$ and hence by Lagrange's theorem [5] can have at most two noncongruent solutions in the field $p$, or equivalently two frequency hop operators can have at most two hits for any time shift. This increases the probability of error in the system, however, for larger primes the error probability remains small and of the order of the error introduced by the noise present in the system. Note that in the special case of the users sending a string of ones or zeros as a message in the way that "1" corresponds to "user active" and "0" to "no transmission" (18) reduces to a linear equation and

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**Table I**

<table>
<thead>
<tr>
<th>User</th>
<th>Address</th>
<th>Message</th>
<th>Hops</th>
</tr>
</thead>
<tbody>
<tr>
<td>u1</td>
<td>145236</td>
<td>1</td>
<td>23</td>
</tr>
<tr>
<td>u2</td>
<td>213465</td>
<td>1</td>
<td>32</td>
</tr>
<tr>
<td>u3</td>
<td>351624</td>
<td>2</td>
<td>05</td>
</tr>
<tr>
<td>u4</td>
<td>426153</td>
<td>2</td>
<td>64</td>
</tr>
</tbody>
</table>

Fig. 3. The address of the second user with message $m = 1$, $p = 7$.
hence the frequency hop patterns can form again at most one hit achieving the minimal probability of error.

IV. MULTI-USER RADAR AND SONAR SYSTEMS

In a typical pulse compression radar or sonar system the transmitted signal must possess an auto-ambiguity function [7] that achieves the perfect, so called, thumb-tack shape allowing thus for both high range and velocity resolution. However, in modern radar and sonar systems where more than one user is operating at the same time occupying the same frequency bands, one user is receiving all the signals in the system. This phenomenon referred to as cross-talk increases the probability of error (false alarm) in the system. The amount of cross-talk between pairs of signals is measured through the cross-ambiguity function, hence one would like to keep the maximum value of the cross-ambiguity function as low as possible. The above requirements on the auto- and cross-ambiguity function can be mathematically expressed as follows.

1) Minimize the signals auto-ambiguity function,

\[ A_u(t, \omega) = \int_{-\infty}^{\infty} u(t) u^*(t - \tau) e^{j\omega\tau} d\tau \]

everywhere except for the (0,0) time-frequency shift in which the normalized auto-ambiguity function has value one, and

2) minimize the cross-ambiguity function,

\[ A_{u_1,u_2}(t,\omega) = \int_{-\infty}^{\infty} u_1(t) u_2^*(t - \tau) e^{j\omega\tau} d\tau \]

of any two signals in the system, for every time-frequency shift.

Signal designers have for a long time tried to construct frequency hop codes whose signals will simultaneously satisfy conditions 1) and 2). For example, linear congruence codes (LCC) [8] have ideal\(^2\) cross-ambiguity functions, but have poor auto-ambiguity functions with as many as \(N - 1\) coincidences for a certain time-frequency shift where \(N\) is the code length. On the other hand, Costas arrays [4] achieve ideal auto-ambiguity functions, but one has difficulty finding a family of such codes with good cross-ambiguity functions, in fact it has been shown [9] that Costas arrays can not have ideal cross-ambiguity properties; only pairs of algebraically constructed Costas arrays have at most two hits\(^4\) in their cross-ambiguity function [10]. Codes based on cubic congruences [11] compromise two ideal requirements having at most two hits in their auto- and at most three hits in their cross-ambiguity function. Note that nonfull frequency hop codes with at most two and one hit in their auto-ambiguity functions were introduced in [12] and [13], respectively. Both of these families of codes have also good cross-ambiguity properties but the fact that they are not full makes them unsuitable for use in radar and sonar systems [3]. In this section we prove that families of hyperbolic frequency hop codes come as close as possible to the ideal case when the auto- and cross-ambiguity functions are considered simultaneously.

Useful tools for analyzing the auto- and cross-ambiguity function of frequency hop codes are the auto- and cross-hit array [14], [11].

**Definition 4:** Let \( C \) denote the \( N \times N \) matrix representing a full frequency hop code, \( y(k) \). The auto-hit array \( H(t,d) \) is a \((2N-1) \times (2N-1)\) matrix where the elements of the matrix, \( h_{t,d} = 2N + 1 < t, d < 2N - 1, \) correspond to the number of hits between the matrix \( C \) and the shifted version by \((t,d)\) of itself. The concept of the auto-hit array is easily extended to the cross-hit array.

**Definition 5:** Let \( C_1 \) and \( C_2 \) denote two \( N \times N \) full frequency hop codes, \( y_1(k) \) and \( y_2(k) \), respectively. The cross-hit array \( C_{1,2}(t,d) \) is a \((2N-1) \times (2N-1)\) matrix where the elements of the matrix, \( c_{1,2} = 2N + 1 < t, d < 2N - 1 \), correspond to the number of hits between matrix \( C_2 \) shifted by \((t,d)\) and the unshifted version of \( C_1 \).

The auto- and cross hit array may be regarded as a discrete analogue of the auto- and cross-ambiguity function, respectively\(^5\) [14]. For illustration in Fig 5 we give the auto-hit arrays\(^6\) of codes shown in Fig. 2, and in Fig. 6 we give the cross-hit array of the two codes. In order to examine the number of hits in the auto- and cross-hit arrays we introduce the concept of the difference function.

**Definition 6:** The difference function for two placement operators \( y_2(k) \) and \( y_1(k) \) is given by

\[ (y_2 \Delta y_1)(k; t, d) = y_2(k + t) + d - y_1(k) \]

(mod \( N \)) for \( 0 \leq k, t, d \leq N - 1 \)

where the parameters, \( t \) and \( d \), correspond to any horizontal (time) or vertical (Doppler) shift, respectively. From definition 6, a hit in the auto- or cross-hit array for a shift \((t,d)\)

\(^2\)Ideal means that the number of coincidences for any time-frequency shift is at most one.

\(^4\)In the paper we use the terms "coincidence" and "hit" interchangeably.

\(^5\)In general this is true only when the time-bandwidth product of the signal is much greater than \(N^2\). In this case the ambiguity functions consist of separated "subambiguities" formed by two subpulses in the signal.

\(^6\)The auto-hit array is odd symmetric hence only the right hand side of the arrays is shown.
corresponds to the zero value of the difference function for that same \((t,d)\). We can now state the theorem concerning the number of hits (coincidences) in the auto- and cross-hit array of hyperbolic frequency hop codes.

**Theorem 1:** A hyperbolic frequency hop code has at most two hits for any time-frequency shift in its auto-hit array and at most two hits for any time-frequency shift in its cross-hit array formed with any other hyperbolic frequency hop code of the same length.

**Proof:** Inserting (6) into the definition of the difference function, we have in the case of cross-hit array

\[
\frac{b}{k+t} + d - \frac{a}{k} \equiv 0 \quad 0 < k + t < p,
\]

\[
0 \leq d < p \quad \text{(mod } p)\quad (22)
\]

or

\[
\frac{bk - a(k + t) + k(k + t)d}{k(k + t)} \equiv 0 \quad \text{(mod } p)\quad (23)
\]

where the condition \(0 < k + t < p\) comes again from the fact that we have only acyclic shifts both in time and frequency. Rearranging the terms we have

\[
k(k + t)d + ak - b(k + t) \equiv 0 \quad \text{(mod } p)\quad (24)
\]

This is a second-order equation in \(k\) and therefore by Lagrange's theorem can have at most two noncongruent solutions in the field, or the cross-hit array has at most two hits. For the auto-hit array \(a = b\), hence, we have from (24)

\[
k(k + t)d - bt \equiv 0 \quad \text{(mod } p)\quad (25)
\]

which is still a second order equation in \(k\) and has at most two noncongruent solutions or equivalently a hyperbolic frequency hop code has at most two hits in its auto-hit array.

Notice that (23) is of the same form as (18) and in fact examining the asynchronous multiple-access spread-spectrum communication system for the number of hits between the received addresses is a similar problem to the one of examining the radar or sonar cross-hit array for the number of hits.

In Fig. 7 we give the auto-ambiguity function of the codes shown in Fig. 2 and in Fig. 8 we give their cross-ambiguity function. Notice how the hits in the ambiguity functions correspond to the hits in the hit arrays, also the maximum value of the cross-ambiguity function is approximately 0.2 which is to be expected since every hit has a height of approximately \(\frac{1}{2} = 0.1\) [3], and there are two hits in the cross-ambiguity function for certain time-frequency shifts (time-frequency shift (2,1) for instance).

**V. CONCLUSION**

In the paper we have introduced a new family of frequency hop codes called hyperbolic frequency hop codes. It is shown that the codes possess ideal properties for use in
both multiple-access spread-spectrum communication systems and multiuser radar and sonar systems. In fact, in multiple-access spread-spectrum communications for the synchronous case hyperbolic codes achieve the minimal probability of error similar to codes based on Reed–Solomon codes [1], or codes based on the properties of congruence equations [15]. In the asynchronous case they have a slightly increased probability of error, at most two hits, but still perform better than the codes introduced in [15], and the same as codes in [1]. In multiuser radar and sonar systems hyperbolic codes possess so far best known simultaneous auto- and cross-ambiguity properties. They achieve at most two hits in the auto-ambiguity function in contrary to Costas arrays that have perfect one hit structure, but they also have at most two hits in their cross-ambiguity functions, while Costas arrays can have as many as \( N - 1 \) hits in their cross-ambiguity functions. In fact considering both the auto- and cross-ambiguity properties the hyperbolic codes are the best possible frequency hop codes, since it has been proved [9] that Costas arrays can not have ideal cross-ambiguity properties.

REFERENCES