POINTS IN GENERIC POSITION AND CONDUCTOR OF VARIETIES WITH ORDINARY MULTIPLE SUBVARIETIES OF CODIMENSION ONE

Ferruccio Orecchia

Abstract. We extend results of our previous papers, on ordinary multiple points of curves [9], and on the computation of their conductor [8], to ordinary multiple subvarieties of codimension one.

1. Introduction.

Let A be the local ring, at a multiple (that is singular) point x, of an algebraic reduced curve C with embedding dimension $emdim(A) = r + 1$ and multiplicity $e(A) = e$. Let $\mathfrak{m}$ be the maximal ideal of A. $Spec(G(A))$ is the tangent cone and $Proj(G(A))$ the projectivized tangent cone to C at x.

The scheme $Proj(G(A))$ is reduced if and only if it consists of e points of $\mathbb{P}^r$, that is, the tangent cone considered as a set consists of e lines of $\mathbb{A}^{r+1}$ through x (the tangents to the curve at x). In this case we say that x is an *ordinary multiple point* (or an *ordinary singularity*) of C [9, Lemma-Definition 2.1]. Clearly if $Spec(G(A))$ is reduced then $Proj(G(A))$ is reduced. The converse, in general, doesn't hold (see [9, Example 1] or [7, Section 4]), but if $Proj(G(A))$ is reduced and consists of points in generic e position, then $Spec(G(A))$ is reduced [9, Theorem 3.3]. Furthermore, if the points of $Proj(G(A))$ are in generic $e-1, e$ position, then the conductor of A in its normalization $\overline{A}$ is $\mathfrak{m}^\nu$ where $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$ [8, Theorem 4.4].

In this paper first we extend these results to any one dimensional reduced local ring B with finite normalization $\overline{B}$. Let $\mathfrak{m}$ be the maximal ideal of B and K be the algebraic closure of the residue field $k(\mathfrak{m})$ of B. Set $e = e(B)$, $emdim(B) = r + 1$. Considering the ring $G(B) \otimes_{k(\mathfrak{m})} K$, instead of $G(A)$, we prove that, if $Proj(G(B) \otimes_{k(\mathfrak{m})} K)$ is reduced and consists of points in generic e position, then $Spec(G(B) \otimes_{k(\mathfrak{m})} K)$ and $Spec(G(B))$ are reduced [Theorem 5]. If in addition the points of $Proj(G(B) \otimes_{k(\mathfrak{m})} K)$ are in generic $e-1$ position then the conductor of B in $\overline{B}$ is $\mathfrak{m}^\nu$ where $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$ [Theorem 11].

Then we apply the previous results to the ring $B = A_\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of codimension one of a local reduced ring A, with finite normalization $\overline{A}$.

Using also the properties of normal flatness we get the following result on the conductor $\mathfrak{b}$ of A (in $\overline{A}$).

Let $e(A_\mathfrak{p}) = e > 1$, $emdim(A_\mathfrak{p}) = r + 1$ and K be the algebraic closure of the residue field $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ of A at $\mathfrak{p}$. Assume $A/\mathfrak{p}$ regular, $\sqrt{\mathfrak{b}} = \mathfrak{p}$ and

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$Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K)$ reduced with points in generic $e-1, e$ position. Given the the following conditions:

(a) A is Cohen-Macaulay, $emdim(A) = emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p})$ and $e(A) = e(A_{\mathfrak{p}})$;

(b) A is S_2 and normally flat along $\mathfrak{p}$;

(c) $\mathfrak{b}$ is primary and A is normally flat along $\mathfrak{p}$;

(d) $\mathfrak{b} = \mathfrak{p}^{\nu}$, where $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$;

then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ [Theorem 15].

In particular we get the following statement.

Let $\mathfrak{p}$ be a prime ideal of codimension one of a reduced Cohen-Macaulay ring A with finite normalization $\overline{A}$. Assume $A/\mathfrak{p}$ regular, $\sqrt{\mathfrak{b}} = \mathfrak{p}$ and $Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K)$ reduced. If $emdim(A) = dim(A)+1$ and $e(A) = e(A_{\mathfrak{p}}) = e$ then $\mathfrak{b} = \mathfrak{p}^{e-1}$ [Corollary 16].

The previous results have the following geometrical consequences.

Let $X = Spec(R)$ be an algebraic variety and $Y = Spec(R/\mathfrak{q})$ be an irreducible codimension one subvariety of X. Suppose that $e(R_{\mathfrak{q}}) = e > 1$ that is Y is a *multiple subvariety* of X, of multiplicity e. Let $emdim(R_{\mathfrak{q}}) = r+1$ and K be the algebraic closure of the residue field $k(\mathfrak{q})$ of R in $\mathfrak{q}$. If $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K)$ is reduced and its points are in generic $e-1, e$ position, then there exists an open nonempty subset U of Y such that for every closed point x of U the conductor of the local ring A, of X at x, is $\mathfrak{p}^{\nu}$, where $\mathfrak{p}$ is the prime ideal in A defining Y and $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$ [Theorem 19].

Note that $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{p})} K)$ is reduced if and only if there exists an open nonempty subset U of Y such that, for every closed point x of U, the tangent cone to X at x is the union, as a set, of e distinct linear spaces of $\mathbb{A}^{r+1}$ [Theorem 21]. Then (extending the definition given for curves) it is natural to say that in this case Y is an *ordinary multiple subvariety* of X.

Let $X = SpecR$ be a reduced non-normal S_2 variety. Assume that the irreducible components of the non-normal locus of X are ordinary multiple subvarieties $Y_i = Spec(R/\mathfrak{q}_i)$ of multiplicity $e_i = e(R_{\mathfrak{q}_i}) > 1$ $(1 \leq i \leq t)$. Set $emdim(R_{\mathfrak{q}_i}) = r_i + 1$ and let K_i be the algebraic closure of the residue field $k(\mathfrak{q}_i)$. Suppose that the varieties Y_i are nonsingular and that the points of $Proj(G(A_{\mathfrak{q}_i}) \otimes_{k(\mathfrak{q}_i)} K_i)$ are in generic $e_i - 1, e_i$ position, for any i. Considering the following conditions:

(a) $X = Spec(R)$ is Cohen-Macaulay, equimultiple of multiplicity e_i along Y_i and has constant embedding dimension along Y_i, for any i;

(b) X is normally flat along Y_i, for any i;

(c) $\mathfrak{b} = \mathfrak{p}_1^{\nu_1} \cap \ldots \cap \mathfrak{p}_t^{\nu_t}$;

then $(a) \Rightarrow (b) \Rightarrow (c)$ [Theorem 24].

In particular if, in (a), X is a hypersurface, then the conductor of R is $\mathfrak{b} = \mathfrak{p}_1^{e_1-1} \cap \ldots \cap \mathfrak{p}_t^{e_t-1}$ [Corollary 26].

Throughout the paper all ring are supposed to be commutative, with identity and noetherian.

Let A be a semilocal ring. If $\mathfrak{p}$ is an ideal of A, $G_{\mathfrak{p}}(A) = \bigoplus_{n\geq 0}(\mathfrak{p}^n/\mathfrak{p}^{n+1})$ is the associated graded ring with respect to $\mathfrak{p}$. By $G(A)$ we denote the associated graded ring with respect to the Jacobson radical $\mathfrak{J}$ of A. If $x \in A$, $x \neq 0$, $x \in \mathfrak{J}^n - \mathfrak{J}^{n+1}$, $n \in \mathbb{N}$, we say that x has degree n and the image $x^* \in \mathfrak{J}^n/\mathfrak{J}^{n+1}$ of x in $G(A)$ is said to be the initial form of x. If $\mathfrak{a}$ is an ideal of A, by $G(\mathfrak{a})$ we denote the ideal of $G(A)$ generated by all the inital forms of the elements of $\mathfrak{a}$.

If A is local with maximal ideal $\mathfrak{m}$, $H^0(A,n) = dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, $n \in \mathbb{N}$, denotes the Hilbert function of A and $e(A)$ the multiplicity of A at $\mathfrak{m}$. The embedding dimension $emdim(A)$ of A is given by $H^0(A,1)$. If $i \in \mathbb{N}$, the functions $H^i(A,n)$ are given by the relations $H^i(A,n) = \sum_{j=0}^{n} H^{i-1}(A,j)$

If $S = \bigoplus_{n\geq 0} S_n$ is a standard graded finitely generated algebra over a field k, of maximal homogeneous ideal $\mathfrak{n}$, $H^0(S,n) = dim_k S_n = H^0(S_{\mathfrak{n}}, n)$ denotes the Hilbert function of S and $emdim(S) = H^0(S,1) = emdim(S_{\mathfrak{n}})$ the embedding dimension of S. The multiplicity of S is $e(S) = e(S_{\mathfrak{n}})$. One has $e(A) = e(G(A))$ and $emdim(A) = emdim(G(A))$.

If B is any ring $dim(B)$ denotes the dimension of B.

2. The one dimensional case.

Let B be a local ring or a standard finitely generated graded k-algebra over a field k. Suppose B has dimension one and is Cohen-Macaulay. Set $emdim(B) = r + 1$ and $e(B) = e$. It is well known that, for any $n \in \mathbb{N}$, $H^0(B,n) \leq Min\{e, \binom{n+r}{r}\}$ and if $H^0(B,m) = e$ then $H^0(B,n) = e$, for any $n \geq m$ [14].

1. Definition. The ring B has *maximal Hilbert function* if, for every $n \in \mathbb{N}$,

$$H^0(B,n) = Min\{e, \binom{n+r}{r}\}$$

2. Definition. A set of points $X = \{P_1, ..., P_e\} \subset \mathbb{P}^r$ is *in generic position* (or the points $P_1, ..., P_e$ are *in generic position*) [8, Definition 3.1] if the Hilbert function of its homogeneous coordinate ring R is maximal. The set X is *in generic t-position*, $t \leq e$, if every t-subset of X is in generic position (then generic e-position is generic position).

Remark. It is proved in [2, Theorem 4] that, for any e and r, "generic position" is an open nonempty condition.

3. Example. It is easily seen that any set of points of $\mathbb{P}^1$ is in generic t-position, for any t.

4. Example. A set of $\binom{n+r}{r}$ points in $\mathbb{P}^r$ $(n > 0, r > 0)$ is in generic position if and only if they do not lie on a hypersurface of degree n [8, Corollary 3.4], in particular six points in $\mathbb{P}^2$ are in generic position if and only if they do not lie on a conic.

5. Theorem. *Let B a one dimensional reduced local ring of maximal ideal $\mathfrak{m}$. Let K be the algebraic closure of the residue field $k(\mathfrak{m}) = B/\mathfrak{m}$. Then:*

(a) the Hilbert functions of B, $G(B)$ and $G(B) \otimes_{k(\mathfrak{m})} K$ are the same, $e(B) = e(G(B)) = e(G(B) \otimes_{k(\mathfrak{m})} K)$ and $emdim(B) = emdim(G(B)) = emdim(G(B) \otimes_{k(\mathfrak{m})} K) = r + 1$;

(b) if $Proj(G(B) \otimes_{k(\mathfrak{m})} K)$ is reduced and consists of points in generic position in $\mathbb{P}^r$ then B has maximal Hilbert function and the rings $G(B) \otimes_{k(\mathfrak{m})} K$ and $G(B)$ are reduced.

Proof. *(a)* If $G(B) \otimes K$ is the K-vector space obtained extending to K the field of scalars of the $k(\mathfrak{m})$-vector space $G(B)$, then the Hilbert functions of $G(B)$ (that is of B) and of $G(B) \otimes K$ are the same. This implies the equalities of the multiplicities and of the embedding dimensions.

(b) If the points of $Proj(G(B) \otimes K)$ are in generic position the Hilbert function of $G(B) \otimes K$ is maximal [Definitions 1 and 2]. Then by *(a)*, the Hilbert function of B is maximal. This implies that $G(B)$ is Cohen-Macaulay [11, Theorem 3.2], that is, there exists $y^* \in \mathfrak{m}/\mathfrak{m}^2$ which is a non zero divisor of $G(B)$ (this is easy to prove if $k(\mathfrak{m})$ is infinite and if $k(\mathfrak{m})$ is finite one can pass to the ring $R[U]_{\mathfrak{m}R[U]}$, U indeterminate). Then, since a field extension is flat, $y^* \otimes 1$ is a non zero divisor of $G(B) \otimes K$ which then is Cohen-Macaulay and reduced, since $Proj(G(B) \otimes K)$ is reduced. Finally, by the flatness of $G(B)$ over $k(\mathfrak{m})$, we have $G(B) \subset G(B) \otimes K$ and $G(B)$ is reduced. □

Remark. If the ring B of Theorem 5 is the local ring A at a multiple point of a curve over an algebrically closed field k, then $K = k(\mathfrak{m}) = k$ and $G(B) \otimes_{k(\mathfrak{m})} K = G(A)$. Hence in this case Theorem 5,(b) proves that, if $Proj(G(A))$ is reduced and consists of points in generic position, then $G(A)$ is reduced which is Theorem 3.3 of [9]. In general the condition $Proj(G(A))$ reduced doesn't imply that $G(A)$ is reduced as has been shown with various examples in [9, Section 3, Example 1] and [7, Section 4]. Another example (pointed out by A. De Paris), which answers also to a question posed in [3, Example 13], is the following.

6. Example. Let $B = \mathbb{C}[g, tg, fg]$, $f = t^5 - 1$, $g = tf$ and A be the local ring of the curve $Spec(B)$ at $\mathfrak{m} = (g, tg, fg)$. We have $e(A) = 6$. Let a_i, $i = 1, ..., 5$, be the fifth roots of the unity. It is proved in [7,Section 4] that $Proj(G(A))$ consists of the points $P_i = (1, a_i, 0), i = 1, ..., 5, P_6 = (1, 0, -1)$ which lie on the conic $yz = 0$ and then they are not in generic position [Example 4]. But $G(A)$ is not reduced as we are going to show. Let $\mathfrak{n}$ be the maximal homogeneous ideal of $G(A)_{red} = G(A)/nil(G(A)) = \mathbb{C}[X, Y, Z]/\mathfrak{a} = \mathbb{C}[x, y, z]$, where $\mathfrak{a} = (Z, Y^5 - X^5) \cap (Y, X + Z)$. There is a natural surjective homomorphism $\phi : \mathfrak{m}^2/\mathfrak{m}^3 \to \mathfrak{n}^2/\mathfrak{n}^3$ given by $\phi(H(g, tg, fg)) = H(x, y, z)$, $H(X, Y, Z) \in k[X, Y, Z]$ homogeneous of degree 2. Let $F(X, Y, Z) = Z^2 + XZ$.Since $z^2 + xz = 0$, if we show that $F(g, tg, fg) = fg(fg + g) \notin \mathfrak{m}^3$ we have that ϕ is not injective and $G(A)$ is not reduced. Now if $fg(fg + g) \in \mathfrak{m}^3$ one has

$$fg(fg + g) = P(1, t, f)g^3 + g^4h \qquad (*)$$

where $h \in \mathbb{C}[t]$ and $P(X, Y, Z)$ is a homogeneous polynomial of degree 3. But $fg(fg + g) = f(f + 1)g^2 = t^4g^3$. Substituting in (*) and dividing by g^3 we have

$t^4 - P(1,t,f) = gh$. But $t^4 - P(1,t,f) = q + fr$ where $q, r \in \mathbb{C}[t]$ and q is monic of degree 4. Thus $q = gh - fr = f(th - r)$ which is impossible because the degree of f is 5.

We recall that the *conductor* $\mathfrak{b}$ of a ring B in its normalization $\overline{B}$ is the ideal (of B and $\overline{B}$) $\mathfrak{b} = \{b \in \mathrm{B} |\ \overline{B}b \subset B\}$.

7. Theorem. *Let B be a one dimensional reduced local ring with finite normalization $\overline{B}$. Let $k(\mathfrak{m}) = B/\mathfrak{m}$ be the residue field of B and K be the algebraic closure of $k(\mathfrak{m})$. If $G(B) \otimes_{k(\mathfrak{m})} K$ is reduced then:*

(a) $G(B)$ is reduced, there is a natural immersion $G(B) \subset G(\overline{B})$ and $G(\overline{B})$ is the normalization of $G(B)$;

(b) there is a natural immersion $G(B) \otimes_{k(\mathfrak{m})} K \subset G(\overline{B}) \otimes_{k(\mathfrak{m})} K$ and $G(\overline{B}) \otimes_{k(\mathfrak{m})} K$ is the normalization of $G(B) \otimes_{k(\mathfrak{m})} K$;

(c) If $\nu = Min\{n \in \mathbb{N} \mid e = H(B,n)\}$, the ideal $G(\mathfrak{m})^\nu$ is contained in the conductor of $G(B)$ in $G(\overline{B})$, and if B (that is $G(B)$) has maximal Hilbert function, then $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$, where $e = e(B)$ and $r+1 = emdim(B)$.

Proof. *(a)* We have $G(B) \subset G(B) \otimes K$ so if $G(B) \otimes K$ is reduced then so is $G(B)$. Then there is a natural immersion $G(B) \subset G(\overline{B})$ [8, Proposition 2.2] and $G(\overline{B})$ is the normalization of $G(B)$ [10, Proposition 1.7].

(b) By (a) and by the flatness of K over $k(\mathfrak{m})$ we have the immersion $G(B) \otimes K \subset G(\overline{B}) \otimes K$. Furthermore $G(\overline{B})$ is the normalization of $G(B)$. From this we want to deduce that $G(\overline{B}) \otimes K$ is the normalization of $G(B) \otimes K$. It is easily seen that $G(\overline{B}) \otimes K$ is integral over $G(B) \otimes K$ and that $G(\overline{B}) \otimes K$ is contained in the total ring of quotients of $G(B) \otimes K$. Since by assumption $G(B) \otimes K$ is reduced, $G(\overline{B}) \otimes K$ is reduced. But, if $D = \overline{B}/\mathfrak{m}\overline{B}$, $G(\overline{B}) \cong D[T]$ and $G(\overline{B}) \otimes K \cong (D \otimes K)[T]$. Thus $D \otimes K$ is an artinian reduced ring, that is a direct sum of fields. Hence $G(\overline{B}) \otimes K$ is normal.

(c) If $\mathfrak{b}$ is the conductor of B in $\overline{B}$ it is well known ([7, Lemma 2.12] and [12, Theorem 1.3]) that $\mathfrak{m}^\nu \subset \mathfrak{b}$. Then $G(\mathfrak{m})^\nu \subset G(\mathfrak{b})$. But, $G(\mathfrak{b})$ is contained in the conductor of $G(B)$ in $G(\overline{B})$. In fact if $b^* \in G(\overline{B})$ and $x^* \in G(\mathfrak{b})$ have degree respectively s and t and are initial forms of elements $b \in \mathfrak{J}^s - \mathfrak{J}^{s+1}$ and $x \in \mathfrak{J}^t \cap \mathfrak{b} - \mathfrak{J}^{t+1}$ (where $\mathfrak{J}$ is the Jacobson radical of $\overline{B}$), then $bx \in \mathfrak{J}^{s+t} \cap A = \mathfrak{m}^{s+t}$ [8, Proposition 2.2] i.e. $b^*x^* \in G(A)$

Finally the last statement is an easy consequence of the definition of maximal Hilbert function [Definition 1]. □

8. Theorem. *Let S be a standard graded finitely generated k-algebra, over an algebraically closed field k, with maximal homogeneous ideal $\mathfrak{n}$. Assume S one dimensional and reduced. Let $e = e(S)$ and $r+1 = emdim(S)$. Then $Proj(S)$ consists of e points of $\mathbb{P}^r$. If these points are in generic $e-1, e$ position, then the conductor of S in its normalization $\overline{S}$ is $\mathfrak{n}^\nu$,where $\nu = Min\{n \mid e \leq \binom{n+r}{r}\}$.*

Proof. [8, Proposition 3.5 and Theorem 4.3].

9. Theorem. *Let A be the local ring of a curve at a singular point with reduced tangent cone. Let $\mathfrak{m}$ be the maximal ideal of A. If, for some integer n, $G(\mathfrak{m}^n) = G(\mathfrak{m})^n$ is the conductor of $G(A)$ in $G(\overline{A})$, then $\mathfrak{m}^n$ is the conductor of A in $\overline{A}$.*

Proof. Let $\mathfrak{J}$ be the Jacobson radical of $G(\overline{A})$. Since by assumption $G(\mathfrak{m}^n)$ is the conductor we have

$$G(\mathfrak{m}^n) = G(\mathfrak{m}^n)G(\overline{A}) = G(\mathfrak{m}^n\overline{A}) = G(\mathfrak{J}^n)$$

and then the result follows from [8, Theorem 2.3]. □

The following is the main result of [8] (see Theorem 4.4).

10. Theorem. *Let A be the local ring of a curve at a singular point, with maximal ideal $\mathfrak{m}$. Let $e = e(A)$ and $emdimA = r + 1$. Assume that $Proj(G(A))$ is reduced and consists of points in generic e-1,e position . Then $\mathfrak{m}^\nu$ is the conductor of A, where $\nu = Min\{n \mid e \leq \binom{n+r}{r}\}$.*

Proof. By Theorem 5, (b) (see also the following Remark) $G(A)$ is reduced. Then, by Theorem 8, the conductor of $G(A)$ in its normalization $G(\overline{A})$ is $G(\mathfrak{m})^\nu = G(\mathfrak{m}^\nu)$ (note that $G(\mathfrak{m})$ is the maximal homogeneous ideal of $G(A)$). Hence, by Theorem 9, $\mathfrak{m}^\nu$ is the conductor of A in $\overline{A}$. □

Theorem 10 can be extended to any one dimensional local ring in the following way:

11. Theorem. *Let B be a one dimensional reduced local ring with finite normalization $\overline{B}$ and maximal ideal $\mathfrak{m}$. Let $k(\mathfrak{m}) = B/\mathfrak{m}$ be the residue field of B and K be the algebraic closure of $k(\mathfrak{m})$. Set $e = e(B)$ and $emdim(B) = r + 1$. If $Proj(G(B) \otimes_{k(\mathfrak{m})} K)$ is reduced and consists of points in generic $e-1, e$ position then the conductor of B in $\overline{B}$ is $\mathfrak{m}^\nu$ where $\nu = Min\{n \mid e \leq \binom{n+r}{r}\}$.*

Proof. If $Proj(G(B) \otimes K)$ is reduced and consists of points in generic $e-1, e$ position then $G(B) \otimes K$ is reduced [Theorem 5, (b)], its normalization is $G(\overline{B}) \otimes K$ [Theorem 7,(b)], and its conductor is $(G(\mathfrak{m}) \otimes K)^\nu = G(\mathfrak{m})^\nu \otimes K$ [Theorem 8 and Theorem 5, (a)]. Furthermore, if $\mathfrak{b}$ is the conductor of $G(B)$, then $\mathfrak{b} \otimes K$ is contained in the conductor of $(G(B) \otimes K)$. In fact if $y \in \mathfrak{b}$, $b \in G(\overline{B})$ and $c, c' \in K$, $(b \otimes c)(y \otimes c') = by \otimes cc' \in G(B) \otimes K$. But, by Theorem 7,(c), $G(\mathfrak{m})^\nu \subset \mathfrak{b}$ and then, by flatness $G(\mathfrak{m})^\nu \otimes K \subset \mathfrak{b} \otimes K$. Thus

$$G(\mathfrak{m})^\nu \otimes K = \mathfrak{b} \otimes K$$

But, again by flatness,

$$dim_k(\mathfrak{b}/G(\mathfrak{m})^\nu) \otimes K = dim_k((\mathfrak{b} \otimes K)/(G(\mathfrak{m})^\nu \otimes K)) = 0$$

that is $\mathfrak{b} = G(\mathfrak{m})^\nu$ and, by Theorem 9, the conductor of B is $\mathfrak{m}^\nu$. □

Remark. Theorem 10 is the main result of a more general statement on the conductor of a curve at an ordinary singularity [8, Theorem 4.4]. This statement can be easily extended to one dimensional rings in the same way of Theorem 11.

3. The codimension one case.

We need some preliminaries.

12. Definition. Le $\mathfrak{p}$ be a prime ideal of a local ring A. A is *normally flat along* $\mathfrak{p}$ if $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is flat over $A/\mathfrak{p}$, for all $n \geq 0$. Note that A is normally flat along $\mathfrak{p}$ if nad only if $G_{\mathfrak{p}}(A)$ is free over $A/\mathfrak{p}$.

13. Theorem. *Let $\mathfrak{p}$ be a prime ideal of a local ring A such that $A/\mathfrak{p}$ is regular and $dim(A/\mathfrak{p}) = d \geq 1$. Then:*

$H^0(A, n) \geq H^d(A_{\mathfrak{p}}, n)$, for any n and equality holds if and only if A is normally flat along $\mathfrak{p}$.

In particular

$$emdim(A) \geq emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p}), e(A) \geq e(A_{\mathfrak{p}})$$

and if A is normally flat along $\mathfrak{p}$, then

$$emdim(A) = emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p}), e(A) = e(A_{\mathfrak{p}})$$

.

Proof. See [5, Theorem (22.24) and Proposition (30.2)]. □

14. Theorem. *Let $\mathfrak{p}$ be a prime ideal of codimension one of a Cohen-Macaulay local ring A such that $A/\mathfrak{p}$ is regular. Assume $e(A) = e(A_{\mathfrak{p}}) = e$ and $emdim(A) = emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p})$. If $H^0(A_{\mathfrak{p}}, n)$ is maximal then A is normally flat along $\mathfrak{p}$ and $G(A)$ is Cohen-Macaulay.*

Proof. Set $dim(A/\mathfrak{p}) = d$, $e(A_{\mathfrak{p}}) = e$ and $emdim(A_{\mathfrak{p}}) = r + 1$. Since A is Cohen-Macaulay there is an A-sequence $x_1, ..., x_d$ of elements of degree 1 (this can always be arranged if the residue field of A is infinite and if it is finite, by passing, if necessary, to the ring $A[U]_{\mathfrak{m}A[U]}$, U indeterminate) and such that the ring $B = A/(x_1, ..., x_d)$ is Cohen-Macaulay. Then, by the assumption, we have $e(B) = e(A) = e(A_{\mathfrak{p}}) = e$, $emdim(B) = emdim(A) - d = emdim(A_{\mathfrak{p}}) = r + 1$. Furthermore B and $A_{\mathfrak{p}}$ are one dimensional and then $H^0(B, n) \leq Min\{e, \binom{n+r}{r}\} = H^0(A_{\mathfrak{p}}, n)$. [Definition 1 and preceding comments] Hence $H^d(B, n) \leq H^d(A_{\mathfrak{p}}, n)$. But $H^d(B, n) \geq H^0(A, n)$ [11, Theorem 3.1 (note that in the statement there is a misprint, namely the exponent s of the Hilbert function H^s should be $s+1$)]. Furthermore $H^0(A, n) \geq H^d(A_{\mathfrak{p}}, n)$ by Theorem 13. We have then $H^d(B, n) = H^0(A, n) = H^d(A_{\mathfrak{p}}, n)$. Hence A is normally flat along $\mathfrak{p}$ [Theorem 13] and the initial forms $x_1^*, ..., x_d^*$ form a regular sequence of $G(A)$ [11, Theorem 3.1]. Furthermore, since $H^0(B, n) = H^0(A_{\mathfrak{p}}, n)$, the one dimensional ring B has maximal Hilbert function and then $G(B)$ is Cohen-Macaulay [11, Theorem 3.2]. But $G(B) \cong G(A)/(x_1^*, ..., x_d^*)$ [14, Chapter 2, Lemma 3.2] hence $G(A)$ is Cohen-Macaulay . □

15. Theorem. *Let $\mathfrak{p}$ be a prime ideal of codimension one of a reduced local ring A with finite normalization $\overline{A}$. Let $e(A_{\mathfrak{p}}) = e > 1$, $emdim(A_{\mathfrak{p}}) = r + 1$ and denote by K the algebraic closure of the residue field $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ of A at $\mathfrak{p}$. Let $\mathfrak{b}$ be the conductor of A in $\overline{A}$. Assume $A/\mathfrak{p}$ regular, $\sqrt{\mathfrak{b}} = \mathfrak{p}$ and $Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K)$ reduced with points in generic $e-1$, e position.*

Given the following conditions:

(a) A is Cohen-Macaulay and $emdim(A) = emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p})$, $e(A) = e(A_{\mathfrak{p}})$*;*

(b) A is S_2 *and normally flat along* $\mathfrak{p}$*;*

(c) $\mathfrak{b}$ *is primary and A is normally flat along* $\mathfrak{p}$*;*

(d) $\mathfrak{b} = \mathfrak{p}^{\nu}$, *where* $\nu = Min\{n \in \mathbb{N} \mid e \leq \binom{n+r}{r}\}$*;*

then $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ A Cohen-Macaulay ring is S_2 [6, 17.I]. By Theorem 5,(b) if $Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K)$ is reduced and consists of points in generic position, the Hilbert function of $A_{\mathfrak{p}}$ is maximal and, by Theorem 14, A is normally flat along $\mathfrak{p}$.

$(b) \Rightarrow (c)$ If A is S_2 then its conductor $\mathfrak{b}$ is unmixed [4,Lemma 7.4] and then primary,since $\sqrt{\mathfrak{b}} = \mathfrak{p}$.

$(c) \Rightarrow (d)$ $\mathfrak{b}A_{\mathfrak{p}}$ is the conductor of $A_{\mathfrak{p}}$ in its normalization $\overline{A}_{\mathfrak{p}}$ and then, by Theorem 11, $\mathfrak{b}A_{\mathfrak{p}}{=}(\mathfrak{p}A_{\mathfrak{p}})^{\nu}{=}\mathfrak{p}^{\nu}A_{\mathfrak{p}}$. Furthermore, if A is normally flat along $\mathfrak{p}$ then $\mathfrak{p}^{\nu}$ is primary [13, Proposition 1.1] and then

$$\mathfrak{b} = A \cap \mathfrak{b}A_{\mathfrak{p}} = A \cap \mathfrak{p}^{\nu}A_{\mathfrak{p}} = \mathfrak{p}^{\nu}$$

□

16. Corollary. *Suppose* $\mathfrak{p}$ *is a prime ideal of codimension one of a Cohen-Macaulay reduced local ring A with finite normalization* $\overline{A}$. *Denote by K the algebraic closure of the residue field* $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.*Let* $\mathfrak{b}$ *be the conductor of A in* $\overline{A}$. *Assume* $A/\mathfrak{p}$ *regular,* $\sqrt{\mathfrak{b}} = \mathfrak{p}$ *and* $Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K)$ *reduced. If* $emdim(A) = dim(A) + 1$ *and* $e(A) = e(A_{\mathfrak{p}}) = e$ *then* $\mathfrak{b} = \mathfrak{p}^{e-1}$.

Proof. Let $emdim(A) = dim(A) + 1$. By assumption $dim(A) = dim(A/\mathfrak{p}) + 1$ and, by Theorem 13, $emdim(A) \geq emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p})$. Then $emdim(A_{\mathfrak{p}}) = 2$ ($emdim(A_{\mathfrak{p}}) > 1$ since $e(A_{\mathfrak{p}}) > 1$) and $emdim(A) = emdim(A_{\mathfrak{p}}) + dim(A/\mathfrak{p})$. Hence $emdim(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K) = 2$ [Theorem 5,a)] that is $Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K) \subset \mathbb{P}^1$ and then its points are always in generic $e-1, e$ position [Example 3]. Hence the claim follows from Theorem 15, $(a) \Rightarrow (d)$ (if $r = 1$ one has $\nu = e - 1$). □

In the following we want to apply the previous results to the geometric case.

Standing notation. *From now on* $X = Spec(R)$ *is a reduced variety over an algebraically closed field k and* $Y = Spec(R/\mathfrak{q})$ *is an irreducible codimension one subvariety Y of X (i.e.* $\mathfrak{q}$ *is a prime ideal of codimension one in R). By K we denote the algebraic closure of the residue field* $k(\mathfrak{q})$ *of R at* $\mathfrak{q}$. *We set* $emdim(R_{\mathfrak{q}}) = r + 1$ *and* $e(R_{\mathfrak{q}}) = e$.

17. Definition. Let x be any closed point of Y and A be the local ring of X at x. Let $\mathfrak{p} = \mathfrak{q}A$ be the prime ideal in A defining the subvariety Y. Then:

(i) Y is a *multiple subvariety* of X of multiplicity e if $e(R_{\mathfrak{q}}) = e > 1$;

(ii) X is *normally flat along* Y *at* x if A is normally flat along $\mathfrak{p}$;

(iii) X is *normally flat along* Y if it is so at every point of Y;

(iv) Y is *nonsingular at* x if $A/\mathfrak{p}$ is regular.

Remark. With the notations of Definition 17 we have $A_{\mathfrak{p}} = R_{\mathfrak{q}}$ for any closed point x of Y.

18. Theorem. *Let Y be a multiple subvariety of X. There exists an open nonempty subset U of Y such that, for every closed point x of U, Y is nonsingular at x and X is normally flat along Y at x.*

Proof. It is well known that the nonsingular points of Y form an open nonempty set and that X is normally flat along Y at the points of an open nonempty subset of Y [5, Corollary (24.5)]. □

19. Theorem. *Set $emdim(R_{\mathfrak{q}}) = r+1$. Suppose that Y is a multiple subvariety of X of multiplicity e. If $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K)$ is reduced and consists of points in generic $e-1, e$ position then there exists an open nonempty subset U of Y such that, for every closed point x of U, the conductor $\mathfrak{b}$ of the local ring A of X at x is primary and equal to $\mathfrak{p}^{\nu}$ where $\mathfrak{p}$ is the prime ideal in A defining Y and $\nu = Min\{n \mid e \leq \binom{n+r}{r}\}$.*

Proof. By the Remark to Definition 17, if x is any point of X and A is the corresponding local ring, one has $e(A_{\mathfrak{p}}) = e(R_{\mathfrak{q}})$, hence, by assumption, $e(A_{\mathfrak{p}}) > 1$ that is, since $dim(A_{\mathfrak{p}}) = 1$, $A_{\mathfrak{p}}$ is not normal. Then it is well known that $\mathfrak{b}$ is contained in $\mathfrak{p}$. But the conductor contains a nonzero divisor. Thus the codimension one ideal $\mathfrak{p}$ is a minimal prime of $\mathfrak{b}$. Now it is easily shown that there exists an open nonempty subset U_1 of Y such that, if x is a closed point of U_1, $\mathfrak{p}$ is the unique prime ideal of A associated to $\mathfrak{b}$. Then $\mathfrak{b}$ is primary and $\sqrt{\mathfrak{b}} = \mathfrak{p}$. Furthermore, by Theorem 18, there exists an open nonempty subset U_2 of Y such that for every closed point x of U_2, Y is nonsingular at x and X is normally flat along Y at x. Then, if we apply Theorem 15, $(c) \Rightarrow (d)$ to the local rings of the points of $U = U_1 \cap U_2$, we have the claim. □

We want now to characterize geometrically the condition: $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K)$ is reduced. We need the following preliminary general result.

20. Theorem. *Let A be the local ring of X at a closed point x of Y and $\mathfrak{p}$ be the prime ideal defining Y in A. Set $dim(A/\mathfrak{p}) = d$. Suppose Y is nonsingular at x and X is normally flat along Y at x. Then there is an isomorphism of graded k-algebras*

$$G(A) \cong (G_{\mathfrak{p}}(A) \otimes_{A/\mathfrak{p}} k)[T_1, ..., T_d]$$

Proof. See [5,Corollary (21.11)]. □

Assume that Y is a multiple subvariety of X and A is the local ring of X at a closed point x of Y. $A/\mathfrak{p}$ is the local ring of Y at x. Then $Spec(G(A))$ is the tangent cone to X at x and $Spec(G(A/\mathfrak{p}))$ is the tangent cone to Y at x. Furthermore, there is a natural surjective homomorphism $G(A) \to G(A/\mathfrak{p})$ and then $Spec(G(A/\mathfrak{p}))$ naturally embeds in $Spec(G(A))$. In this setting we have the following result:

21. Theorem. *Let K be the algebraic closure of the residue field $k(\mathfrak{q})$ of R in $\mathfrak{q}$. Then $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K)$ is reduced (that is consists of e points) if and only if there exists an open nonempty subset U of Y such that, for every closed point x of U, the tangent cone to X at x is the union, as a set, of e distinct linear varieties,whose intersection is the tangent cone to Y at x.*

Proof. If $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K)$ has e points by [1, Theorem 1.1] there exists an open

$Spec(G(A))$ to X at x , has e irreducible components, that is $G(A)$ has e minimal primes. Furthermore, by Theorems 18 and 20, there exists an open nonempty subset U_2 of Y such that, for every closed point x of U_2, $G(A)$ is a polynomial ring over $G_{\mathfrak{p}}(A) \otimes_{A/\mathfrak{p}} k$. Then if x is a point of U_2 the minimal primes of $G(A)$ are extensions of the minimal primes of the one dimensional finitely generated graded k-algebra $G_{\mathfrak{p}}(A) \otimes_{A/\mathfrak{p}} k$, hence they are generated by linear forms. Since, at every closed point $x \in U = U_1 \cap U_2$ the tangent cone to Y is contained in the tangent cone to X, we have the claim. Vice versa if, at every point of an open nonempty subset of Y, the tangent cone to X has e irreducible components, again by [1, Theorem 1.1], also the zero dimensional scheme $Proj(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K)$ has e irreducible components, that is e points, and then, since by Theorem 5,(a) $e(G(A_{\mathfrak{p}}) \otimes_{k(\mathfrak{p})} K) = e(G(A_{\mathfrak{p}})) = e(A_{\mathfrak{p}}) = e$, is reduced. $\square$

Remark. Theorem 21 extends Corollary 1.4 of [1] which states the only if part in the particular case in which $G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K$ is reduced.

22. Definition. Let Y be a multiple subvariety of X. If $Proj(G(R_{\mathfrak{q}}) \otimes_{k(\mathfrak{q})} K)$ is reduced Y is said to be an *ordinary multiple subvariety* of X.

Remark. If Y is a multiple point of a curve X the previous definition agrees the one given in [9, Lemma-Definition 2.1].

23. Definition. If A is the local ring of the variety $X = Spec(R)$ at a closed point x, then $e(A)$ and $emdim(A)$ are called respectively the *multiplicity* and the *embedding dimension of X at x*. Let $Y = Spec(R/\mathfrak{q})$ be a multiple subvariety of X. Then

(i) X is *equimultiple along* Y if X has the same multiplicity $e(R_{\mathfrak{q}})$ at every point of Y;

(ii) X has *constant embedding dimension along* Y if X has the same embedding dimension $emdim(R_{\mathfrak{q}}) + dim(R/\mathfrak{q})$ at every point of Y.

Remark. By Theorem 18 there exists one point x of Y for which Y is nonsingular at x and X is normally flat along Y at x, then by Theorem 13 the multiplicity of X at x is $e(R_{\mathfrak{q}})$ and the embedding dimension of X at x is $emdim(R_{\mathfrak{q}}) + dim(R/\mathfrak{q})$ (see also the Remark to Definition 17); hence if X is equimultiple and has constant embedding dimension along Y these have to be the multiplicity and embedding dimension at all points of Y.

It is well known that, if $X = Spec(R)$ is a reduced non-normal S_2 variety, the non-normal locus of X (that is the set of closed points in X whose local ring is not normal) consists of the union of irreducible subvarieties of codimension one, $Y_i = Spec(R/\mathfrak{q}_i)$, $(1 \le i \le t)$. Furthermore the $\mathfrak{q}_i$ are the primes associated to the conductor of R, that is the primes for which the one dimensional rings $R_{\mathfrak{q}_i}$ are not normal or equivalently not regular.

24. Theorem. *Let $X = SpecR$ be a reduced non-normal S_2 variety. Assume that the irreducible components of the non-normal locus of X are ordinary multiple subvarieties $Y_i = Spec(R/\mathfrak{q}_i)$ $(1 \le i \le t)$ of multiplicity $e_i = e(R_{\mathfrak{q}_i}) > 1$. Set $emdim(R_{\mathfrak{q}_i}) = r_i + 1$ and let K_i be the algebraic closure of the residue field $k(\mathfrak{q}_i)$. If $Proj(G(R_{\mathfrak{q}_i}) \otimes_{k(\mathfrak{q}_i)} K_i)$ has points in generic $e_i - 1, e_i$ position, for any i, then the conductor of R is $\mathfrak{b} = \mathfrak{q}^{(\nu_1)} \cap \dots \cap \mathfrak{q}^{(\nu_t)}$ where $\nu_i = Min\{n \mid e_i \le \binom{n+r_i}{\dots}\}$ and*

$\mathfrak{q}_i^{(\nu_i)}$ *denotes the* $\nu_i - th$ *symbolic power of* $\mathfrak{q}_i$. *Furthermore assume that the* Y_i *are nonsingular varieties, for any* i, *and consider the following conditions:*

(a) $X = Spec(R)$ *is Cohen-Macaulay, equimultiple along* Y_i *and has constant embedding dimension along* Y_i *, for any* i*;*

(b) X *is normally flat along* Y_i*, for any* i*;*

(c) $\mathfrak{b} = \mathfrak{q}_1^{\nu_1} \cap ... \cap \mathfrak{q}_t^{\nu_t}$*;*

then $(a) \Rightarrow (b) \Rightarrow (c)$.

Proof. Let $\mathfrak{b} = \mathfrak{a}_1 \cap ... \cap \mathfrak{a}_t$ be a minimal primary decomposition, $\sqrt{\mathfrak{a}_i} = \mathfrak{q}_i$ $(1 \leq i \leq t)$. By Theorem 11, for any $i = 1, ..., t$, $\mathfrak{a}_i R_{\mathfrak{q}_i} = \mathfrak{b} R_{\mathfrak{q}_i} = {\mathfrak{q}_i}^{\nu_i} R_{\mathfrak{q}_i} = {\mathfrak{q}_i}^{(\nu_i)} R_{\mathfrak{q}_i}$. Then

$$\mathfrak{a}_i = R \cap \mathfrak{a}_i R_{\mathfrak{q}_i} = R \cap \mathfrak{q}_i^{\nu_i} R_{\mathfrak{q}_i} = \mathfrak{q}_i^{(\nu_i)}$$

which is the first claim. The implications of the second claim follow easily from the analogous local implications of Theorem 15. □

If we apply Theorem 24 to the particular case $t = 1$ we get that, under the assumption (a), Theorem 19 holds for any point x of the subvariety Y (not only on an open subset of Y).

25. Corollary. *Let* $X = SpecR$ *be a reduced non-normal Cohen-Macaulay variety. Assume that the non-normal locus of* X *is an ordinary multiple nonsingular subvariety* $Y = Spec(R/\mathfrak{q})$ *of multiplicity* $e = e(R_\mathfrak{q}) > 1$ *and that* X *is equimultiple and has constant embedding dimension along* Y*. Then, for every point* x *of* Y*, the conductor* $\mathfrak{b}$ *of the local ring* A *of* X *at* x *is primary and equal to* $\mathfrak{p}^\nu$ *where* $\mathfrak{p}$ *is the prime ideal in* A *defining* Y *and* $\nu = Min\{n \mid e \leq \binom{n+r}{r}\}$

Proof. Setting $t = 1$ in Theorem 24 we have that $\mathfrak{b} R_\mathfrak{q} = \mathfrak{q}^\nu R_\mathfrak{q} = (\mathfrak{q} R_\mathfrak{q})^\nu = \mathfrak{p}^\nu$ is the conductor of A. □

26. Corollary. *Let* H *be a non-normal hypersurface. Assume that the irreducible components of the non-normal locus of* X *are ordinary multiple subvarieties* $Y_i = Spec(R/\mathfrak{q}_i)$ *of multiplicity* $e_i = e(R_{\mathfrak{q}_i}) > 1$ $(1 \leq i \leq t)$*. If the* Y_i *are nonsingular varieties and* X *is equimultiple along* Y_i*, for any* i*, then the conductor of* R *is* $\mathfrak{b} = \mathfrak{q}_1^{e_1 - 1} \cap ... \cap \mathfrak{q}_t^{e_t - 1}$.

Proof. If $H = Spec(R)$ is a hypersurface, then R is Cohen-Macaulay and its embedding dimension at any point is constant. Furthermore $Proj(G(R_{\mathfrak{q}_i}) \otimes_{k(\mathfrak{q}_i)} K_i) \subset \mathbb{P}^1$ has points in generic $e_i - 1, e_i$ position [Example 3]. Then the claim follows by Theorem 24, $(a) \Rightarrow (c)$, in which $r_i = 1$. □

27. Example. Let $H = SpecR$, $R = \mathbb{C}[T_0, ..., T_r]/(L_1 ... L_n)$ $(r \geq 2, n \geq 2)$ be the union of n hyperplanes of $\mathbb{P}^r$. Let $\mathfrak{q}_i$ $(1 \leq i \leq t)$ denote the codimension one primes of R of the form $(\overline{L}_p, \overline{L}_q)$, $p, q \in \{1, ..., n\}$, $p \neq q$, and $Y_1, ..., Y_t$ be the corresponding linear varieties (that is the irreducible components of the non-normal locus of X). Then $e(R_{\mathfrak{q}_i}) = e_i$ is the number of hyperplanes passing through Y_i. Furthermore since Y_i are linear it is easily shown that $\mathfrak{q}_i^m = \mathfrak{q}_i^{(m)}$, for any m. Hence from Theorems 21 and 24, it follows that the conductor of R is $\mathfrak{b} = \mathfrak{q}_1^{e_1 - 1} \cap ... \cap \mathfrak{q}_t^{e_t - 1}$.

28. Example. Let $R = \mathbb{C}[X,Y,Z]/(XY^n - Z^n) = \mathbb{C}[x,y,z]$ $(n \geq 2)$. The non-normal locus of the hypersurface $H = SpecR$ is the line $L : y = 0, z = 0$ of multiplicity n , that is $e(R_{\mathfrak{q}}) = n$, where $\mathfrak{q} = (y,z)$. Moreover it is easily seen that H is equimultiple along L. Furthermore, if $a \in \mathbb{C}$, $a \neq 0$, the tangent cone, at the point $(a,0,0)$ of L, consists, as a set, of the n distinct planes $z = by$, where $b^n = a$, and then, by Theorem 21 and Corollary 26 and, the conductor of R in $\overline{R}$ is $(y,z)^{n-1}$.

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Ferruccio Orecchia, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Complesso Universitario di Monte S. Angelo - Via Cintia , 80126 Napoli-Italy
E-mail address: ORECCHIA@MATNA2.DMA.UNINA.IT.