A NOTE ON TOTAL COLORINGS OF PLANAR GRAPHS WITHOUT 4-CYCLES

PING WANG*

Department of Mathematics, Statistics and Computer Science
St. Francis Xavier University, Antigonish, Nova Scotia, Canada

e-mail: pwang@stfx.ca

AND

JIAN-LIANG WU†

School of Mathematics, Shandong University
Jinan, Shandong, 250100, P.R. China

Abstract

Let $G$ be a 2-connected planar graph with maximum degree $\Delta$ such that $G$ has no cycle of length from 4 to $k$, where $k \geq 4$. Then the total chromatic number of $G$ is $\Delta + 1$ if $(\Delta, k) \in \{(7, 4), (6, 5), (5, 7), (4, 14)\}$.

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We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph $G$ respectively. Let $d(v)$ denote the degree of vertex $v$. A $k$-vertex is a vertex of degree $k$.

A total $k$-coloring of a graph $G$ is a coloring of $V(G) \cup E(G)$ using $k$ colors such that no two adjacent or incident elements receive the same color. The total chromatic number $\chi_T(G)$ is the smallest integer $k$ such that $G$ has a total $k$-coloring. Behzad and Vizing (see page 86 in [8]) conjectured independently that any graph $G$ is totally $(\Delta(G) + 2)$-colorable in 1965.

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Various coloring techniques have been introduced in effort to prove this conjecture for some special graph classes (see survey papers [7] and [11]). In 1989, Sanchez-Arroyo [10] proved that for any graph \( G \) it is NP-complete to decide if \( \chi_T(G) = \Delta(G) + 1 \). In 1997, Borodin et al [3] proved that a planar graph \( G \) with maximum degree \( \Delta \geq 11 \) has \( \chi_T(G) = \Delta(G) + 1 \), and they also obtained several related results by adding girth restrictions [4]. Note that the added girth requirement in [4] prohibits the appearance of triangles. The forbidden cycle or the girth restriction plays an important role in considering list-coloring planar graphs. For example, Kratochv’il and Tuza showed that every triangle-free planar graph is 4-choosable and Thomassen observed that a planar graph is 3-choosable if the girth of the graph is at least 5 (both results can be found in Section 2.13 of [8]). Recently, Lam, Xu and Liu [9] proved that every \( C_4 \)-free planar graph is 4-choosable. We shall adopt a similar approach and prove the following theorem. Note that triangles are allowed in the graph \( G \) in our theorem.

Let a planar graph \( G \) be charged by an initial charge \( w(v) = d(v) - 4 \) if \( v \in V(G) \) and \( w(f) = r(f) - 4 \) if \( f \in F \), where \( r(f) \) is the degree of the face \( f \). Euler’s formula implies that \( \sum_{x \in V \cup F} w(x) < 0 \). The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to \( \sum_{x \in V \cup F} w(x) > 0 \). A contradiction follows and this shows the unavoidability of a set of special elements in \( G \) (see Claims 2, 3 and 4).

**Theorem.** Let \( G \) be a connected planar graph with maximum degree \( \Delta \) such that \( G \) has no cycle of length from 4 to \( k \), where \( k \geq 4 \). If

1. \( \Delta \geq 7 \) and \( k \geq 4 \)
2. \( \Delta \geq 6 \) and \( k \geq 5 \)
3. \( \Delta \geq 5 \) and \( k \geq 7 \)
4. \( \Delta \geq 4 \) and \( k \geq 14 \),

then \( \chi_T(G) = \Delta(G) + 1 \).

**Lemma 1** [6]. Every region of a planar imbedding of a graph has a simple cycle for its boundary if and only if \( G \) is 2-connected.

This lemma is equivalent to the assertion that no three edges incident with any vertex \( v \) lie on the same face. It implies that each vertex \( v \) is incident with \( d(v) \) faces. We shall use this fact often in the proof of the Theorem.
An edge coloring of a graph $G$ is a coloring of $E(G)$ such that no two adjacent edges receive the same color. A graph $G$ is said to be edge-$f$-choosable if, whenever we give lists $A_e$ of $f(e)$ colors to each edge $e \in E(G)$, there exists an edge coloring of $G$ where each edge is colored with a color from its own list.

Lemma 2 [5]. A bipartite graph $G$ is edge-$f$-choosable where $f(e) = \max\{d(u), d(v)\}$ for $e = uv \in E(G)$.

Proof of Theorem. Let $G = (V, E, F)$ be a minimal counterexample to any of (1) – (4) in the Theorem. Then

(a) $G$ is 2-connected and

(b) any vertex is incident with at most $\left\lfloor \frac{d(v)}{2} \right\rfloor$ 3-faces, and

(c) $G$ contains no even cycle $v_1v_2\cdots v_{2t}v_1$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$, and

(d) $G$ contains no edge $uv$ with $\min\{d(u), d(v)\} \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor$ and $d_G(u) + d_G(v) \leq \Delta(G) + 1$.

(a) and (b) are obvious. The proofs of (c) and (d) can be found in [2] and [5], respectively.

Let $G_2$ be the subgraph induced by the edges incident with the 2-vertices of $G$. Since $\Delta(G) \geq 4$ in all four cases in the Theorem, (d) implies that $G$ does not contain two adjacent 2-vertices. Hence, $G_2$ does not contain any odd cycle. It follows from (c) that $G_2$ does not contain any even cycle. Therefore, any component of $G_2$ is a tree. For any component in $G_2$ that is a path of even length, one can easily find a set of edges saturating all 2-vertices. For any component that is not a path of even length, we can select a vertex $t$ with $d_{G_2}(t) \geq 3$ as the root of the tree. We denote edges of distance $i$ from the root to be at level $i + 1$ where $i = 0, 1, \ldots, d$ and $d$ is the depth of the tree. Since $G$ does not contain two adjacent 2-vertices, the distance from any leaf to the root is even. We can select all the edges at even level to form a matching saturating all 2-vertices in this component. Thus, there exists a matching $M$ such that all 2-vertices in $G_2$ are saturated. If $uv \in M$ and $d(u) = 2$, $v$ is called the 2-master of $u$ and $u$ is called the dependent of $v$. Each 2-vertex has a 2-master and each vertex of degree $\Delta$ can be the 2-master of at most one 2-vertex.

Since $G$ is a planar graph, by Euler’s formula, we have
where \( r(f) \) is the degree of the face \( f \), that is, the number of edges around \( f \). A \( k \)-face is a face of degree \( k \). Now we define the initial charge function \( w(x) \) for each \( x \in V \cup F \). Let \( w(x) = d(x) - 4 \) if \( x \in V \) and \( w(x) = r(x) - 4 \) if \( x \in F \). It follows from (E) that \( \sum_{x \in V \cup F} w(x) < 0 \).

We begin the proof of (1) in the Theorem. First we prove a claim

Claim 1. If \( X \neq \emptyset \), then \( G \) contains a bipartite subgraph \( B = (X,Y) \) such that \( d_B(x) = 1 \) and \( d_B(y) \leq 2 \) whenever \( x \in X \) and \( y \in Y \).

**Proof of Claim 1.** Let \( H = (X',Y) \), where \( X' \subseteq X \), be a maximum bipartite subgraph such that \( d_H(x) = 1 \) and \( d_H(y) \leq 2 \) whenever \( x \in X' \) and \( y \in Y \). Note that there may be some isolated vertices in \( Y \). Clearly, \( H \) is not empty since there is at least one edge from \( X \) to \( Y \). Suppose that \( X \setminus X' \neq \emptyset \). Let \( v \in X \setminus X' \). An alternating path, \( P_v \), in \( G \) is a path whose origin is \( v \) and edges are alternating between \( E(K) \setminus E(H) \) and \( E(H) \). By the maximality of \( H \), there exists no alternating path that will terminate at a vertex \( v' \in Y \) with \( d_H(v') \leq 1 \). Let \( Z \) denote the set of all vertices connected to \( v \) by alternating paths. Set \( X'' = Z \cap X' \) and \( Y'' = Z \cap Y \) (see Figure 1).

Clearly, \( Y'' \subseteq \cup_{x \in X''} N(x) \). Suppose \( \cup_{x \in X''} N(x) \not\subseteq Y'' \). It follows that there exists a vertex \( x \in X'' \) such that \( xy \in E(G) \) and \( y \not\in Y'' \). This implies that an alternating path \( P_v \) terminates at a vertex \( y \in Y \), a contradiction. Hence, \( Y'' = \cup_{x \in X''} N(x) \).

Now we show that \( d_H(y) \geq 2 \) for any \( y \in Y'' \). Suppose, on the contrary, there exists a vertex \( y_i \in Y'' \) where \( vy_1x_1 \ldots x_{i-1}y_i \) is an alternating path such that \( d_H(y_i) = 1 \). Let \( H' = H - \{y_1x_1, \ldots, y_{i-1}x_{i-1}\} + \{vy_1, x_1y_2, \ldots, y_ix_i\} \) if \( i \geq 2 \) and let \( H' = H + \{vy_1\} \) if \( i = 1 \). It follows that \( |E(H')| > |E(H)| \), a contradiction to \( H \) being maximum.
Let $F = (X'', Y'')$. It follows that $d_F(y) \geq d_H(y) + 1 \geq 3$ for any $y \in Y''$. Note that $d_G(x) = d_F(x) \leq 3$ for any $x \in X''$.

Now $G - X''$ has a total $(\Delta + 1)$-coloring by the minimality of $G$. By Lemma 2, we can color all edges in $F$ using the same set of colors by choosing the colors unused on $y \in Y''$. Since the maximum degree in $X''$ is 3, all vertices in $X''$ can be easily colored by $(\Delta+1)$ colors. Therefore, $G$ has a total $(\Delta + 1)$-coloring, a contradiction with the fact that $G$ is a counterexample. This implies $X = X'$, and which in turn, proves Claim 1.

We call $y$ the $3$-master of $x$ if $xy \in B$ and $x \in X$. It follows from this claim that each vertex of degree at most 3 has a 3-master. Each vertex of degree at least $\Delta - 1$ can be a 3-master of at most two vertices.

Claim 2. If $\Delta \geq 7$, then $G$ does not contain a 3-face $uvw$ such that $d(u) = d(v) = 4$.

**Proof of Claim 2.** Suppose it does contain such a 3-face. Let $G' = G - uv$. By the minimality of $G$, $G'$ has a total $(\Delta + 1)$-coloring $\varphi$. Since $d_{G'}(u) = d_{G'}(v) = 3$ and $\Delta \geq 7$, we may assume that $\varphi(u) \neq \varphi(v)$. Let $C$ be the set of colors used to color edges adjacent to $uv$. If $\varphi(w) \notin C$, then color $uv$ with $\varphi(w)$. Otherwise, without loss of generality, we may assume that an edge $e$ incident with $u$ is colored with $\varphi(w)$. Then we erase the color on $u$. It follows that at least one color is available for $uv$, and then we re-color $u$. This is possible because $d(u) = 4$ and both $e$ and $w$ share the same color. Now, $G$ has a total coloring with $(\Delta + 1)$ colors, a contradiction with the fact that $G$ is a counterexample.
Claim 2 and (d) imply that every 3-face is incident with at least two vertices of degree at least 5. To prove (1), we are ready to construct a new charge $w^*(x)$ on $G$ as follows:

R11: Each $r(\geq 5)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R12: Each 2-vertex receives $\frac{2}{5}$ from its 3-master, and receives $\frac{16}{15}$ from its 2-master if it is incident with a 3-face and receives 1 from its 2-master otherwise.

R13: Each 3-vertex receives $\frac{8}{15}$ from its 3-master. In addition, if $v$ is incident with a 3-face $f$, then each 3-vertex $v$ receives $\frac{1}{15}$ from $u$ where $u$ is a neighbor of $v$ but not incident with $f$.

R14: Each 3-face receives $\frac{1}{2}$ from its incident vertices of degree at least 5.

By (d), $d(v) = \Delta \geq 7$ if a vertex $v$ is the 2-master of some vertex, $d(v) \geq \Delta - 1 \geq 6$ if $v$ is the 3-master of some vertices, and $d(u) \geq 6$ if a vertex $u$ gives $\frac{1}{15}$ via R13. Note that a vertex can be the 3-master of most two vertices and, in turn, it may give at most $2 \times \max\{\frac{3}{5}, \frac{8}{15}\} = \frac{6}{5}$. Let $f$ be a face of $G$. Clearly, $w^*(f) = 0$ if $r(f) \geq 5$. By Claim 2, each 3-face $f$ is incident with at least two vertices of degree at least 5. Hence, $w^*(f) \geq w(f) + 1 = 0$. Let $v$ be an arbitrary vertex of $G$. First, we consider the case of $d(v) = 2$. It will receive $\frac{2}{5}$ from its 3-master. By Lemma 1, $v$ is incident with two faces. If $v$ is incident with a 3-face, then the other incident face of $v$ must have degree at least 6 since $G$ is a $C_4$-free graph.

This implies that $v$ receives at least $\frac{1}{3}$ from the face of degree $\geq 6$. If $v$ is not incident with a 3-face, then $v$ receives at least $2 \times \frac{1}{5}$ from its incident faces. So $w^*(v) \geq w(v) + \min\{\frac{2}{5} + \frac{10}{15}, \frac{1}{5} + \frac{3}{5} + 1 + \frac{2}{5}\} = 0$. Consider $d(v) = 3$. If it is incident with a 3-face, then the other two vertices on the same face must be of degree at least 5 and this implies that $v$ receives at least $\frac{2}{5}$ from its incident faces. If $v$ is not incident with a 3-face, then it must be incident with three $r$-faces where $r \geq 5$. It follows that it receives at least $\frac{2}{5}$ from its incident faces. Hence, $w^*(v) \geq w(v) + \min\{\frac{8}{15} + \frac{1}{15}, \frac{2}{5} + \frac{8}{15} + \frac{3}{5}\} = 0$.

If $d(v) = 4$, then it is incident with at most two 3-faces and its other two incident faces must be of degree $\geq 5$. Hence, $w^*(v) \geq w(v) + \frac{2}{5} \geq 0$. If $d(v) = 5$, then $v$ is incident with at least three $r$-faces where $r \geq 5$ and at most two 3-faces. Hence, $w^*(v) \geq w(v) + \frac{3}{5} - 2 \times \frac{1}{2} \geq 0$. If $d(v) = 6$, it can be 3-master of at most two vertices. Consider any two neighbors of $v$, say $u_1$ and $u_2$. If they form a 3-face, then $v$ gives $\frac{1}{2}$ to the 3-face. If each of them is a 3-vertex on some 3-face, then $v$ gives $2 \times \frac{1}{15}$. However, these two cases can not happen simultaneously; that is, $vu_1u_2$ is a 3-face and $u_1$,
u_2 have another common neighbor w \neq v, such that either d(u_1) = 3 or d(u_2) = 3 since G is C_4-free graph. In the evaluation of the lower bound of w^*(v), it suffices to consider the case when v gives $3 \times \frac{1}{3}$ to its incident 3-faces. It follows that $w^*(v) \geq w(v) + \frac{3}{5} - 2 \times \frac{8}{15} - 3 \times \frac{1}{9} > 0$. Now consider $d(v) = 7$. Suppose v is a 2-master of a vertex u. If u and v are incident with the same 3-face, then v receives at least $3 \times \frac{1}{3} + (1 - \frac{4}{5})$ from its incident faces and gives $\frac{10}{9}$ to u. Otherwise v receives at least $4 \times \frac{1}{9}$ from its incident faces and gives 1 to u. Vertex v may be incident with at most three 3-faces and the remaining neighbor of v not incident with any three 3-faces may be a 3-vertex and in another 3-face, in turn, v may give $\frac{1}{15}$ to the 3-vertex. Vertex v may also be the 3-master of a vertex u. Hence, $w^*(v) \geq w(v) + \min\{3 \times \frac{1}{3} + \frac{1}{5} - \frac{16}{15}, \frac{4}{5} - 1\} - \left(3 \times \frac{1}{3} + \frac{1}{5} + 2 \times \frac{3}{5}\right) > 0$. In general, if $d(v) \geq 8$, then $w^*(v) \geq w(v) + \min\{\lfloor \frac{d(v)}{5}\rfloor \times \frac{1}{5} + 1 - \frac{16}{15}, \lfloor \frac{d(v)}{5}\rfloor \times \frac{1}{5} - 1\} - \left(\lfloor \frac{d(v)}{5}\rfloor \times \frac{1}{2} + \frac{1}{15} + 2 \times \frac{3}{5}\right) > 0$. It follows that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) \geq 0$, a contradiction with (E). This completes the proof of (1).

Note that (1) implies that (2) is true if $\Delta \geq 7$. Hence, it is sufficient to prove (2) by assuming $\Delta = 6$. Similarly, we may assume that $\Delta = 5$ in the proof of (3) and $\Delta = 4$ in the proof of (4).

**Claim 3.** If $\Delta \geq 5$, then G does not contain a 3-face uvw such that $d(u) = d(v) = 3$.

The proof of Claim 3, which we omit, is the same as Claim 2. Claim 3 implies that each 3-face is incident with at least two vertices of degree at least 4. To prove (2), we construct the new charge $w^*(x)$ on G as follows:

R21: Each r(\geq 6)-face gives $1 - \frac{4}{7}$ to its incident vertices.

R22: Each 2-vertex receives $\frac{1}{7}$ from its 2-master if it is incident with a 3-face and receives $\frac{4}{7}$ from its 2-master otherwise.

R23: Each 3-vertex v receives $1/3$ from u if v is incident with a 3-face f and u is a neighbor of v but not incident with f.

R24: Each 3-face receives $\frac{1}{7}$ from its incident vertex v if $d(v) \geq 5$ and receives $\frac{1}{3}$ if $d(v) = 4$.

Clearly, we have $w^*(f) \geq 0$ for any face f. Let v be an arbitrary vertex of G. Consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face must have degree at least 7 since G is a C_4-free and C_5-free graph. It follows that v receives at least $1 - \frac{4}{7} = \frac{3}{7}$ from the incident face and $\frac{11}{7}$ from its 2-master; that is, $w^*(v) \geq w(v) + \frac{3}{7} + \frac{11}{7} = 0$. Otherwise if
v is not incident with any 3-face, then it receives at least \(2 \times (1 - \frac{4}{6}) = \frac{2}{3}\) from its two incident faces of degree at least 6 and \(\frac{4}{3}\) from its 2-master. Hence, \(w^*(v) \geq w(v) + \frac{2}{3} + \frac{4}{3} = 0\). Suppose \(d(v) = 3\). If \(v\) is incident with a 3-face, then \(v\) receives at least \(\frac{2}{3}\) from its two incident faces and \(\frac{1}{3}\) from its 3-master not lying on the same 3-face. Otherwise if \(v\) is not incident with any 3-face, then \(v\) receives at least 1 from its three incident faces. Hence, \(w^*(v) \geq w(v) + 1 = 0\). Note that \(v\) gives either \(\frac{1}{6}\) if \(d(v) = 4\) or \(\frac{1}{2}\) if \(d(v) \geq 5\) to an incident 3-face, say \(uvw\) where \(u, w \in N(v)\), or gives \(\frac{3}{2}\) to \(u\) and \(\frac{1}{7}\) to \(w\) by R23 but \(v\) will then receive at least \(1 - \frac{4}{6} = \frac{1}{3}\) from the face whose partial boundary contains \(u, v, w\) sequentially if \(uw \notin E(G)\). In the evaluation of the lower bound of \(w^*(v)\), it suffices to consider the case when \(v\) gives \(\frac{1}{3}\) or \(\frac{1}{2}\) to its incident 3-faces. If \(d(v) = 4\), then it receives at least \(\frac{2}{3}\) from its two incident faces of degree \(\geq 6\) and gives at most \(\frac{2}{3}\) to its incident 3-faces since any 4-vertex is incident with at most two 3-faces. It follows that \(w^*(v) \geq w(v) + \frac{3}{4} - \frac{2}{3} = 0\). If \(d(v) = 5\), then there are five faces incident with \(v\) by Lemma 1. It follows that \(v\) is incident with at most two 3-faces and at least three \(r\)-faces \((r \geq 6)\). If four neighbors of \(v\) form two 3-faces and a 3-face is pending on the remaining neighbor of \(v\), then \(v\) discharges at most \(2 \times \frac{1}{2} + \frac{1}{3}\) via R23. This implies that \(w^*(v) \geq w(v) + 3 \times \frac{1}{3} - (2 \times \frac{1}{2} + \frac{1}{3}) > 0\). Suppose \(d(v) = 6\). It follows that \(v\) can be the 2-master of some vertex \(u\). In this case, either \(u\) is on 3-face \(vu'u'\) (and it follows that \(v\) gives \(\frac{1}{4} + \frac{1}{3}\), or \(v\) is the 2-master of \(u\) and 3-master of \(u'\) where \(v\) and \(u\) are not on the same 3-face, and it follows that \(v\) gives \(\frac{4}{3} + \frac{1}{3}\). To find a low bound for \(w^*(v)\), it suffices to consider the first case when \(v\) is the 2-master of \(u\) and \(vu'u'\) forms a 3-face. If \(v\) is the 3-master of some 3-vertex \(u_1\), then \(v\) gives at most \(2 \times \frac{1}{3}\) to its dependents and \(\frac{1}{2}\) to another 3-face. In this case, \(v\) receives \(\frac{3}{7} + 3 \times \frac{1}{3}\) from its incident faces. If \(v\) is not a 3-master of any 3-vertex, then \(v\) gives at most \(2 \times \frac{1}{2}\) to its two incident 3-faces. In this case, \(v\) receives \(\frac{3}{7} + 2 \times \frac{1}{3}\) from its incident faces. Hence, \(w^*(v) \geq w(v) + \min\left(\frac{3}{7} + 1 - \left(\frac{1}{4} + \frac{1}{3} + \frac{1}{3}\right), \frac{3}{7} + 2 - \frac{1}{2} - \frac{1}{2} - 1\right) = 2 - \frac{33}{22} = \frac{17}{22} > 0\). It follows that \(\sum_{x \in V(U) \cup F} w^*(v) = \sum_{x \in V(U) \cup F} w^*(x) > 0\), a contradiction. This completes the proof of (2).

To prove (3), we construct a new charge \(w^*(x)\) on \(G\) as follows:

R31: Each \((r \geq 8)\)-face gives \(1 - \frac{1}{r}\) to its incident vertices.

R32: Each 2-vertex receives \(\frac{13}{3}\) from its 2-master if it is incident with a 3-face and receives 1 from its 2-master otherwise.

R33: Each 3-face receives \(\frac{1}{2}\) from its incident vertices of degree at least 4.
We also have $w^*(f) \geq 0$ for any face $f$. Let $v$ be an arbitrary vertex of $G$. If $d(v) = 2$, then $w^*(v) \geq w(v) + \min\{\frac{13}{18} + \frac{3}{2}, 2 \times \frac{1}{2} + 1\} = 0$. If $d(v) = 3$, then $v$ is incident with at most one 3-face and at least two faces of degree $\geq 8$. It follows that $v$ receives at least $2 \times \frac{1}{2} = 1$ from its incident faces, and in turn, $w^*(v) \geq w(v) + 1 = 0$. If $d(v) = 4$, then it receives at least $2 \times \frac{1}{2} = 1$ from its incident faces and gives at most $2 \times \frac{1}{2} = 1$ to its incident 3-faces, that is, $w^*(v) \geq w(v) + 1 - 1 = 0$. Suppose $d(v) = 5$. If $v$ is the 2-master of a 2-vertex $u$, and $u$ is incident with a 3-face, then $v$ receives at least $3 \times \frac{1}{2}$ from its incident faces and gives at most $\frac{15}{18} + 2 \times \frac{1}{2}$ to its dependent and 3-faces. Otherwise $v$ receives at least $3 \times \frac{1}{2}$ from its incident faces and gives at most $1 + 2 \times \frac{1}{2}$ to its dependent and 3-faces. It follows that $w^*(v) \geq w(v) + \min\{\frac{3}{2} - \frac{15}{18} - 1, \frac{3}{2} - 2\} = \frac{1}{18} > 0$. This implies that $\sum_{x \in V} w^*(x) \geq \sum_{x \in V} w^*(x) > 0$, a contradiction. This completes the proof of (3).

We will prove the following claim before we prove (4).

**Claim 4.** If $\Delta = 4$, then $G$ contains no 4-vertex $z$ where $z$ is incident with two 3-faces $zux, zvy$ and $d(x) = d(y) = 2$.

**Proof of Claim 4.** Suppose, on the contrary, such vertex $z$ does exist. We can totally color the edges and vertices of $G - \{xz, yz\}$ with a set of five colors, say $C$, by the minimality of $G$. First, we erase the colors assigned on $x$ and $y$. Let $c_1, c_2, c_3, c_4, c_5$ be colors used on $xu, zu, zv, yv, z$, respectively.

We will show that $c_1 \neq c_4$. Otherwise if $c_1 = c_4$, we claim that $c_1 \neq c_5$. If $c_1 = c_5$, then we can color $xz$ by $\alpha \in C \setminus \{c_1, c_2, c_3\}$ and $yz$ by a color in $C \setminus \{c_1, c_2, c_3, \alpha\}$. It easy to see that $x$ and $y$ can be colored because they are only adjacent to two vertices and incident with two edges. This implies that $G$ can be totally colored by five colors, a contradiction. Now we show it is impossible that $c_1 = c_4$ and $c_1 \neq c_5$. If $c_1 = c_4$, then we can interchange colors $c_3$ and $c_1$ at $v$ and color $zx$ by $c_3$. It follows that we can also color $zy$ by a color in $C \setminus \{c_1, c_2, c_3, c_5\}$. Similarly we can color vertices $x$ and $y$ since they are both vertices of degree 2. This implies that $G$ can be totally colored by five colors, a contradiction.

Similarly, we can show that $c_1 \neq c_3$ and $c_1 \neq c_5$. Since $c_1 \notin \{c_2, c_3, c_4, c_5\}$, $c_1$ can be assigned to $zy$ and there is a color available for $zx$, $x$ and $y$. This implies that $G$ can be totally colored by five colors, a contradiction. ■

To prove (4), construct a new charge $w^*(x)$ on $G$ as follows:

R41: Each $(\geq 15)$-face gives $1 - \frac{4}{r}$ to its incident vertices.
R42: Each 2-vertex receives $\frac{19}{24}$ from its neighbors if it is incident with a 3-face and receives $\frac{8}{15}$ from its 2-master otherwise.

R43: Each 3-face receives $\frac{1}{3}$ from its incident vertices.

It is obvious that $w^*(f) = 0$ for any face $f$. Let $v$ be an arbitrary vertex of $G$. First consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face $f$ must have degree at least 16. From (d), any neighbor of $v$ should be of degree at least $(\Delta + 2) - 2 = 4$. Hence, they can not be 2-vertices. It follows that $v$ receives at least $1 - \frac{4}{15} = \frac{11}{15}$ from $f$ and $2 \times \frac{19}{24} = \frac{19}{12}$ from its neighbors, and gives $\frac{1}{3}$ to its incident 3-face. Otherwise $v$ receives at least $2 \times \frac{11}{15} = \frac{22}{15}$ from its incident faces and $\frac{8}{15}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \min\left(\frac{11}{15}, \frac{22}{15} + \frac{8}{15}\right) = 0$.

Now consider the case of $d(v) = 3$. $v$ receives at least $2 \times \frac{11}{15} = \frac{22}{15}$ from its incident faces. Hence, $w^*(v) = w(v) + \frac{22}{15} - \frac{1}{3} = \frac{20}{15} > 0$. If $d(v) = 4$ and it is incident with two 3-faces, then $v$ is adjacent to at most one 2-vertex by Claim 4. It follows that $w^*(v) \geq w(v) + \frac{22}{15} - \left(\frac{2}{3} + \frac{19}{24}\right) = \frac{13}{20} > 0$. Otherwise it receives at least $3 \times \frac{11}{15}$ from its incident faces, and gives at most $\frac{1}{3}$ to its incident 3-face and $\frac{19}{24} + \frac{8}{15}$ to its adjacent 2-vertices. It follows that $w^*(v) \geq w(v) + \frac{33}{15} - \left(\frac{1}{3} + \frac{19}{24} + \frac{8}{15}\right) = \frac{13}{20} > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (4).

In the proof of the Theorem, we showed that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$. It implies the following corollary.

**Corollary 1.** Let $G$ be a graph with maximum degree $\Delta$ embedded in a surface of nonnegative characteristic, and $G$ has no cycle of length from 4 to $k$, where $k \geq 4$. Then $\chi_T(G) = \Delta + 1$ if $(\Delta, k) \in \{(7, 4), (6, 5), (5, 7), (4, 14)\}$.

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**References**


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