EXTREME POINTS OF SETS OF RANDOMIZED STRATEGIES IN CONstrained OPTIMIZATION AND CONTROL PROBLEMS∗

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Abstract. This paper concerns the existence and characterization of optimal randomized strategies for some constrained optimization and control problems. We first present a characterization of the extreme points of a set of randomized strategies that satisfy \( n \) moment-like constraints. Conditions are given under which those extreme points are randomizations of at most \( n + 1 \) deterministic strategies. This result is then applied to obtain the existence and characterization of optimal strategies for a class of deterministic, allocation-like, optimization problems and their Young relaxations. Similar results are obtained for constrained Markov control processes in Borel spaces.

Key words. randomized strategies (relaxed controls), extreme points of convex sets, Young’s relaxation technique, allocation problems, constrained Markov control processes

AMS subject classifications. 90C25, 46A55, 90C40

DOI. 10.1137/040605345

1. Introduction. In some areas—e.g., optimal control, game theory, and statistical decision theory, to name just a few—there are optimization problems in which one has to deal with “randomized” decisions or control actions, known as randomized strategies. (Other names: Young measures, parametrized measures, relaxed controls, randomized control policies, etc.; see [3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23, 27, 28, 29, 30].) There are, on the other hand, “deterministic” optimization problems which can be relaxed via randomized strategies. For these classes of optimization problems, an important issue is of course to find conditions for the existence and characterization of optimal randomized strategies, which are the questions addressed in this paper.

To motivate our general approach and main results, we shall first present a brief description of Young’s relaxation technique, which can be traced back to the 1930s (see, e.g., [9, 30, 42] for earlier references), originally introduced for problems in the calculus of variations.

Young’s relaxation technique. Consider two topological spaces \( X \) and \( A \), endowed with their Borel \( \sigma \)-algebras \( \mathcal{B}(X), \mathcal{B}(A) \), and let \( c_0 \) be a real-valued cost function on \( X \times A \). Moreover, let \( \nu \) be a probability measure (p.m.) on \( \mathcal{B}(X) \), and let \( F_0 \) be the set of all (Borel) measurable functions from \( X \) to \( A \). A function \( f \) in \( F_0 \) will be referred to as a deterministic strategy or decision rule. The problem then is

\[
\text{minimize } \int_X c_0(x, f(x))\nu(dx) \text{ over all } f \in F_0.
\]

Note that this simple-looking problem is in fact far from being trivial because, without suitable (restrictive) hypotheses, one can use none of the usual continuity-compactness arguments for optimization problems.

∗Received by the editors March 17, 2004; accepted for publication August 23, 2004; published electronically July 26, 2005. This research was partially supported by Consejo Nacional de Ciencia y Tecnología (CONACYT) grant 37355-E.
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Young’s relaxation technique to solve (1.1) consists basically of two steps. The first one is to replace each control or decision function \( f \in F_0 \) with the Dirac measure (or “unit mass”) \( \delta_{f(x)}(\cdot) \) concentrated at \( f(x) \) for each \( x \in X \) (i.e., \( \delta_{f(x)}(D) := 1 \) if \( f(x) \) is in \( D \), and 0 otherwise, for all \( D \in \mathcal{B}(A) \)). Actually, the set of all these Dirac measures is not sufficiently large and so, more precisely, we have to replace \( F_0 \) with the set \( \Phi_0 \) defined as follows.

**Definition 1.1.** A randomized strategy \( \varphi \) from \( X \) to \( A \) (or on \( A \) given \( X \)) is a mapping \((x, D) \mapsto \varphi(D|x)\) from \( X \times \mathcal{B}(A) \) to the interval \([0, 1]\) such that

(a) \( \varphi(\cdot|x) \) is a p.m. on \( \mathcal{B}(A) \) for every fixed \( x \in X \), and

(b) \( \varphi(D|\cdot) \) is a measurable function on \( X \) for every fixed \( D \in \mathcal{B}(A) \).

We shall denote by \( \Phi_0 \) the set of all these randomized strategies.

**Remark.** For the reader unfamiliar with “deterministic” and “randomized” decision strategies, we mention that a deterministic strategy \( f \in F_0 \) prescribes the action \( f(x) \in A \) whenever the state of nature is \( x \in X \), whereas for a randomized strategy \( \varphi \in \Phi_0 \) the corresponding action is an \( A \)-valued random variable with probability distribution \( \varphi(\cdot|x) \) for each \( x \in X \).

Given a deterministic strategy \( f \in F_0 \), we may identify each \( f(x) \in A \) with the Dirac measure \( \delta_{f(x)}(\cdot) \), which of course defines a randomized strategy. With this identification we may then write

\[ \mathbb{F}_0 \subset \Phi_0. \]

Moreover, defining

\[ c_0(x, \varphi(x)) := \int_A c_0(x, a)\varphi(da|x) \quad \forall \varphi \in \Phi_0, \ x \in X, \]

the relaxation of problem (1.1) can be expressed as

\[ \minimize \int_X c_0(x, \varphi(x))\nu(dx) \quad \text{over all} \ \varphi \in \Phi_0. \]

We may now note that (1.4) is a convex problem because it is evident that the mapping

\[ \varphi \mapsto \int_X c_0(x, \varphi(x))\nu(dx) \]

is linear on the convex set \( \Phi_0 \). We have thus transformed, or “relaxed”, (1.1) into the convex problem (1.4), which completes the first step in Young’s relaxation technique.

In the second (and final) step, which is the difficult one, we have to show that (under some assumptions)

(a) problem (1.4) is solvable and that

(b) the existence of optimal solutions to (1.4) yields the existence of optimal solutions (in \( \mathbb{F}_0 \)) to the original problem (1.1).

These are the questions we are concerned with in this paper. Part (a) is more or less standard. On the other hand, if (a) holds, we then obtain (b) as follows: we show that (1.4) has an optimal solution that is an extreme point of \( \Phi_0 \), which, therefore, is in \( \mathbb{F}_0 \) because it turns out that \( \text{ex}(\Phi_0) \), the set of extreme points of \( \Phi_0 \), coincides with \( \mathbb{F}_0 \). See Lemma 3.3 for a more general version of the latter statement. (In passing, we note that the problem of minimizing the so-called Bayes risk [29] can be expressed as in (1.4). In this case, \( c_0 \) is called a “loss” function rather than a “cost.”)
Here, we are in fact interested in a constrained version of (1.1) which is essentially the typical allocation problem in economics [6, 41], namely,

\[
(1.6) \quad \text{minimize } \int_X c_0(x, f(x))\nu(dx)
\]

over the set of deterministic strategies \( f \in F_0 \) for which

\[
(1.7) \quad \int_X c_i(x, f(x))\nu(dx) \leq b_i \quad \forall i = 1, \ldots, n,
\]

where \( c_1, \ldots, c_n \) are given cost functions on \( X \times A \), and \( b_1, \ldots, b_n \) are given numbers. The corresponding relaxation is obtained as in (1.3), (1.4), replacing \( f \in F_0 \) with \( \varphi \in \Phi_0 \); in particular, (1.7) becomes

\[
(1.8) \quad \int_X c_i(x, \varphi(x))\nu(dx) \leq b_i \quad \forall i = 1, \ldots, n, \quad \varphi \in \Phi_0.
\]

**Contributions of this paper.** The remainder of this paper is divided in two main parts. The first one, which consists of sections 2 and 3, concerns the characterization of the extreme points of the family, call it \( \Delta \), of randomized strategies \( \varphi \in \Phi_0 \) that satisfy (1.8). The main result in this part, Theorem 2.6, gives conditions under which a randomized strategy \( \varphi \) in \( \text{ex}(\Delta) \) is a randomization of at most \( n+1 \) functions \( f_j \) in \( F_0 \); i.e., there is a positive integer \( m \leq n+1 \) such that

\[
(1.9) \quad \varphi(\cdot| x) = \sum_{j=1}^{m} \alpha_j \delta_{f_j(x)}(\cdot) \quad \forall x \in X,
\]

where \( \alpha_1, \ldots, \alpha_m \) are positive numbers with \( \alpha_1 + \cdots + \alpha_m = 1 \).

The second part of the paper, sections 4, 5, and 6, deals with applications of Theorem 2.6. In section 4 we study the unconstrained problems (1.1) and (1.4), in which one can already appreciate the main technical tools to obtain (a) and (b) above. For instance, to get (a) it all boils down to reducing (1.4) to a problem of minimizing a lower semicontinuous (l.s.c.) function (see (4.22) or (4.23)) on a (relatively) compact set of p.m.’s. On the other hand, to get (b) one can either use an “elimination of randomization” result (Lemmas 4.8 and 4.9) or directly analyze the set of optimal solutions for (1.4) (see Theorem 4.12). Essentially the same approach and tools are used in section 5 to study the constrained problem (1.6), (1.7) and its relaxation.

In section 6 we turn our attention to a class of constrained stochastic control problems called Markov control processes (MCPs). It turns out that a constrained MCP can be transformed into an optimization problem with an objective function as in (1.4), with constraints of the form (1.8) together with an additional, suitable, constraint (see (6.7)). We then use the approach briefly described in the previous paragraph to show, under very mild assumptions, the existence of optimal randomized strategies of the form (1.9).

**Related literature.** Our Theorem 2.6 extends to randomized strategies a result by Winkler [40, Theorem 2.1], which in turn is an extension of Karr’s [24, Theorem 2.1]. Winkler considers a set \( \overline{P} \) of p.m.’s on a measurable space \( (S, \mathcal{S}) \), and the subset \( P_0 \) of p.m.’s \( \mu \) in \( P \) for which

\[
(1.10) \quad \int_S |g_i| \, d\mu < \infty \quad \text{and} \quad \int_S g_i \, d\mu \leq b_i \quad \forall i = 1, \ldots, n,
\]
where \( g_1, \ldots, g_n \) are given measurable functions on \( S \), and \( b_1, \ldots, b_n \) are fixed real numbers. He then proceeds to characterize the extreme points of \( P \) assuming that \( P \) is a (Choquet-) simplex of p.m.’s whose extreme points are Dirac measures. Here, we show that the set \( \Phi_0 \) of randomized strategies is a simplex whose set of extreme points is \( \mathbb{P}_0 \) (Lemma 3.3), and so specializing our Theorem 2.6 to “constant” kernels \( \varphi(\cdot | x) \equiv \mu(\cdot) \) we obtain Winkler’s, and hence Karr’s, main results. For applications of Winkler’s Theorem 2.1 to problems in optimal control and decision analysis see [8] and [36], respectively. The latter reference, and also (e.g.) [11, 26], considers problems in which, for some given function \( g_0 \) on \( S \), one wishes to optimize integrals \( \int g_0 \, d\mu \) over all \( \mu \in P_0 \). These problems include versions of the well-known “Markov moment problem” [26].

On the other hand, the allocation problem (1.6), (1.7) is well known in the economics literature—see [6, 41], for instance. Our approach here, via its Young relaxation and Theorem 2.6, is new.

Finally, our main result (Theorem 6.2) on constrained MCPs extends some facts for MCPs with a countable state space \( X \) [3, 14]. In this paper, the state and control (or action) spaces \( X \) and \( A \) are both Borel spaces (a Borel space is a Borel subset of a Polish—i.e., a complete and separable metric—space) and, moreover, all of the cost functions \( c_0, \ldots, c_n \) can be unbounded. Hence our Theorem 6.2 also extends recent results in, e.g., [28, Theorem 8] in which \( X \) is a Borel space, but the control set \( A \) is required to be compact, the constraint functions \( c_1, \ldots, c_n \) are bounded, and, in addition, either \( c_0 \) is bounded or a Slater condition is satisfied. Our hypotheses are considerably weaker (see Assumption 6.1). For results related to our Theorem 6.2 see [12, 13].

We conclude in section 7 with some comments on related open problems.

2. Constrained sets of randomized strategies. In many applications—e.g., in optimal control and game theory—the feasible actions \( a \in A \) depend on the current state \( x \in X \); that is, \( a \) is restricted to a subset \( A(x) \) of \( A \). Therefore, in most of what follows, instead of the sets \( \mathbb{F}_0 \) and \( \Phi_0 \) in section 1 we shall consider subsets \( \mathbb{F} \subset \mathbb{F}_0 \) and \( \Phi \subset \Phi_0 \) defined below. Moreover, we need \( \Phi \) to be a convex set in a suitable linear space, and so we will also consider signed transition measures or “signed kernels” (as in [20, section 7.2] or [35], for instance).

Let \( X \) and \( A \) be topological spaces with Borel \( \sigma \)-algebras \( \mathcal{B}(X) \) and \( \mathcal{B}(A) \), respectively. For each \( x \in X \), we denote by \( A(x) \in \mathcal{B}(A) \) the (nonempty) set of feasible actions in \( x \). We shall assume that the set

\[
(2.1) \quad \mathbb{K} := \{(x,a)|x \in X, a \in A(x)\}
\]

of feasible state-action pairs is a closed (hence a Borel) subset of \( X \times A \), and that, in addition, \( \mathbb{K} \) contains the graph of a measurable function from \( X \) to \( A \). (The latter condition ensures that the set \( \mathbb{F} \) in Definition 2.2, below, is nonempty.) Observe that if \( A(x) = A \) for all \( x \in X \), then \( \mathbb{K} = X \times A \).

**Definition 2.1.** A randomized strategy \( \varphi \) (restricted to \( \mathbb{K} \)) is a mapping \( (x,D) \mapsto \varphi(D|x) \) from \( X \times \mathcal{B}(A) \) to the interval \([0,1] \) such that

(a) for every fixed \( x \in X \), \( \varphi(\cdot | x) \) is a p.m. on \( \mathcal{B}(A) \) concentrated on \( A(x) \), i.e.,

\[
(2.2) \quad \varphi(A(x)^c|x) = 0 \quad \forall x \in X,
\]

where \( A(x)^c \) denotes the complement of \( A(x) \); and

(b) \( \varphi(D|\cdot) \) is a measurable function on \( X \) for every fixed \( D \in \mathcal{B}(X) \).

We shall denote by \( \Phi \) the family of randomized strategies (restricted to \( \mathbb{K} \)).
We will also consider the convex cone $\mathcal{D}_+ := \mathbb{R}_+ \cdot \Phi$ of so-called \textit{transition measures} (restricted to $\mathbb{K}$), and the linear space $\mathcal{D} := \mathcal{D}_+ - \mathcal{D}_+$ (with the obvious definitions of sum and scalar multiplication) of \textit{signed} transition measures.

**Definition 2.2.** $\mathcal{F}$ denotes the family of measurable functions $f$ from $X$ to $A$ such that $f(x)$ is in $A(x)$ for all $x \in X$. The functions in $\mathcal{F}$ are called deterministic strategies from $X$ to $A$ (restricted to $\mathbb{K}$). (A function in $\mathcal{F}$ is sometimes called a selector of the set-valued mapping $x \mapsto A(x)$.)

If $A(x) = A$ for all $x \in X$, then $\mathbb{K} = X \times A$ and the sets $\Phi$ and $\mathcal{F}$ coincide, respectively, with the sets $\Phi_0$ and $\mathbb{F}_0$ in section 1. Further, as in (1.2), identifying $f(x) \in A(x)$ with $\varphi(\cdot | x) := \delta_{f(x)}(\cdot)$, we have

\begin{equation}
\mathcal{F} \subset \Phi.
\end{equation}

Incidentally, if $A(x) = A$ for all $x \in X$ and $A = X$, then a randomized strategy is usually called a \textit{Markov transition probability}.

**Definition 2.3.** A randomized strategy $\varphi \in \Phi$ is said to be a \textit{randomization} of at most $n + 1$ deterministic strategies, or simply an $(n + 1)$-randomization, if there exist a positive integer $m \leq n + 1$, functions $f_1, \ldots, f_m$ in $\mathcal{F}$, and positive numbers $\alpha_1, \ldots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = 1$ for which

\begin{equation}
\varphi(D | x) = \sum_{j=1}^{m} \alpha_j \delta_{f_j(x)}(D) \quad \forall D \in \mathcal{B}(A), \ x \in X.
\end{equation}

We denote by $\mathcal{R}_{n+1}$ the family of $(n + 1)$-randomizations.

We need the following regularity concepts that are briefly discussed in Remarks 2.5 and 3.4. (For further details see, e.g., Topsoe [37, section 1] or Vakhania, Tarieladze and Chobanyan [38, section I.3].)

**Definition 2.4.** Let $\mu$ be a finite (nonnegative) measure on $\mathcal{B}(A)$. Then $\mu$ is said to be

(a) regular if $\mu(D) = \sup\{\mu(F) | F \subset D \text{ is closed}\}$ for each Borel set $D \in \mathcal{B}(A)$.

(b) $\tau$-smooth if for every decreasing net $\{F_\alpha\}$ of closed subsets of $A$ we have

$\mu(\cap_\alpha F_\alpha) = \inf_\alpha \mu(F_\alpha)$.

**Remark 2.5.** (a) If $A$ is a Hausdorff (or $T_2$) space, then every Radon measure on $A$ is $\tau$-smooth, and if $A$ is regular (or $T_3$), then every $\tau$-smooth measure is regular (see, e.g., Proposition I.3.1 in [38]).

(b) If $A$ is strongly Lindelöf (which is the case, e.g., if $A$ is a Suslin space—see Schwartz [34, p. 104]), then every finite measure on $\mathcal{B}(A)$ is $\tau$-smooth. The latter fact and (a) yield, in particular, the following.

(c) As an example, if $A$ is a locally compact and separable metric space, then each p.m. on $\mathcal{B}(A)$ is both $\tau$-smooth and regular. (See also Remark 3.4.)

We can finally state our first main result, which is proved in section 3, and applied to optimization and control problems in sections 4, 5, 6. We will use the notation (1.3) replacing $\Phi_0$ and $A$ with $\Phi$ and $A(x)$, respectively. Concerning the hypothesis on the space $A$ in the following theorem see Remark 3.4.

**Theorem 2.6.** Let $X$ be an arbitrary topological space, and $A$ a topological space such that every p.m. on $\mathcal{B}(A)$ is $\tau$-smooth and regular. Let $\mathbb{K}$ be as in (2.1). Fix an arbitrary p.m. $\nu$ on $\mathcal{B}(X)$, real-valued measurable functions $c_1, \ldots, c_n$ on $\mathbb{K}$, and real numbers $b_1, \ldots, b_n$. Consider the set $\Delta \subset \Phi$ that consists of all the randomized
strategies \( \varphi \in \Phi \) for which

\[
(2.5) \quad \int_X \int_{A(x)} |c_i(x,a)| \varphi(da|x) \nu(dx) < \infty \quad \forall i = 1, \ldots, n
\]

and

\[
(2.6) \quad \int_X c_i(x, \varphi(x)) \nu(dx) \leq b_i \quad \forall i = 1, \ldots, n.
\]

Let \( \text{ex}(\Delta) \) be the set of extreme points of \( \Delta \). Then

(a) \( \Delta \) is convex and

\[
(2.7) \quad \text{ex}(\Delta) \subset \mathbb{R}^0_{n+1},
\]

where \( \mathbb{R}^0_{n+1} \) is the set of all the \((n+1)\)-randomizations \( \varphi \in \mathbb{R}_{n+1} \) as in (2.4) for which the vectors

\[
(2.8) \quad \left( \int c_1(x, f_j(x)) \nu(dx), \ldots, \int c_n(x, f_j(x)) \nu(dx), 1 \right) \in \mathbb{R}^{n+1}
\]

for \( j = 1, \ldots, m \) are linearly independent.

(b) If equality holds in (2.6), then the sets in (2.7) are equal.

Remark 2.7. A randomized strategy \( \varphi \) in \( \Phi \) (or in \( \Phi_0 \)) is said to be constant if there is a p.m. \( \mu \) on \( A \) such that \( \varphi(\cdot|x) \equiv \mu(\cdot) \) for all \( x \in X \). Now let \( g_1, \ldots, g_n \) be given measurable functions on \( A \), and suppose that the functions \( c_i \) in (2.5), (2.6) are of the form \( c_i(x, a) := g_i(a) \) for all \( i = 1, \ldots, n \). Then comparing (2.5), (2.6) with (1.10), we can view Winkler’s [40, Theorem 2.1] as a special case of our Theorem 2.6 restricted to constant kernels, and similarly for Theorem 2.1 in Karr [24].

3. Proof of Theorem 2.6. We begin by noting the following.

Remark 3.1. The convex set \( \Phi \) of randomized strategies from \( X \) to \( A \) (restricted to \( K \)) is the base of the convex cone \( \mathcal{D}_+ = \mathbb{R}_+ \cdot \Phi \); that is, for each \( q \in \mathcal{D}_+ \) there exist a unique \( \alpha \geq 0 \) and \( \varphi \in \Phi \) such that \( q = \alpha \varphi \). On the other hand, the linear space \( \mathcal{D} = \mathcal{D}_+ - \mathcal{D}_+ \) with the partial order induced by \( \mathcal{D}_+ \) (i.e., \( q_1 \geq q_2 \) if and only if \( q_1 - q_2 \) is in \( \mathcal{D}_+ \)) is a vector lattice.

Definition 3.2. (see Phelps [31, p. 59]) Let \( S \) be a convex subset of a real linear space \( L \), and suppose that \( S \) is contained in a closed hyperplane that misses the origin. Then \( S \) is called a (Choquet-)simplex if it is the base of a cone \( \tilde{S} \) such that the space \( \tilde{S} - \tilde{S} \) is a vector lattice in the ordering induced by \( \tilde{S} \).

The above definition of a simplex coincides with the usual one in case \( S \) is finite dimensional. (See [31, Proposition 9.11].)

Lemma 3.3. The convex set \( \Phi \) is a simplex, and the set of its extreme points coincides with \( \mathcal{F} \), i.e.,

\[
(3.1) \quad \text{ex}(\Phi) = \mathcal{F}.
\]

Proof. The first statement follows from Remark 3.1 (and Definition 3.2). To prove (3.1) we will show in fact that

\[
(3.2) \quad \mathcal{F} = \Phi^d = \text{ex}(\Phi),
\]
where $\Phi^d \subset \Phi$ is the set of randomized strategies $\varphi \in \Phi$ such that $\varphi(\cdot \mid x)$ is a Dirac measure for each $x \in X$. Clearly $F \subset \Phi^d \subset ex(\Phi)$, and so to get (3.2) it remains to show that

$$ex(\Phi) \subset \Phi^d \subset F.$$  

To this end, suppose first that $ex(\Phi)$ is not contained in $\Phi^d$, and let $\varphi$ be a randomized strategy in $ex(\Phi)$ that is not in $\Phi^d$. Then, by the hypothesis on $A$ and Theorem 11.1 in Topsoe [37], there exists $z \in X$ such that $\varphi(\cdot \mid z)$ is not a Dirac measure; that is, there is a set $C$ in $B(A)$ with $0 < \varphi(C \mid z) < 1$. Now let $\beta := \varphi(C \mid z)$, and for each $x \in X$ and $D \in B(A)$ define

$$\varphi_1(D \mid x) := \varphi(D \mid x) \text{ if } x \neq z$$

and

$$\varphi_2(D \mid x) := \varphi(D \mid x) / \beta \text{ if } x = z,$$

where $C^c$ stands for the complement of $C$. We have thus stochastic kernels $\varphi_1$ and $\varphi_2$ such that $\varphi = \beta \varphi_1 + (1 - \beta) \varphi_2$, and, therefore, $\varphi$ is not an extreme point of $\Phi$. This contradiction yields the first relation in (3.3).

To prove the second relation in (3.3) choose an arbitrary $\varphi$ in $\Phi^d$. It follows that for each $x \in X$ there exists $a \in A$ such that $\varphi(D \mid x) = \delta_a(D)$ for all $D \in B(X)$ and, moreover, by (2.2), $a$ is in $A(x)$. Writing $a$ as $f(x)$ we thus obtain a function $f$ from $X$ to $A$, with $f(x)$ in $A(x)$ for each $x \in X$, and so to prove that $f$ is in $F$ we need to show that $f$ is measurable. This, however, is obvious because by part (b) of Definition 2.1, for any set $D \in B(A)$

$$f^{-1}(D) = \{x \in X \mid \varphi(D \mid x) = 1\} = \varphi^{-1}(D \mid \cdot)(1)$$

is in $B(X)$. Hence, as $\varphi \in \Phi^d$ was arbitrary, we conclude that $\Phi^d \subset F$. This completes the proof of (3.3), and (3.1) follows.

Completion of the proof. Having Lemma 3.3, the proof of Theorem 2.6 can now be completed exactly as the proof of Winkler’s [40, Theorem 2.1], with the appropriate changes. Namely, we define

- a convex set $C \subset \Phi$ as the collection of randomized strategies $\varphi \in \Phi$ that satisfy (2.5);
- an affine map $J : C \to \mathbb{R}^n$ as $J(\varphi) = (J_1(\varphi), \ldots, J_n(\varphi))$ with components

$$J_i(\varphi) := \int_X c_i(x, \varphi(x)) \nu(dx) \text{ for } i = 1, \ldots, n;$$

and a convex set $W \subset J(C)$ as

$$W := J(C) \cap \prod_{i=1}^n (-\infty, b_i] \text{ for part (a),}$$

$$W := J(C) \cap \{(b_1, \ldots, b_n)\} \text{ for part (b).}$$
Moreover, we note that if $\varphi - \varphi'$ is in the cone $\mathbb{R}_+ \cdot C$, with $\varphi$ in $\mathbb{R}_+ \cdot C$ and $\varphi'$ in $\mathcal{D}_+ = \mathbb{R}_+ \cdot \Phi$, then
\[
\int \int |c_i(x,a)| \varphi' (da|x) \nu(dx) \leq \int \int |c_i(x,a)| \varphi (da|x) \nu(dx) < \infty
\]
for each $i = 1, \ldots, n$, and so $\varphi'$ is in $\mathbb{R}_+ \cdot C$; in other words, $\mathbb{R}_+ \cdot C$ is a hereditary subcone of $\mathcal{D}_+$. This implies (by Lemma 9.4 in [31]) that $\mathbb{R}_+ \cdot C$ is a lattice cone in its own order, which in turn (by the main theorem and paragraph 4 (p. 369) in Kendall [25]) yields that $C$ is linearly compact. Therefore, the conclusions (a) and (b) in Theorem 2.6 follow from Proposition 2.1 in Winkler [40]. (The number 1 that appears in the vectors (2.8) ensures linear independence rather than the affine independence in [40, Proposition 2.1]; see also Karr [24, Theorem 2.1].)

Remark 3.4. (a) Theorem 2.6 requires the action set $A$ to be a topological space such that
\[
\text{every p.m. on } \mathcal{B}(A) \text{ is } \tau\text{-smooth and regular.}
\]
This condition ensures that, as in the proof of Lemma 3.3, the set of extreme points of $\mathbb{P}(A)$, the space of p.m.'s on $A$, coincides with the set of Dirac measures $\delta_a$ for all $a \in A$ [37, Theorem 11.1]. Without (3.7) we can only guarantee that the extreme points of $\mathbb{P}(A)$ are zero-one p.m.'s [1], that is, nontrivial p.m.'s $\mu$ such that $\mu(B) = 0$ or 1 for all $B \in \mathcal{B}(A)$. For examples of zero-one p.m.'s that are not Dirac measures see [15] or the references in [1]. On the other hand, we can obtain (3.7) as follows (see also Remark 2.5).

(b) A p.m. $\mu$ on $A$ is said to be tight if for every $\varepsilon > 0$ there exists a compact subset $K$ of $A$ such that $\mu(K) \geq 1 - \varepsilon$. Moreover, if a p.m. is tight, then it is $\tau$-smooth and regular (see p. xiii in [37]). It follows that to obtain (3.7) it suffices to give conditions on $A$ so that every p.m. on $\mathcal{B}(A)$ is tight. This is the case if, for instance, $A$ is (i) a $\sigma$-compact Hausdorff space, or (ii) a Polish space, or (iii) a locally compact separable metric space. (See [2, 29, 34, 37].)

4. The unconstrained optimization problems. In this section we consider the unconstrained problem (1.1) and its Young relaxation (1.4) but replacing $F_0$ and $\Phi_0$ with the sets $\mathbb{F}$ and $\Phi$ in Definitions 2.2 and 2.1, respectively. For notational ease, let (as in (3.6))
\[
J_0(\varphi) := \int_X c_0(x, \varphi(x)) \nu(dx) \quad \forall \varphi \in \Phi \supset \mathbb{F}, \; x \in X,
\]
where (as in (1.3))
\[
c_0(x, \varphi(x)) := \int_{A(x)} c_0(x,a) \varphi(da|x).
\]
Thus the problems addressed in this section are (using the abbreviation “s.t.” for “subject to”)
\[
\begin{align*}
\text{(4.3)} \quad \text{minimize } \quad J_0(f) & \quad \text{s.t. } f \in \mathbb{F}, \\
\text{(4.4)} \quad \text{minimize } \quad J_0(\varphi) & \quad \text{s.t. } \varphi \in \Phi.
\end{align*}
\]
If $A(x) = A$ for all $x \in X$, then (4.3), (4.4) reduce to the problems (1.1), (1.4).
To fix ideas we shall impose the following assumptions but in Remark 4.10 we mention other possibilities.

**Assumption 4.1.** (a) The spaces $X$ and $A$ are both Borel spaces. (Recall that the set $K$ in (2.1) is supposed to be closed.)

(b) $c_0$ is nonnegative.

(c) $c_0$ is inf-compact; i.e., the level set $\{(x, a) \in K | c_0(x, a) \leq r\}$ is compact for each $r \in \mathbb{R}$.

Let $\rho^*$ and $\rho^\#$ be the values of problems (4.3) and (4.4), respectively; that is,

$$\rho^* := \inf \{ J_0(f) | f \in \mathcal{F} \}$$

and similarly for $\rho^\#$. By (2.3),

$$\rho^\# \leq \rho^*.$$  

The next theorem shows, in particular, that the equality holds in (4.6).

**Theorem 4.2.** Under Assumption 4.1,

(a) the unconstrained relaxed problem (4.4) is solvable; i.e., there exists $\varphi^*$ in $\Phi$ such that

$$\rho^\# = J_0(\varphi^*) = \int_X c_0(x, \varphi^*(x)) \nu(dx);$$

(b) (4.3) is also solvable and, moreover, $\rho^* = \rho^\#$.

Before proving Theorem 4.2 we will recall some well-known concepts and facts from probability theory, which are stated here for completeness and ease of reference.

**Definition 4.3.** Let $Y$ be a metric space, and $\mathcal{P}(Y)$ the family of p.m.'s on $Y$. A subset $M$ of $\mathcal{P}(Y)$ is said to be

(a) tight if for each $\epsilon > 0$ there is a compact set $K = K_\epsilon$ in $Y$ such that

$$\inf_{\mu \in M} \mu(K) \geq 1 - \epsilon;$$

(b) relatively compact if for each sequence $\{\mu_n\}$ in $M$ there is a subsequence $\{\mu_m\}$ of $\{\mu_n\}$ and a p.m. $\mu$ on $Y$ (but not necessarily in $M$) such that $\mu_m$ converges weakly to $\mu$, i.e.,

$$\int_Y u \, d\mu_m \to \int_Y u \, d\mu \quad \forall u \in C_b(Y),$$

where $C_b(Y)$ denotes the space of continuous bounded functions on $Y$.

Throughout the following $\mathcal{P}(Y)$ is endowed with the topology of weak convergence (as in (4.8)). It follows in particular that if $Y$ is a Borel space, then so is $\mathcal{P}(Y)$.

By Prohorov’s Theorem [9, 32, 35],

$$\text{tightness } \Rightarrow \text{ relative compactness.}$$

(The converse of (4.9) holds if $Y$ is a Polish space.) The Assumption 4.1—in particular the inf-compactness condition in part (c)—will be used in combination with the following lemma.

**Lemma 4.4.** Let $Y$ and $M \subset \mathcal{P}(Y)$ be as in Definition 4.3. If there exists an inf-compact function $u : Y \to \mathbb{R}_+$ such that

$$\sup_{\mu \in M} \int_Y u \, d\mu < \infty,$$
then $M$ is tight, and hence (by (4.9)) relatively compact.

Lemma 4.4 is a direct consequence of the definitions of inf-compactness and tightness. (Again, the converse of the first statement in Lemma 4.4 holds under suitable assumptions—e.g., if the metric space $Y$ is $\sigma$-compact.)

**Lemma 4.5.** Let $Y$ be a metric space, and $u : Y \to \mathbb{R}$ l.s.c. and bounded below. If a sequence $\{\mu_n\}$ in $P(Y)$ converges weakly to $\mu$ (i.e., as in (4.8)), then

$$\liminf_{n \to \infty} \int_Y u \, d\mu_n \geq \int_Y u \, d\mu.$$  

Lemma 4.5 is well known; see, e.g., statement (12.3.37) in [20].

If $\mu$ is a p.m. on $X \times A$, we shall denote by $\hat{\mu}$ its marginal (or projection) on $X$, i.e., $\hat{\mu}(B) := \mu(B \times A)$ for all $B$ in $B(X)$.

**Lemma 4.6 (disintegration of measures).** Let $X$, $A$, and $K$ be as in Assumption 4.1(a). If $\mu$ is a p.m. on $X \times A$ concentrated on $K$ (i.e., $\mu(K^c) = 0$), then there exists a randomized strategy $\phi \in \Phi$ such that

$$\mu(B \times D) = \int_B \varphi(D|x) \hat{\mu}(dx) \quad \forall B \in B(X), \ D \in B(A).$$  

(4.10)

For a proof of Lemma 4.6 see, e.g., Hinderer [23, p. 89]. Note, on the other hand, that if $\phi$ is in $\Phi$ and $\nu$ is a p.m. on $X$, then

$$\mu(B \times D) := \int_B \varphi(D|x) \nu(dx) \quad \forall B \in B(X), \ D \in B(A)$$  

(4.11)

defines a p.m. on $X \times A$ that (by (2.2)) is concentrated on $K$. The measures $\mu$ in (4.10) and (4.11) will be written as $\mu = \hat{\mu} \cdot \varphi$ and $\mu = \nu \cdot \varphi$, respectively. (The p.m. $\nu \cdot \varphi$ is sometimes called the $\nu$-mixture of $\varphi$ [35].)

**Definition 4.7.** $P(K)$ denotes the family of p.m.’s on $X \times A$ that are concentrated on $K$. If $\nu$ is a p.m. on $X$ and $\Phi'$ is a subfamily of randomized strategies in $\Phi$, then

$$\nu \cdot \Phi' := \{\nu \cdot \varphi | \varphi \in \Phi'\} \subset P(K).$$

Throughout the following we use a notation similar to (4.2), namely,

$$v(x, \varphi(x)) := \int_{A(x)} v(x, a) \varphi(da|x) \quad \forall \varphi \in \Phi, \ x \in X,$$

(4.12)

for any measurable (possibly extended) real-valued function $v$ for which the integral in (4.12) is well defined. Moreover, as usual, $v^- := - \min\{v, 0\} \geq 0$ stands for the “negative part” of $v$.

**Lemma 4.8 (see [23, Lemma 15.1]).** Suppose that $X$ and $A$ are as in Assumption 4.1(a). Let $\varphi$ be a randomized strategy in $\Phi$, and $v$ an extended real-valued measurable function on $K$. If the function $x \mapsto v^-(x, \varphi(x))$ on $X$ is finite-valued, then there exists a deterministic strategy $f \in \Phi$ such that

$$v(x, \varphi(x)) \geq v(x, f(x)) \quad \forall x \in X.$$  

(4.13)

In statistical decision theory, a result such as Lemma 4.8 is called an elimination of randomization theorem [7, 10, 39] because of (4.13). We will use Lemma 4.8 in the proof of Theorem 4.2, which concerns an unconstrained problem. In the constrained
case, however, we shall need a different setting, as in the following “multidimensional” elimination of randomization result.

**Lemma 4.9** (see [7, Theorem 2.3 and Corollary 2.5]). Let \((X, \mathcal{X}, \nu)\) be a probability space in which the p.m. \(\nu\) is nonatomic, and let \(A\) be a Polish space. Suppose that \(A(x) = A\) for all \(x \in X\), so that the set \(K\) in (2.1) coincides with \(X \times A\) and, moreover, \(\mathcal{F} = \mathcal{F}_0\) and \(\Phi = \Phi_0\).

(a) If the functions \(c_0, \ldots, c_n\) in (1.6)–(1.8) are nonnegative and each \(c_i(x, \cdot)\) is l.s.c. on \(A\) for each \(x \in X\), then for every \(\varphi \in \Phi_0\) there exists \(f \in \mathcal{F}_0\) such that

\[
\int_X c_i(x, \varphi(x))\nu(dx) \geq \int_X c_i(x, f(x))\nu(dx) \quad \forall i = 0, 1, \ldots, n.
\]

(b) If the functions \(c_0, \ldots, c_n\) are uniformly bounded and each \(c_i(x, \cdot)\) is continuous on \(A\) for each \(x \in X\), then for every \(\varphi \in \Phi_0\) there exists \(f \in \mathcal{F}_0\) for which (4.14) holds with equality.

**Proof of Theorem 4.2.** Theorem 4.2 is trivial if \(J_0(\varphi) = +\infty\) for all \(\varphi \in \Phi\). Hence, without loss of generality, we may—and will—assume that

\[
J_0(\tilde{\varphi}) = \int_X c_0(x, \tilde{\varphi}(x))\nu(dx) < \infty
\]

for some randomized strategy \(\tilde{\varphi}\) in \(\Phi\). Let \(\rho^\# := \inf\{J_0(\varphi) | \varphi \in \Phi\}\) be the value of problem (4.4), and let \(\{\varphi_n\} \subset \Phi\) be a minimizing sequence, i.e.,

\[
J_0(\varphi_n) \downarrow \rho^\#.
\]

Thus for each \(\epsilon > 0\) there exists \(N(\epsilon)\) such that

\[
J_0(\varphi_n) \leq \rho^\# + \epsilon \quad \forall n \geq N(\epsilon).
\]

For each \(n\), let \(\mu_n = \nu \cdot \varphi_n\) be the p.m. in \(\mathbb{P}(K)\) defined as in (4.11), i.e.,

\[
\mu_n(B \times D) := \int_B \varphi_n(D|x)\nu(dx)
\]

for all \(B \in \mathcal{B}(X), D \in \mathcal{B}(A)\), and \(n = 1, 2, \ldots\). Therefore we can express \(J_0(\varphi_n)\) as

\[
J_0(\varphi_n) = \int_K c_0(x, a)\mu_n(d(x, a)).
\]

We can now see (from (4.16), Assumption 4.1—in particular part (c)—and Lemma 4.4) that

\[
\text{the sequence } M := \{\mu_n, n \geq N(\epsilon)\} \text{ is relatively compact.}
\]

Therefore there is a subsequence \(\{\mu_m\}\) of \(M\) and a p.m. \(\mu^*\) on \(X \times A\), which is concentrated on \(K\) (by the closedness of \(K \subset X \times A\), i.e., \(\mu^*\) is in \(\mathbb{P}(K)\), such that

\[
\mu_m \to \mu^* \text{ weakly,}
\]

that is, as in (4.8). In turn (4.20) implies the following.
(i) The marginals $\hat{\mu}_m \rightharpoonup \mu^*$ weakly; hence $\mu^* = \nu$ because (by (4.17)) $\hat{\mu}_m = \nu$ for all $m$. Thus, by Lemma 4.6, there exists $\varphi^*$ in $\Phi$ such that
\begin{equation}
\mu^* = \hat{\mu}^* \cdot \varphi^* = \nu \cdot \varphi^*.
\end{equation}

(ii) As Assumptions 4.1(b), (c) yield that $c_0$ is nonnegative and l.s.c., from (4.20) and Lemma 4.5 we have that
\begin{equation}
\liminf_{n \to \infty} \int c_0 \, d\mu_m \geq \int c_0 \, d\mu^*;
\end{equation}
equivalently, by (4.21) and (4.18),
\begin{equation}
\liminf_{m \to \infty} J_0(\varphi_m) \geq J_0(\varphi^*) \geq \rho^*.
\end{equation}
This concludes the proof of part (a) because (4.23) and (4.15) give (4.7).

(b) By (4.7) and Lemma 4.8, there exists $f^*$ in $\mathcal{F}$ such that
\begin{equation}
\rho^* = \int_X c_0(x, f^*(x)) \nu(dx) = J_0(f^*) \geq \rho^*.
\end{equation}
The latter inequality and (4.6) yield that $f^* \in \mathcal{F}$ is an optimal solution for problem (4.3) and that $\rho^* = \rho^*$. This completes the proof of Theorem 4.2. \hfill \Box

Remark 4.10. It is clear that the nonnegativity of $c_0$ in Assumption 4.1(b) can be replaced with the following: $c_0$ is bounded below. It is just as clear that the inf-compactness of $c_0$ (Assumption 4.1(c)) was used in the above proof to get that (a) $c_0$ is l.s.c., which, together with (4.20), gives (4.22)–(4.23); and (b) to obtain (4.19). The condition (a) seems to be unavoidable: either we assume it holds or we get it from some stronger condition, as in our case. On the other hand, to get the relative compactness in (b) we can replace the inf-compactness of $c_0$ with, for instance, the following condition:

(b.1) There exists $\hat{\varphi}$ in $\Phi$ such that the set \{\[\mu = \nu \cdot \varphi | J_0(\varphi) \leq J_0(\hat{\varphi})\} \subset \mathcal{P}(\mathcal{K}) is tight (or just relatively compact; recall (4.9)).

Of course, a trivial case to get (b) would be

(b.2) $X$ and $A$ are both compact metric spaces,

because then the closed set $\mathcal{K} \subset X \times A$ would be compact, and so would be $\mathcal{P}(\mathcal{K})$ in the weak topology defined by (4.8).

Remark 4.10, with obvious changes, is valid for the problems addressed in sections 5 and 6.

In the following sections we use the fact that part (b) in Theorem 4.2 can be obtained from part (a) and Lemma 3.3. To show that this is indeed the case we need to introduce some terminology and a preliminary result.

Let $\mathcal{D}$ be the linear space introduced after Definition 2.1, and $M(\mathcal{K})$ the linear space of finite signed measures on $X \times A$ concentrated on $\mathcal{K}$. Consider the linear mapping $\ell : \mathcal{D} \to M(\mathcal{K})$ defined by $\ell(\varphi) := \nu \cdot \varphi$ and the quotient space $\tilde{\mathcal{D}} := \mathcal{D}/\text{Ker}(\ell)$, where $\text{Ker}(\ell) := \{\varphi \in \mathcal{D} | \ell(\varphi) = 0\}$ is the kernel of $\ell$. For each $\varphi \in \mathcal{D}$, let $\varphi^* := \{\varphi' \in \mathcal{D} | \nu \cdot \varphi' = \nu \cdot \varphi\}$ be the corresponding equivalence class in $\tilde{\mathcal{D}}$. The quotient sets $\tilde{\mathcal{F}}, \tilde{\Phi}, \tilde{\Delta}, \tilde{\Phi}_{n+1}$ are defined similarly. For instance, $\tilde{\Delta} := \{\varphi | \varphi \in \Delta\}$.

Lemma 4.11. (a) If $\nu \cdot \varphi$ is an extreme point of $\nu \cdot \Phi$ (resp., $\nu \cdot \Delta$), then $\varphi$ is an extreme point of $\tilde{\Phi}$ (resp., $\tilde{\Delta}$).
(b) If $\varphi$ is an extreme point of $\tilde{\Phi}$, then $\varphi$ has an extreme point $f$ in $\mathcal{F} \cap \varphi$. 
(c) If \( \varphi \) is an extreme point of \( \overline{\Delta} \), with \( \Delta \) as in Theorem 2.6, then \( \varphi \) has an extreme point \( \varphi^* \) of \( \Delta \) with \( \varphi^* \) in \( \overline{\mathbb{R}}_+^n \cap \varphi \).

Proof. (a) Let \( \nu \cdot \varphi \) be an extreme point of \( \nu \cdot \Phi \) and suppose that \( \varphi = \beta \varphi_1 + (1-\beta)\varphi_2 \) for some \( \beta \in (0,1) \) and \( \varphi_1, \varphi_2 \in \Phi \). It follows that \( \nu \cdot \varphi = \beta \nu \cdot \varphi_1 + (1-\beta)\nu \cdot \varphi_2 \) and by the extremality of \( \nu \cdot \varphi \) we get \( \nu \cdot \varphi_1 = \nu \cdot \varphi_2 = \nu \cdot \varphi \). Hence \( \varphi_1 = \varphi_2 = \varphi \). The same argument holds if \( \Phi \) is replaced with \( \Delta \).

(b) This part follows from Theorem 2 in [32], or Theorem 10 in [33, pp. 83–89].

(c) This is obtained exactly as the proof of Theorem 2.6 but dealing with equivalence classes \( \varphi \in \overline{\Delta} \) rather than randomized strategies \( \varphi \in \Delta \). \( \Box \)

Now let \( C \) be a convex set, and recall that an extremal subset of \( C \) is any nonempty subset \( F \) of \( C \) with the property that if \( x = \alpha y + (1-\alpha)z \) is in \( F \), where \( 0 < \alpha < 1 \) and \( y, z \in C \), then \( y, z \) are in \( F \). A convex extremal subset of \( C \) is called a face of \( C \) [2]. Finally, let

\[
(4.24) \quad \Phi^# := \{ \varphi \in \Phi \mid J_0(\varphi) = \rho^# \}
\]

be the set of optimal solutions for problem (4.4), and let \( \nu \cdot \Phi^# \subset \mathbb{F}(\mathbb{K}) \) be as in Definition 4.7, i.e., \( \nu \cdot \Phi^# := \{ \nu \cdot \varphi \mid \varphi \in \Phi^# \} \). By Theorem 4.2(a), \( \nu \cdot \Phi^# \) is nonempty. We also have the following.

**Theorem 4.12.** Under Assumption 4.1, the set \( \nu \cdot \Phi^# \) is

(a) a face of \( \nu \cdot \Phi \).

Moreover, \( \nu \cdot \Phi^# \) is

(b) relatively compact and

c) weakly closed (i.e., closed in the weak topology defined by (4.8)); hence

(d) \( \nu \cdot \Phi^# \) is weakly compact.

(e) \( \nu \cdot \Phi^# \) has an extreme point \( \nu \cdot \varphi \) with \( \varphi \in \text{ez}(\Phi) = \overline{\mathbb{F}} \); that is (by Lemma 3.3),

\( \varphi(x) = \delta_{f^#}(x) \) for some \( f^# \) in \( \overline{\mathbb{F}} \), and \( J_0(f^#) = \rho^# = \rho^* \).

Proof. Part (a) follows from the linearity of the mapping \( \varphi \mapsto J_0(\varphi) \), whereas (b) and (c), which give (d), can be obtained as in (4.19)–(4.21).

(e) By (a)–(d), \( \nu \cdot \Phi^# \) is a (weakly) compact extremal subset of \( \nu \cdot \Phi \) and, therefore, \( \nu \cdot \Phi^# \) contains an extreme point, say \( \nu \cdot \varphi \), of \( \nu \cdot \Phi \) (see Lemma 5.114 in [2]). Consequently (by Lemma 4.11(a)), \( \varphi \) is an extreme point of \( \overline{\Phi} \) and so (by Lemma 4.11(b)) there exists \( f^# \) in \( \mathbb{F} \cap \overline{\Phi} \), that is, \( \nu \cdot \varphi = \nu \cdot f^# \). Finally, as \( \rho^# = J_0(f^#) = \rho^* \), the last statement in (e) follows from (4.6). \( \Box \)

**Remark 4.13.** In the proof of Theorems 4.2 and 4.12 we have shown “indirectly” the relative compactness of some subsets of \( \nu \cdot \Phi \) via their relative compactness in \( \mathbb{F}(\mathbb{K}) \). However, there are ways of defining “directly” a topology on \( \Phi \) with respect to which sets such as \( \Phi^# \) in (4.24) are compact; see, e.g., Yushkevich [43] and the references therein. (This typically requires the sets \( A(x) \subset A \) to be compact.) Therefore, as usually it is not difficult to get that \( \varphi \mapsto J_0(\varphi) \) is l.s.c. (see (4.22), (4.23)), the last part of both Theorems 4.2 and 4.12 would be a direct consequence of Lemma 3.3 and Bauer’s extremum principle, which states the following [2]: If \( \mathbb{K} \) is a compact convex subset of a locally convex Hausdorff topological vector space, then every l.s.c. concave function on \( \mathbb{K} \) achieves its minimum at an extreme point.

5. The constrained optimization problems. We next consider the constrained optimization problems in (1.6)–(1.8) with \( \mathbb{F} \) and \( \Phi \) in lieu of \( \mathbb{F}_0 \) and \( \Phi_0 \), respectively. That is, extending the notation (4.1), (4.2) to all the cost functions \( c_0, \ldots, c_n \), we have the “deterministic” problem

\[
(5.1) \quad \text{minimize} \ J_0(f) \ \text{s.t.} \ J_i(f) \leq b_i \ \forall i = 1, \ldots, n, \ f \in \mathbb{F},
\]
and its Young relaxation

\begin{equation}
\minimize J_0(\varphi) \text{ s.t. } J_i(\varphi) \leq b_i \quad \forall i = 1, \ldots, n, \varphi \in \Phi.
\end{equation}

In the remainder of this section we suppose the following.

**Assumption 5.1.**
(a) \(X\) and \(A\) are Borel spaces (and \(\mathbb{K} \subset X \times A\) is closed).
(b) The functions \(c_i : \mathbb{K} \to \mathbb{R} \quad (i = 0, 1, \ldots, n)\) are nonnegative and l.s.c.
(c) At least one of the functions \(c_0, \ldots, c_n\) is inf-compact.
(d) There exists \(\varphi^*_0\) in \(\Phi\) such that \(J_0(\varphi^*_0) < \infty\) and \(J_i(\varphi) \leq b_i\) for all \(i = 1, \ldots, n\).

Let \(\rho^*_c\) and \(\rho^\#_c\) be the values of the problems (5.1) and (5.2), i.e.,

\[ \rho^*_c := \inf \{ J_0(f) | J_i(f) \leq b_i \quad \forall i = 1, \ldots, n ; \ f \in \mathcal{F} \}, \]

and similarly for \(\rho^\#_c\). As in (4.6), in the present case we also have (by (2.3))

\begin{equation}
\rho^\#_c \leq \rho^*_c.
\end{equation}

**Theorem 5.2.** Under Assumption 5.1, we have the following.

(a) The constrained problem (5.2) is solvable; i.e., there exists \(\varphi^*\) in \(\Phi\) such that

\begin{equation}
J_0(\varphi^*) = \rho^\#_c
\end{equation}

and

\begin{equation}
J_i(\varphi^*) \leq b_i \quad \forall i = 1, \ldots, n.
\end{equation}

Let \(\Delta^* := \{ \varphi \in \Phi | \varphi \text{ satisfies (5.4), (5.5)} \}\) be the set of optimal solutions for problem (5.2).

(b) If in addition \(A\) is a locally compact and separable metric space, then there exists \(\varphi^* \in \Delta^*\) that is also in \(\mathcal{R}_{n+1}^0\), the set of \((n+1)\)-randomizations in Theorem 2.6(a).

(c) Suppose that in addition to Assumption 5.1 we have

(c.1) The p.m. \(\nu\) is nonatomic; and

(c.2) \(A\) is a Polish space and \(A(x) = A\) for all \(x \in X\).

Then the (deterministic) constrained problem (5.1) is solvable, and \(\rho^*_c = \rho^\#_c\).

**Proof.** (a) By Assumption 5.1(d), problem (5.2) is feasible. Let \(\{\varphi_k\} \subset \Phi\) be a minimizing sequence for (5.2); that is, each \(\varphi_k\) satisfies the constraints

\begin{equation}
J_i(\varphi_k) \leq b_i \quad \forall i = 1, \ldots, n
\end{equation}

and

\begin{equation}
J_0(\varphi_k) \downarrow \rho^\#_c \quad \text{as } k \to \infty.
\end{equation}

By Assumption 5.1(c), at least one of the nonnegative functions \(c_0, \ldots, c_n\) is inf-compact, which, together with (5.6), (5.7) and Lemma 4.4, yields that the sequence \(\{\mu_k := \nu \cdot \varphi_k, k \geq N\}\) is relatively compact for some \(N\) sufficiently large. Observe that the latter statement is analogous to (4.19); then the remainder of the proof follows exactly the same arguments used in (4.20)–(4.23).

(b) Let \(\Delta \subset \Phi\) be as in Theorem 2.6. As the cost functions \(c_1, \ldots, c_n\) are nonnegative (Assumption 5.1(b)), \(\Delta\) coincides with the set of randomized strategies \(\varphi \in \Phi\) that satisfy (2.6), which is the same as the set of \(\varphi\)'s that satisfy the constraints \(J_i(\varphi) \leq b_i\) in (5.2). Let \(\Delta^* \subset \Delta\) be as in part (a) and consider the set \(\nu \cdot \Delta^*\); see
Definition 4.7. Then (as in Theorem 4.12) it is easily seen that \( \nu \cdot \Delta^\ast \) is a weakly compact face of \( \nu \cdot \Delta \), and so (as in the proof of Theorem 4.12(e)) \( \nu \cdot \Delta^\ast \) contains an extreme point \( \nu \cdot \varphi \), say, of \( \nu \cdot \Delta \). Hence, by Lemma 4.11(a), \( \varphi \) is an extreme point of \( \Delta \). Finally, by the hypothesis on \( A \) and Remark 2.5(c) (or Remark 3.4(b)), the desired conclusion follows from Lemma 4.11(c).

(c) Under Assumptions 5.1, (c.1) and (c.2), the hypotheses of Lemma 4.9 are satisfied. Consequently if \( \varphi^\ast \in \Phi \) is as in (5.4) and (5.5), then there exists \( f^\ast \in F \) such that

\[
\rho^\ast d \geq \int_X c_0(x, f^\ast(x)) \nu(dx) = J_0(f^\ast) \geq \rho^\ast c
\]

and, similarly,

\[
b_i \geq J_i(\varphi^\ast) \geq J_i(f^\ast) \quad \forall i = 1, \ldots, n.
\]

These inequalities and (5.3) yield (c).

6. Constrained MCPs. In this section we consider constrained MCPs with discounted cost criteria, but it will be clear from the following discussion that similar results can be obtained for other cost criteria, e.g., average cost, total cost, etc. On the other hand, as constrained MCPs are quite standard, we shall introduce only the minimum necessary material required to state the corresponding control problem and its connection with the results in sections 2 and 5. (For further details, if necessary, the reader may consult, e.g., [3, 14, 16, 17, 27, 28, 32, 33].)

In a constrained MCP we are given, first, a control model

\[
(X, A, K, Q, c_0, \ldots, c_n, b_1, \ldots, b_n)
\]

with \( X, A, K, \) and \( c_i \) (\( i = 0, \ldots, n \)) as in Assumption 5.1(a), (b), (c), and \( b_i \geq 0 \) (\( i = 1, \ldots, n \)). Thus with respect to the previous sections the only new component in (6.1) is the so-called system’s transition law \( Q \), which is a stochastic kernel from \( K \) to \( X \), i.e.,

- \( Q(B|\cdot) \) is a measurable function on \( K \) for each fixed \( B \in \mathcal{B}(X) \), and
- \( Q(\cdot|x, a) \) is a p.m. on \( X \) for each fixed pair \((x, a) \in K\).

Denoting by \( x_t \) and \( a_t \) (\( t = 0, 1, \ldots \)) the state and control variables, the transition law \( Q \) is interpreted as the transition probability

\[
Q(B|x, a) = \text{Prob}(x_{t+1} \in B \mid x_t = x, a_t = a) \quad \forall B \in \mathcal{B}(X), (x, a) \in K.
\]

**Control policies.** A control policy is a sequence \( \pi = \{\pi_t\} \) of stochastic kernels \( \pi_t \) that satisfy the constraints

\[
\pi_t(A(x)|x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t = x) = 1
\]

for all \( t = 0, 1, \ldots \), with \((x_i, a_i) \) in \( K \) for all \( i = 0, \ldots, t-1 \). Let \( \Pi \) be the set of all control policies.

Let \( \Phi \) and \( F \) be as in Definitions 2.1 and 2.2; recall (2.3). A policy \( \pi = \{\pi_t\} \) is said to be randomized stationary if there exists \( \varphi \in \Phi \) such that

\[
\pi_t(\cdot|x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t = x) = \varphi(\cdot|x) \quad \forall t = 0, 1, \ldots, x \in X.
\]

and deterministic stationary if there exists \( f \in F \) for which (6.2) holds with \( \varphi(\cdot|x) = \delta_{f(x)}(\cdot) \). We shall identify \( \Phi \) and \( F \) with the families of randomized stationary policies and deterministic stationary policies, respectively.
Performance criteria. Let \(0 < \alpha < 1\) be a given “discount factor,” and let \(\gamma\) be a given p.m. (the “initial distribution”) on \(X\). For each \(i = 0, 1, \ldots, n\) and each policy \(\pi \in \Pi\) let

\[
V_i(\pi, \gamma) := (1 - \alpha)E_\gamma^\pi \left[ \sum_{t=0}^{\infty} \alpha^t c_i(x_t, a_t) \right]
\]

be the \(\alpha\)-discounted cost when using the control policy \(\pi\), given the initial distribution \(\gamma\). We may then express the constrained MCP we are concerned with as

\[
\text{CP} : \begin{align*}
\text{minimize} & \quad V_0(\pi, \gamma) \\
\text{s.t.} & \quad V_i(\pi, \gamma) \leq b_i \quad \forall i = 1, \ldots, n, \pi \in \Pi.
\end{align*}
\]

To prove, among other things, that CP is solvable (i.e., there exists \(\pi^* \in \Pi\) that satisfies (6.4) and it attains the minimum in (6.3)—in which case \(\pi^*\) is said to be an optimal policy for CP) we will use the fact that the CP is equivalent to a constrained optimization problem similar to (5.2); see (6.6), (6.7), below. To do this we impose the following conditions.

**Assumption 6.1.** Parts (a), (b), and (c) are the same as in Assumption 5.1. Moreover,

(d) the transition law is weakly continuous, i.e., for each \(u \in C_b(X)\), the function

\[
(x, a) \mapsto \int_X u(y)Q(dy|x, a)
\]

is in \(C_b(K)\);

(e) CP is feasible; i.e., there exists \(\tilde{\pi} \in \Pi\) such that \(V_0(\tilde{\pi}, \gamma) < \infty\), and \(\pi = \tilde{\pi}\) satisfies (6.4).

Under Assumption 6.1, there are several ways to prove that CP is solvable; see, for instance, Corollary 5.1 in [13] or Theorem 3.2 in [16]. In particular, a key step in the proof in the latter reference, [16], is to show that CP is equivalent to the problem

\[
\text{CP}_0 : \begin{align*}
\text{minimize} & \quad \int_X c_0(x, \varphi(x))\tilde{\mu}(dx) \\
\text{s.t.} & \quad \int_X c_i(x, \varphi(x))\tilde{\mu}(dx) \leq b_i \quad \forall i = 1, \ldots, n,
\end{align*}
\]

\[
\tilde{\mu}(B) = (1 - \alpha)\gamma(B) + \alpha \int_X Q(B|x, \varphi(x))\tilde{\mu}(dx) \quad \forall B \in \mathcal{B}(X),
\]

\[
\mu = \tilde{\mu} \cdot \varphi \in \mathbb{P}_0(K), \quad \varphi \in \Phi,
\]

where \(\mathbb{P}_0(K)\) is the subset of measures \(\mu\) in \(\mathbb{P}(K)\) for which \(\int c_0 d\mu < \infty\) (see Lemma 3.3 in [16]). The equivalence between CP and \(\text{CP}_0\) is in the sense that one of these problems is solvable if and only if so is the other, and both problems have the same value. Moreover, if \(\rho\) is the optimal value of \(\text{CP}_0\) and \(\mu_0 = \tilde{\mu}_0 \cdot \varphi_0 \in \mathbb{P}_0(K)\) is an optimal solution, i.e.,

\[
\rho = \int c_0 \, d\mu_0 = \inf \left\{ \int c_0 \, d\mu | \mu \in M \right\},
\]
where $M$ is the (convex) set of feasible solutions for CP$_0$, then the randomized stationary policy (or randomized strategy) $\varphi_0 \in \Phi$ is an optimal policy for CP.

To extend this result to an analogue of Theorem 5.2(b) we shall require, among other things, the transition law $Q$ to be nonatomic, which means that $Q(\cdot|x,a)$ is nonatomic for each pair $(x,a)$ in $\mathbb{K}$.

**Theorem 6.2.** Under Assumption 6.1 we have the following.

(a) CP$_0$ (equivalently, CP) is solvable.

(b) Suppose that, in addition, $A$ is a locally compact separable metric space and $\gamma$ and $Q$ are both nonatomic. Furthermore, let $\mu_0 = \hat{\mu}_0 \cdot \varphi_0$ be an optimal solution for CP$_0$ (see (6.8)). Then there is an optimal stationary strategy $\varphi^* \in \Phi$ for CP that is in $\mathcal{R}_{n+1}^0$ (the set of $(n+1)$-randomizations in Theorem 2.6).

**Proof.** As was already noted above, part (a) has been established in (e.g.) [13, 16]. To prove (b), let $\mu_0 = \hat{\mu}_0 \cdot \varphi_0$ be the given optimal solution for CP$_0$ and consider the convex set $\Delta_0$ of randomized strategies $\varphi \in \Phi$ such that $\hat{\mu}_0 \cdot \varphi$ is in $M$, i.e., $\varphi$ satisfies

\begin{equation}
\int_X c_0(x, \varphi(x)) \hat{\mu}_0(dx) < \infty,
\end{equation}

\begin{equation}
\int_X c_i(x, \varphi(x)) \hat{\mu}_0(dx) \leq b_i \quad \forall i = 1, \ldots, n,
\end{equation}

\begin{equation}
\hat{\mu}_0(\cdot) = (1 - \alpha)\gamma(\cdot) + \alpha \int_X Q(\cdot|x, \varphi(x)) \hat{\mu}_0(dx).
\end{equation}

Observe that $\rho_0 := \inf \{ \int c_0 d\mu | \mu = \hat{\mu}_0 \cdot \varphi \in \hat{\mu}_0 \cdot \Delta_0 \}$ coincides with $\rho$, i.e.,

\begin{equation}
\rho_0 = \rho.
\end{equation}

Indeed, by (6.8) it is clear that $\rho \leq \rho_0$, and, on the other hand, the reverse inequality is obvious because $\mu_0 = \hat{\mu}_0 \cdot \varphi_0$ is in $\hat{\mu}_0 \cdot \Delta_0$. Moreover, as in Theorem 4.12, it can be seen that $\hat{\mu}_0 \cdot \Delta_0$ is a convex compact subset of $M$. Hence, there is an extreme point $\mu^* = \hat{\mu}_0 \cdot \varphi^*$ of $\hat{\mu}_0 \cdot \Delta_0$, which, by (6.12), is an optimal solution for CP$_0$, i.e.,

\begin{equation}
\int c_0 d\mu^* = \rho_0 = \rho.
\end{equation}

On the other hand, from Lemma 4.11(a) with $\hat{\mu}_0$ and $\Delta_0$ in lieu of $\nu$ and $\Delta$, respectively, i.e.,

\[ \overline{\Delta}_0 := \{ \varphi \in \Delta_0 \} \text{ with } \overline{\varphi} := \{ \varphi' \in \Delta_0 | \hat{\mu}_0 \cdot \varphi' = \hat{\mu}_0 \cdot \varphi \}, \]

it follows that $\overline{\varphi}^*$ is an extreme point of $\overline{\Delta}_0$. This in turn gives that, as in Lemma 4.11(c), $\overline{\varphi}^*$ has an extreme point, which we shall denote $\varphi^*$ again, of $\overline{\Delta}_0$. To complete the proof of the theorem we will next show that such a $\varphi^* \in ex(\Delta_0)$ is precisely as required in part (b).

To this end, we will use Theorem 2.6 with $\nu = \hat{\mu}_0$ as above and $\Delta$ being the set of randomized strategies $\varphi \in \Phi$ that satisfy (6.10). Note that $\hat{\mu}_0 \cdot \Delta$ contains $\hat{\mu}_0 \cdot \Delta_0$, and so $\varphi^*$ is in $\Delta$. Therefore we may use (2.7) to conclude (b) if we can show that

\begin{equation}
\varphi^* \in ex(\Delta).
\end{equation}
To prove (6.14), let us suppose that \( \varphi^* \) is not an extreme point of \( \Delta \). Then, as in the proof of (3.3), there exist \( z \in X \) and \( C \in \mathcal{B}(A) \) such that \( 0 < \varphi^*(C|z) < 1 \). Moreover, let \( \beta := \varphi^*(C|z) \), and for \( i = 1, 2 \) define \( \varphi_i \in \Delta \) as in (3.4), (3.5) with \( \varphi^* \) in lieu of \( \varphi \), so that

\[
(6.15) \quad \varphi^* = \beta \varphi_1 + (1 - \beta) \varphi_2.
\]

Now, to prove (6.14), we will next show that \( \varphi_1, \varphi_2 \) are in fact in \( \Delta_0 \), which, together with (6.15), contradicts the fact that \( \varphi^* \) is in \( \text{ex} \Delta_0 \). To prove this, note first that \( \varphi_1 \) and \( \varphi_2 \) satisfy (6.9) because so does \( \varphi^* \). On the other hand, by (6.11) and the hypothesis that \( \gamma \) and \( Q \) are nonatomic, so is \( \mu_0 \); in particular \( \mu_0(\{z\}) = 0 \). Hence, by the definition of \( \varphi_1 \) and \( \varphi_2 \), they both satisfy (6.11); hence they belong to \( \Delta_0 \). \( \square \)

**Remark 6.3.** (a) A key step in the previous proof was to reduce \( \text{CP}_0 \) to a problem similar to (5.2) with \( \nu = \mu_0 \); see (6.10). However, because of the additional constraint (6.11), to this date (as far as we can tell) there are no general conditions ensuring the existence of optimal deterministic strategies for \( \text{CP}_0 \), that is, for the constrained MCP (6.3), (6.4). In other words, for constrained control problems there is no analogue of, say, Theorem 5.2(c).

(b) A MCP is said to be unconstrained if in (6.4) we have \( c_i(x, a) \equiv b_i \) for all \( i = 1, \ldots, n \) and \( (x, a) \) in \( K \). Theorem 6.2 can be simplified in the obvious manner to the unconstrained case. The results will be, of course, similar to those in section 4, but incorporating the constraint (6.7); see, e.g., [18, Chapter 6] for details.

7. **Concluding remarks and open problems.** In the previous sections we have presented a fairly complete, reasonably self-contained analysis of some classes of constrained (and unconstrained) optimization and control problems that use randomized strategies—or, in other words, problems in which the decisions or control actions may or need to be “randomized”. For these problems, our results ensure the existence of optimal solutions, as well as the characterization of these solutions as extreme points of certain convex sets of randomized strategies. A key feature of our paper is that such a characterization is obtained via the general results in Theorem 2.6 (and Lemma 3.3).

But, on the other hand, there is an important question conspicuously missing in our results, namely, how to compute optimal solutions. Concerning this question there are several possibilities that might be worth exploring. First, one can try to apply some approximation procedure for general optimization problems—see, e.g., [4] and the references therein.

Second, one may note that our problems (see (4.4), (5.2) or (6.6), (6.7)) are in fact infinite-dimensional linear programs; therefore, one could try to adapt some of the existing approximation schemes, as in [19], for instance.

Third, it is well known that (at least some) constrained optimization problems can be seen as multiobjective problems in which one is looking for some particular Pareto (or efficient or nondominated) solutions—see, e.g., [14, 22, 27]. For example, consider the constrained problem (5.2) and let \( J(\varphi) \) be the cost vector

\[
J(\varphi) := (J_0(\varphi), J_1(\varphi), \ldots, J_n(\varphi)) \in \mathbb{R}^{n+1},
\]

and \( \Gamma := \{ J(\varphi) \mid \varphi \in \Phi \} \subset \mathbb{R}^{n+1} \) the corresponding achievable (or performance) set. As \( \Gamma \) is a convex set, one can easily characterize the set \( P^* \) of Pareto solutions of the multiobjective problem (MP), i.e., the set of randomized strategies \( \varphi \in \Phi \) for which the cost vector \( J(\varphi) \) is in the Pareto set of \( \Gamma \). Moreover, under mild assumptions...
(see the above references), there is a Pareto solution $\varphi^* \in P^*$ to MP that is also a solution to the constrained problem (5.2). The key fact is that for such a $\varphi^*$ the constraints (2.6) hold with equality. Therefore, by part (b) of Theorem 2.6, to solve (5.2) we would essentially be looking for Pareto solutions in the set of extreme points $\text{ex}(\Delta) = R_{n+1}^0$. In other words, to find a solution for (5.2) it suffices to look at $P^* \cap R_{n+1}^0$. This should certainly simplify the search of optimal randomized strategies.

To conclude, note that the “Pareto approach” briefly described above is referred to as the “geometric approach” by some authors (e.g., [11]).

REFERENCES

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