Inputs/outputs estimation in DEA when some factors are undesirable

G.R. Jahanshahloo a, A. Hadi Vencheh b,c,*, A.A. Foroughi a, R. Kazemi Matin d

a Department of Mathematics, Teacher Training University, 15614 Tehran, Iran
b Department of Mathematics, Azad University, P.O. Box 81595-158, Khorasgan, Isfahan, Iran
c Department of Mathematics, Azad University, Science and Research Campus, P.O. Box 14515-775, Tehran, Iran
d Department of Mathematics, Azad University, P.O. Box 31485-313, Karaj, Tehran, Iran

Abstract

In this paper we investigate inputs/outputs estimate in the presence of undesirable factors. We discuss such questions: suppose some outputs (inputs) are undesirable and among a group of decision making units (DMUs) a DMU changes (increases and decreases simultaneously) some of its input (output) levels, how much outputs (inputs) could the unit change such that its efficiency index remains unchanged? To solve the problem we propose a multiple objective linear programming (MOLP). Our MOLP model could estimate input/output levels, regardless of the efficiency or inefficiency of the DMU. We propose necessary and sufficient conditions for inputs/outputs estimation. A numerical example is given.

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1. Introduction

In their seminal paper, Charnes et al. [3] coined the data envelopment analysis (DEA) to estimate the relative efficiencies of decision making unit
(DMUs) that use multiple inputs to produce multiple outputs. Classical DEA models rely on the assumption that inputs have to be minimized and outputs have to be maximized. However, it was mentioned already in [6] that the production process may also generate undesirable outputs like smoke pollution or waste. Motivated by public and governmental environment, ecological efficiency measurement has recently attached much interest [1]. Undesirable outputs may as well appear in nonecological applications like health care (complications of medical operations) and business (tax payment), cf. Smith [10]. A symmetric case of inputs which should be maximized may also occur (see [1]). For example, the aim of a recycling process is to use maximal quantity of the input waste. For further discussion one can read Refs. [4,5,7,8].

Recently Wei et al. [11] proposed inverse DEA, to answer the questions as follows: if among a group of DMUs, we increase certain inputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other DMUs, how much should the outputs of the DMU increase, or if the outputs need to be increased to a certain level and the efficiency of the unit remains unchanged, how much more inputs should be provided to the unit? These questions can be considered more generally, that is, assume that some factors are undesirable and we change (increase and decrease simultaneously) some or all input (output) levels of a given DMU and assume that the DMU maintains its current efficiency level, how much should the outputs (inputs) of the DMU change. In this paper we consider arbitrary changing in input (output) levels and we propose an multiple objective linear programming (MOLP) model for outputs (inputs) estimate. Our model could estimate input/output levels, regardless of the efficiency level of the DMU. As a special case we can consider the case in which inputs/outputs are increased (decreased) which, as we will see, covers [11].

The rest of this paper is as follows: in the following section, first we review BCC–DEA model when some factors are undesirable and then state our problem. In Section 3 we propose an MOLP for inputs/outputs estimate. Section 4 is devoted to investigate a special case. In Section 5 we illustrate our method by a numerical example. Conclusions are given in Section 6.

2. Background

Suppose we have \( n \) observations on \( n \) DMUs with input and output vectors \((x_j, y_j)\) for \( j = 1, 2, \ldots, n \). Let \( x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^t \) and \( y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^t \). All \( x_j \in \mathbb{R}^m \) and \( y_j \in \mathbb{R}^s \) and \( x_j > 0, y_j > 0 \) for \( j = 1, 2, \ldots, n \). The input matrix \( X \) and output matrix \( Y \) can be represented as

\[
X = [x_1, \ldots, x_j, \ldots, x_n], \quad Y = [y_1, \ldots, y_j, \ldots, y_n],
\]
where $X$ is an $(m \times n)$ matrix and $Y$ an $(s \times n)$ matrix. As outlined in [2] we assume that the production possibility set is estimated by

$$
T = \left\{ (x,y) : \sum_{j=1}^{n} x_j \lambda_j \leq x, \sum_{j=1}^{n} y_j \lambda_j \geq y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, n \right\}.
$$

In the absence of undesirable factors when a DMU, $o \in \{1, 2, \ldots, n\}$, is under evaluation, we can use the following BCC model:

$$
\begin{align*}
\text{max} & \quad \eta, \\
\text{s.t.} & \quad \sum_{j=1}^{n} x_j \lambda_j \leq x_o, \\
& \quad \sum_{j=1}^{n} y_j \lambda_j \geq y_o \eta, \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \lambda_j \geq 0 \quad j = 1, \ldots, n,
\end{align*}
$$

similarly, one can use an input-oriented model

$$
\begin{align*}
\text{min} & \quad \gamma, \\
\text{s.t.} & \quad \sum_{j=1}^{n} x_j \lambda_j \leq x_o \gamma, \\
& \quad \sum_{j=1}^{n} y_j \lambda_j \geq y_o, \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \lambda_j \geq 0 \quad j = 1, \ldots, n.
\end{align*}
$$

Now suppose that some outputs are undesirable so the output matrix $Y$ can be represented as follows:

$$
Y = \begin{bmatrix}
Y^g \\
Y^b
\end{bmatrix},
$$

where $Y^g$ and $Y^b$ represent the desirable (good) and undesirable (bad) outputs, respectively. Hence according to [9] the output-oriented BCC model is as follows:
\[ \begin{align*}
(P_o^O) \quad & \max \quad \varphi, \\
& \text{s.t.} \quad \sum_{j=1}^{n} x_j \lambda_j \leq x_o, \\
& \quad \sum_{j=1}^{n} y_j^g \lambda_j \geq y_o^g \varphi, \\
& \quad \sum_{j=1}^{n} y_j^b \lambda_j \geq y_o^b \varphi, \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \lambda_j \geq 0 \quad j = 1, \ldots, n,
\end{align*} \]

where \( y_j^b = y_o^b + v, j = 1, \ldots, n, \) and \( v \) is a chosen vector such that \( y_j^b > 0 \) for all \( j \).

**Definition 1.** The optimal value \( \varphi_o \) of problem \( (P_o^O) \) is called the efficiency index of \( \text{DMU}_o \). If \( \varphi_o = 1 \), we say \( \text{DMU}_o \) is (at least) weakly efficient.

Consider the following question: if the efficiency index \( \varphi_o \) remains unchanged, but the inputs change, how much should the outputs of the DMU change? To solve our problem, suppose the inputs of \( \text{DMU}_o \) are changed from \( x_o \) to \( x_o + \Delta x_o \), where the vector \( \Delta x_o \in \mathbb{R}^m \). We need to estimate the output vector \( \beta_o \) provided that the efficiency index of \( \text{DMU}_o \) is still \( \varphi_o \). Here

\[ \beta_o = (\beta_o^g, \beta_o^b)^t = (y_o^g + \Delta y_o^g, y_o^b + \Delta y_o^b)^t, \quad (\Delta y_o^g, \Delta y_o^b)^t \in \mathbb{R}^t. \]

For convenience, suppose \( \text{DMU}_{n+1} \) represents \( \text{DMU}_o \) after changing the inputs and outputs. Hence, to measure the efficiency of \( \text{DMU}_{n+1} \), we use the following model:

\[ \begin{align*}
(P_{o+O}) \quad & \max \quad \varphi, \\
& \text{s.t.} \quad \sum_{j=1}^{n+1} x_j \lambda_j \leq x_o, \\
& \quad \sum_{j=1}^{n+1} y_j^g \lambda_j \geq \beta_o^g \varphi, \\
& \quad \sum_{j=1}^{n+1} y_j^b \lambda_j \geq \beta_o^b \varphi, \\
& \quad \sum_{j=1}^{n+1} \lambda_j = 1, \\
& \quad \lambda_j \geq 0 \quad j = 1, \ldots, n + 1,
\end{align*} \]

where \( x_{n+1} = x_o, \) \( y_{n+1}^g = \beta_o^g \) and \( y_{n+1}^b = \beta_o^b = -\beta_o^b + v. \)
**Definition 2.** If the efficiency index of the DMU under the new inputs/outputs, \((x_o, \beta_o)\), is equal to the efficiency index under the old inputs/outputs, \((x_o, y_o)\), then we say that the efficiency is unchanged and we write \(\text{score}(x_o, y_o) = \text{score}(x_o, y_o)\).

**Remark 1.** In fact the new inputs/outputs for the DMU is \((x_o, \beta_o)\) where \(\beta_o = (\beta_o^e, \beta_o^b)^t\) satisfies

\[
\begin{align*}
\sum_{j=1}^{n} y_j^b \lambda_j &\geq \varphi_o \beta_o^b, \\
\sum_{j=1}^{n} y_j^b \lambda_j &\geq \varphi_o \bar{\beta}_o^b, \\
\sum_{j=1}^{n} \lambda_j & = 1, \\
\lambda_j & \geq 0 \quad j = 1, \ldots, n,
\end{align*}
\]

where \(\varphi_o\) is the optimal value of problem \((P_o^O)\) and \(\bar{\beta}_o^b = -\beta_o^b + v\).

**Definition 3.** Let \((\beta, \lambda) = (\beta^e, \beta^b, \lambda)\) be a feasible solution of problem \((V_o)\). If there is no feasible solution \((\beta^*, \lambda^*)\) of \((V_o)\) such that \(\beta_r \leq \beta_r^*\) for all \(r = 1, 2, \ldots, s\) and \(\beta_r < \beta_r^*\) for at least one \(r\), then we say \((\beta, \lambda)\) is a strongly efficient solution of problem \((V_o)\).

**Definition 4.** Suppose \((\beta, \lambda) = (\beta^e, \beta^b, \lambda)\) is a feasible solution of problem \((V_o)\). If there is no feasible solution \((\beta^*, \lambda^*)\) of \((V_o)\) such that \(\beta < \beta^*\), that is, \(\beta_r < \beta_r^*\) for all \(r = 1, 2, \ldots, s\), then we say \((\beta, \lambda)\) is a weak efficient solution of problem \((V_o)\).

For convenience, we say \(\beta\) is a strongly (weak) efficient solution to \((V_o)\).

**Theorem 1.** Let \((\beta_o, \lambda) = (\beta_o^e, \beta_o^b, \lambda)\) be a feasible solution of problem \((V_o)\). Then, \(\text{score}(x_o, \beta_o^e, v - \beta_o^b) = \text{score}(x_o, y_o)\) if and only if \(\beta_o\) is a weak efficient solution to problem \((V_o)\).
Proof. First assume that \( \beta_o \) is a weak efficient solution of problem \((V_o)\) and \((\varphi_o^+, \lambda^+)\) is an optimal solution of problem \((P_o^{+o})\). We must show that \( \varphi_o^+ = \varphi_o \). Since \((\beta_o, \lambda)\) is a feasible solution of \((V_o)\) we have

\[
\sum_{j=1}^{n} x_j \lambda_j \leq \alpha_o, \tag{1}
\]

\[
\sum_{j=1}^{n} y_e^j \lambda_j \geq \varphi_o \beta_e^g, \tag{2}
\]

\[
\sum_{j=1}^{n} \tilde{y}_j^b \lambda_j \geq \varphi_o \tilde{\beta}_o^b, \tag{3}
\]

\[
\sum_{j=1}^{n} \lambda_j = 1, \tag{4}
\]

\[
\lambda_j \geq 0 \quad j = 1, \ldots, n. \tag{5}
\]

Let \( \lambda^* = (\lambda, 0)^t \), it is easy to see that \((\varphi_o, \lambda^*)\) is a feasible solution of problem \((P_o^{+o})\) and hence we have \( \varphi_o \leq \varphi_o^+ \). Now if \( \varphi_o < \varphi_o^+ \), since \((\varphi_o^+, \lambda^+\) is a feasible solution of problem \((P_o^{+o})\) we have

\[
\sum_{j=1}^{n+1} x_j \lambda_j^+ \leq \alpha_o, \tag{6}
\]

\[
\sum_{j=1}^{n+1} y_e^j \lambda_j^+ \geq \beta_e^g \varphi_o^+, \tag{7}
\]

\[
\sum_{j=1}^{n+1} \tilde{y}_j^b \lambda_j^+ \geq \tilde{\beta}_o^b \varphi_o^+ \tag{8}
\]

and since \( \varphi_o \geq 1 \) and from (2), (3), (5), and (6) we have

\[
\varphi_o^+ \left( \begin{array}{c} \beta_e^g \\ \tilde{\beta}_o^b \end{array} \right) \leq \left( \begin{array}{c} \sum_{j=1}^{n+1} y_e^j \lambda_j^+ \\ \sum_{j=1}^{n+1} \tilde{y}_j^b \lambda_j^+ \end{array} \right) = \left( \begin{array}{c} \sum_{j=1}^{n} y_e^j \lambda_j^+ + \beta_e^g \lambda_{n+1}^+ \\ \sum_{j=1}^{n} \tilde{y}_j^b \lambda_j^+ + \tilde{\beta}_o^b \lambda_{n+1}^+ \end{array} \right) \leq \left( \begin{array}{c} \sum_{j=1}^{n} y_e^j \lambda_j^+ + \lambda_{n+1}^+ \sum_{j=1}^{n} y_e^j \lambda_j \\ \sum_{j=1}^{n} \tilde{y}_j^b \lambda_j^+ + \lambda_{n+1}^+ \sum_{j=1}^{n} \tilde{y}_j^b \lambda_j \end{array} \right) = \left( \begin{array}{c} \sum_{j=1}^{n} y_e^j (\tilde{\lambda}_j + \lambda_{n+1}^+ \lambda_j) \\ \sum_{j=1}^{n} \tilde{y}_j^b (\tilde{\lambda}_j^+ + \lambda_{n+1}^+ \lambda_j) \end{array} \right) = \left( \begin{array}{c} \sum_{j=1}^{n} y_e^j \tilde{\lambda}_j \\ \sum_{j=1}^{n} \tilde{y}_j^b \tilde{\lambda}_j \end{array} \right), \tag{9}
\]

where \( \tilde{\lambda}_j = \lambda_j^+ + \lambda_{n+1}^+ \lambda_j \) for all \( j = 1, \ldots, n \). Therefore we have
\[
\varphi_o \left( \frac{\beta^g_o}{\beta^b_o} \right) < \varphi_o^+ \left( \frac{\beta^g_o}{\beta^b_o} \right) \leq \left( \sum_{j=1}^{n} x_j \lambda^+_j \right) \]

or

\[
\varphi_o \left( \frac{\beta^g_o}{\beta^b_o} \right) < \left( \sum_{j=1}^{n} y^g_j \tilde{\lambda}_j \right) \leq \left( \sum_{j=1}^{n} y^b_j \tilde{\lambda}_j \right).
\]

So there exists a \( k > 1 \) such that

\[
\varphi_o \left( \frac{k \beta^g_o}{k \beta^b_o} \right) \leq \left( \sum_{j=1}^{n} y^g_j \tilde{\lambda}_j \right) \leq \left( \sum_{j=1}^{n} y^b_j \tilde{\lambda}_j \right).
\]

Also from (4) and (1) we have

\[
\alpha_o \geq \sum_{j=1}^{n} x_j \lambda^+_j + \alpha_o \lambda^+_n + \sum_{j=1}^{n} x_j \lambda^+_j + \lambda^+_n = \sum_{j=1}^{n} x_j \tilde{\lambda}_j,
\]

where \( \tilde{\lambda}_j \) is defined as before. It is clear that \( \sum_{j=1}^{n} \tilde{\lambda}_j = 1 \). Hence \( (k \beta_o, \tilde{\lambda}) \) is a feasible solution to \((V_o)\). But it is impossible since \( \beta_o \) is a weak efficient solution to \((V_o)\). Conversely, let \( \varphi_o^+ = \varphi_o \). We show that \( \beta_o \) is a weak efficient solution of problem \((V_o)\). By contradiction assume \( \beta_o \) is not a weak efficient solution of problem \((V_o)\), so there exists a feasible solution \((\beta', \lambda') = (\beta^g, \beta^b, \lambda')\) to problem \((V_o)\) such that \( \beta_o < \beta' \). Since \( \beta_o = (\beta^g_o, \beta^b_o) < \beta' = (\beta^g, \beta^b) \) and \( (\beta', \lambda') \) is a feasible solution of problem \((V_o)\) we have

\[
\sum_{j=1}^{n} x_j \lambda'_j \leq \alpha_o,
\]

\[
\sum_{j=1}^{n} y^g_j \lambda'_j \geq \varphi_o \beta^g > \varphi_o \beta^g_o,
\]

\[
\sum_{j=1}^{n} y^b_j \lambda'_j \geq \varphi_o \beta^b > \varphi_o \beta^b_o,
\]

so there exists a \( k > 1 \) such that

\[
\left( \sum_{j=1}^{n} y^g_j \lambda'_j \sum_{j=1}^{n} y^b_j \lambda'_j \right) \geq \varphi_o \left( \frac{\beta^g_o}{\beta^b_o} \right) \geq (k \varphi_o) \left( \frac{\beta^g_o}{\beta^b_o} \right).
\]

Now let \( \lambda^+ = (\lambda', 0) \), it is obvious that \( (k \varphi_o, \lambda^+) \) is a feasible solution to \((P_o^{+0})\) and \( k \varphi_o > \varphi_o \). But it is against the assumption that \( \varphi_o \) is the optimal value of problem \((P_o^{+0})\).
The above discussion can also be applied to situations when some inputs need to be increased rather than decreased to improve the performance. Therefore the input matrix $X$ can be represented as follows:

$$X = \begin{bmatrix} X^d \\ X^i \end{bmatrix},$$

where $X^i$ and $X^d$ represent inputs to be increased and decreased respectively. In this case to measure the efficiency index of $\text{DMU}_o$ we can use the following model [9]:

$$\begin{bmatrix} (P'_o) \end{bmatrix} \begin{array}{c} \min \\
\text{s.t.} \end{array} \rho, \begin{array}{l} \sum_{j=1}^{n} x^d_j \lambda_j \leq x^d_o \rho, \\
\sum_{j=1}^{n} \bar{x}^d_j \lambda_j \leq \bar{x}^d_o \rho, \\
\sum_{j=1}^{n} y_j \lambda_j \geq y_o, \\
\sum_{j=1}^{n} \lambda_j = 1, \\
\lambda_j \geq 0 \quad j = 1, \ldots, n, \end{array}$$

where $\bar{x}^d_j = -x^d_j + t$ and $t$ is a proper translation vector such that $\bar{x}^d_j > 0$ for all $j = 1, \ldots, n$. Suppose the DMU attempts to change the output levels from $y_o$ to $\beta_o = y_o + \Delta y_o$, $\Delta y_o \in \mathbb{R}^s$, and assume that the efficiency index $\rho_o$ of the unit remains unchanged, how much inputs, say $\alpha_o$, should the DMU provide? To solve this problem we can use the following theorem.

**Theorem 2.** Let $(\alpha_o, \lambda) = (x^d_o, \bar{x}^i_o, \lambda)$ be a feasible solution of the following MOLP:

$$\begin{bmatrix} (V_i) \end{bmatrix} \begin{array}{c} \min \\
\text{s.t.} \end{array} \begin{array}{l} \alpha = (x^d, \bar{x}^i), \\
\sum_{j=1}^{n} x^d_j \lambda_j \leq \rho_o x^d, \\
\sum_{j=1}^{n} \bar{x}^d_j \lambda_j \leq \rho_o \bar{x}^i, \\
\sum_{j=1}^{n} y_j \lambda_j \geq \beta_o, \\
\sum_{j=1}^{n} \lambda_j = 1, \\
\lambda_j \geq 0 \quad j = 1, \ldots, n, \end{array}$$

where $\bar{x}^i_j = -x^i_j + t$ and $t$ is a proper translation vector such that $\bar{x}^i_j > 0$ for all $j = 1, \ldots, n$. Suppose the DMU attempts to change the output levels from $y_o$ to $\beta_o = y_o + \Delta y_o$, $\Delta y_o \in \mathbb{R}^s$, and assume that the efficiency index $\rho_o$ of the unit remains unchanged, how much inputs, say $\alpha_o$, should the DMU provide? To solve this problem we can use the following theorem.

**Theorem 2.** Let $(\alpha_o, \lambda) = (x^d_o, \bar{x}^i_o, \lambda)$ be a feasible solution of the following MOLP:

$$\begin{bmatrix} (V_i) \end{bmatrix} \begin{array}{c} \min \\
\text{s.t.} \end{array} \begin{array}{l} \alpha = (x^d, \bar{x}^i), \\
\sum_{j=1}^{n} x^d_j \lambda_j \leq \rho_o x^d, \\
\sum_{j=1}^{n} \bar{x}^d_j \lambda_j \leq \rho_o \bar{x}^i, \\
\sum_{j=1}^{n} y_j \lambda_j \geq \beta_o, \\
\sum_{j=1}^{n} \lambda_j = 1, \\
\lambda_j \geq 0 \quad j = 1, \ldots, n, \end{array}$$

where $\bar{x}^d_j = -x^d_j + t$ and $t$ is a proper translation vector such that $\bar{x}^d_j > 0$ for all $j = 1, \ldots, n$. Suppose the DMU attempts to change the output levels from $y_o$ to $\beta_o = y_o + \Delta y_o$, $\Delta y_o \in \mathbb{R}^s$, and assume that the efficiency index $\rho_o$ of the unit remains unchanged, how much inputs, say $\alpha_o$, should the DMU provide? To solve this problem we can use the following theorem.
where \( \rho_o \) is given as the optimal value of problem \((P_o')\) and \( \bar{x} = -x^1 + t \). Then, score \((x_o, t - x^1_o, \beta_o) = \text{score} (x_o, y_o)\) if and only if \( x_o \) is a weak efficient solution to \((V_i)\).

**Proof.** It is similar to the proof of the last theorem only with some changes. \(\square\)

4. A special case

In this section we consider a special case. Assume that some inputs are undesirable. Now suppose that our problem is as follows: if the output levels are increased from \( y_o \) to \( \beta_o = y_o + \Delta y_o \), \( \Delta y_o \geq 0 \) and \( \Delta y_o \neq 0 \), and we assume that the efficiency index \( \rho_o \) remains unchanged, how much should the inputs of the DMU increase? Like before assume that DMU \( n+1 \) represents DMU \( o \) after changing of the inputs and outputs. So to measure the efficiency level of DMU \( n+1 \) we can use the following model:

\[
(P_{o+1}^ {+1}) \quad \min_{\rho, \lambda, \bar{\lambda}} \quad \sum_{j=1}^{n+1} x^d_j \lambda_j \leq \alpha^d_o \rho, \\
\sum_{j=1}^{n+1} \bar{x}^i_j \bar{\lambda}_j \leq \bar{\alpha}^i_o \rho, \\
\sum_{j=1}^{n+1} y_j \lambda_j \geq \beta_o, \\
\sum_{j=1}^{n+1} \bar{\lambda}_j = 1, \\
\lambda_j \geq 0 \quad j = 1, \ldots, n+1,
\]

where \( x^d_{n+1} = x^d_o, \bar{x}^i_{n+1} = \bar{x}^i_o = -x^i_o + t \) and \( y_{n+1} = \beta_o \).

Consider the following MOLP:

\[
(V) \quad \min_{\lambda} \quad x = (x^d, \bar{x}^i), \\
\text{s.t.} \quad \sum_{j=1}^{n} x^d_j \lambda_j \leq \rho_o x^d, \\
\sum_{j=1}^{n} \bar{x}^i_j \bar{\lambda}_j \leq \rho_o \bar{x}^i, \\
\sum_{j=1}^{n} y_j \lambda_j \geq \beta_o, \\
\sum_{j=1}^{n} \bar{x}^d_j \geq x^d_o, \\
-x^i \geq -x^i, \\
\sum_{j=1}^{n} \lambda_j = 1, \\
\lambda_j \geq 0 \quad j = 1, \ldots, n,
\]
where \( \rho_o \) is given as the optimal value of problem \((P^i_o)\) and \( \bar{x}^i = -x^i + t \). It seems that if \( x_o = (x_o^1, \bar{x}_o^i) \) be an arbitrary weak efficient solution of problem \((V)\) then \( \text{score} \ (x_o^1, t - \bar{x}_o^i, \beta_o) = \text{score} \ (x_o, y_o) \). Here by an example we show that it is not true in general.

**Example 1.** Consider the following table:

<table>
<thead>
<tr>
<th>DMU</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_1 )</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Output</td>
<td>( y )</td>
<td>2</td>
</tr>
</tbody>
</table>

Here we have two DMUs A and B with three inputs \( x_1, x_2 \) and \( x_3 \) and one output \( y \). Let the first input be undesirable and \( t = 6 \). We take B into consideration, and solve problem \((P^i_o)\)

\[
\begin{align*}
\min \quad & \rho, \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_2 \leq 2\rho, \\
& \lambda_1 + 6\lambda_2 \leq 6\rho, \\
& 2\lambda_1 + 5\lambda_2 \leq 5\rho, \\
& 2\lambda_1 + \lambda_2 \geq 1, \\
& \lambda_1 + \lambda_2 = 1, \\
& \lambda_1, \lambda_2 \geq 0.
\end{align*}
\]

The optimal value is \( \rho_B = 0.5 \). Now suppose that the output is increased from 1 to 1.25 and we are interested to know how much more inputs could the unit provide. Therefore we solve problem \((V)\)

\[
\begin{align*}
\min \quad & (\bar{x}_1^1, x_2^d, x_3^d), \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_2 \leq 0.5x_2^d, \\
& \lambda_1 + 6\lambda_2 \leq 0.5x_2^d, \\
& 2\lambda_1 + 5\lambda_2 \leq 0.5\bar{x}_1^1, \\
& 2\lambda_1 + \lambda_2 \geq 1.25, \\
& x_2^d \geq 2, \\
& x_3^d \geq 6, \\
& -\bar{x}_1^1 \geq -5, \\
& \lambda_1 + \lambda_2 = 1, \\
& \lambda_1, \lambda_2 \geq 0.
\end{align*}
\]
It can be seen that $x_0 = (2, 5, 6)$, $\lambda = (1, 0)$ is a weak efficient solution of the above problem. Now using $x_B = (2, 5, 6)$, we solve problem $\left( P_o^{+1} \right)$.  

$$\begin{align*}
\min & \quad \rho, \\
\text{s.t.} & \quad \lambda_1 + 2\lambda_2 + 5\lambda_3 \leq 5\rho, \\
& \quad \lambda_1 + 6\lambda_2 + 6\lambda_3 \leq 6\rho, \\
& \quad 2\lambda_1 + 5\lambda_2 + 2\lambda_3 \leq 2\rho, \\
& \quad 2\lambda_1 + \lambda_2 + \lambda_3 \geq 1.25, \\
& \quad \lambda_1 + \lambda_2 + \lambda_3 = 1, \\
& \quad \lambda_1, \lambda_2, \lambda_3 \geq 0.
\end{align*}$$

The optimal value is 1, so $\text{score} (4, 5, 6, 1.25) \neq \text{score} (1, 2, 6, 1)$. Let $z$ be a feasible solution of problem $(V)$ and define  

$$E = \{x \mid \text{there is not feasible solution } z' \text{ of } (V) \text{ such that } x^d \leq x^d, \bar{x}^d < \bar{x}^d\}.$$  

Now we have

**Theorem 3.** Let $(x_o, \lambda) = (x_o^d, \bar{x}_o^d, \lambda)$ be a feasible solution of problem $(V)$ and one of the following requirements is satisfied:

(i) $x_o$ is a weak efficient solution to $(V)$ and $x_o^d > x_o^d$.

(ii) $x_o \in E$.

Then, $\text{score} \left( x_o^d, t - \bar{x}_o^d, \beta_o \right) = \text{score} \left( x_o, y_o \right)$.

**Proof.** Suppose $\rho_o$ and $\rho_o^+$ are the optimal values of $(P_o^l)$ and $(P_o^{+l})$, respectively. We must prove that $\rho_o^+ = \rho_o$. Similar to Theorem 1 it is easy to see that $\rho_o^+ \leq \rho_o$. By contradiction assume that $\rho_o^+ < \rho_o$.

(i) If $x_o$ is a weak efficient solution of $(V)$ and $x_o^d > x_o^d$, then similar to theorem 1 it is easy to see that there exists a $k < 1$ such that $(kx_o, \bar{\lambda})$ is a feasible solution of $(V)$ which is contradiction since $x_o$ is a weak efficient solution of $(V)$. Note that this $k$ is selected such that $kx_o^d \geq x_o^d$.

(ii) Let $x_o$ be a feasible solution of $(V)$ and $x_o \in E$. If $x_o^d = x_o^d$ then, like theorem 1 it is obvious that $(\rho_o^+, \bar{\lambda})$ is a feasible solution to problem $(P_o^l)$ and hence $\rho_o \leq \rho_o^+$ which is contradiction. Otherwise, there is a $k > 0$ such that $(\tilde{x}, \bar{\lambda})$ is a feasible solution of $(V)$, where

$$\tilde{x} = x_o - kw, w = (e_i, e).$$

$e_i$ is the $i$th unit vector in $\mathbb{R}^p$, $p$ is the number of the desirable inputs, and $i$ is the index in which $x_i^d < x_i^d$ and $e = (1, 1, \ldots, 1) \in \mathbb{R}^{m-p}$. But this is against the assumption that $x_o \in E$. $\square$
Remark 2. Consider the following set:

\[ E' = \{ x \mid \text{there is not feasible solution } x' \text{ of } (V) \text{ such that } x' \preceq x \}. \]

In fact \( E' \) is the set of all strongly efficient solutions to problem \((V)\) and it is clear that \( E' \subseteq E \). Therefore, the above theorem is abiding when \( x_0 \in E' \).

5. An illustrative example

In this section we illustrate our method by an example.

Example 2. Consider the following table:

<table>
<thead>
<tr>
<th>DMU</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td>( x_1 )</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>( x_2 )</td>
<td>15</td>
<td>10</td>
<td>25</td>
<td>10</td>
</tr>
<tr>
<td>Output</td>
<td>( y_1 )</td>
<td>60</td>
<td>90</td>
<td>80</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>( y_2 )</td>
<td>7</td>
<td>11</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

In this table we have five DMUs with two inputs \( x_1 \) and \( x_2 \) and two outputs \( y_1 \) and \( y_2 \). Assume that the second output is undesirable hence

\[ Y^g = (60, 90, 80, 90, 75), \quad Y^b = (7, 11, 6, 5, 9). \]

Suppose \( v = 15 \) and \( C \) is under evaluation.

\[
\begin{align*}
\text{max} & \quad \varphi, \\
\text{s.t.} & \quad 5\lambda_1 + 10\lambda_2 + 15\lambda_3 + 20\lambda_4 + 7\lambda_5 \leq 15, \\
& \quad 15\lambda_1 + 10\lambda_2 + 25\lambda_3 + 10\lambda_4 + 4\lambda_5 \leq 25, \\
& \quad 60\lambda_1 + 90\lambda_2 + 80\lambda_3 + 90\lambda_4 + 75\lambda_5 \geq 80\varphi, \\
& \quad 8\lambda_1 + 4\lambda_2 + 9\lambda_3 + 10\lambda_4 + 6\lambda_5 \geq 9\varphi, \\
& \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\
& \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0
\end{align*}
\]

the optimal value is \( \varphi_c = 1.01308 \). Now suppose the input vector is changed from \( x_C = (15, 25)^t \) to \( x_C = (17, 20)^t \). To estimate the output levels we must solve the following MOLP:
Let $\Omega$ be the feasible region of the above problem. The following weighted sum model solves the above model:

\[
\begin{align*}
\max & \quad w_1 \beta_1^e + w_2 \beta_2^b, \\
\text{s.t.} & \quad (\lambda, \beta) \in \Omega
\end{align*}
\]

without loss of generality, let $w_1 = 4$ and $w_2 = 0.25$. Hence we solve the following model:

\[
\begin{align*}
\max & \quad 4\beta_1^e + 0.25\beta_2^b, \\
\text{s.t.} & \quad 5\lambda_1 + 10\lambda_2 + 15\lambda_3 + 20\lambda_4 + 7\lambda_5 \leq 17, \\
& \quad 15\lambda_1 + 10\lambda_2 + 25\lambda_3 + 10\lambda_4 + 4\lambda_5 \leq 20, \\
& \quad 60\lambda_1 + 90\lambda_2 + 80\lambda_3 + 90\lambda_4 + 75\lambda_5 \geq 1.01308\beta_1^e, \\
& \quad 8\lambda_1 + 4\lambda_2 + 9\lambda_3 + 10\lambda_4 + 6\lambda_5 \geq 1.01308\beta_2^b, \\
& \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\
& \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.
\end{align*}
\]

the optimal solution is $\beta_1^e = 88.83800, \beta_2^b = 8.09413, \lambda_2 = 0.30000, \lambda_4 = 0.70000$ and other variables are equal to zero. Thus the DMU should produce output vector $\beta_C = (88.83800, 6.90587)^T$ by utilizing input vector $\alpha_C = (17, 20)^T$.

6. Conclusions

In this paper we discuss these problems: in the presence of undesirable factors, how should we control the changes in input/output levels of a given DMU such that the efficiency index of the DMU is preserved. To solve the problem we propose an MOLP model. We provide the necessary and sufficient conditions for the input and the output changes under the same efficiency index. In comparison with current methods our method has some advantages as follows:

(1) Other methods estimate inputs (outputs) for a given DMU when some or all outputs (inputs) are increased, while we consider the general, viz., when some factors are undesirable and some input (output) levels are increased and at the same time some of inputs (outputs) are decreased.
(2) To estimate input/output levels Wei et al. use an MOLP for the inefficient DMUs and a linear programming when the DMU is weakly efficient, while we propose an MOLP which could estimate input/output levels, regardless of the efficiency or inefficiency of the DMU.

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References