SOME RESULTS ON FARTHEST POINTS IN 2-NORMED SPACES

M. Iranmanesh\(^1\) and F. Soleimany\(^2\)

Abstract. In this paper, we consider the problem of the farthest point for bounded sets in a real 2-normed spaces. We investigate some properties of farthest points in the setting of 2-normalised spaces and present various characterizations of b-farthest point of elements by bounded sets in terms of b-linear functional. We also provide some applications of farthest points in the setting of 2-inner product spaces.

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1. Introduction

The concepts of 2-metric spaces and linear 2-normed spaces were first introduced by Gähler in 1963\(^8\) and have been developed extensively in different subjects by others authors (see \(\cite{1, 2, 3, 4, 9, 10, 12}\)). Elumalai, Vijayaragavan and Sistani, Moghaddam in \(\cite{6, 14}\) gave some results on the concept best approximation in the context of bounded linear 2-functionals on real linear 2-normed spaces. They established various characterizations of the best approximation elements in these spaces. The concepts of farthest point in normed spaces have been studied by many authors (see \(\cite{1, 2, 5, 7, 13}\)). In this paper we study this concept in 2-normed spaces, and obtain some results on characterization and existence of farthest points in normed linear spaces in terms of bounded b-linear functionals. In section \(\S 2\), we give some preliminary results. In section \(\S 3\), we give some fundamental concepts of b-farthest points and give characterization of farthest points in 2-normed linear spaces and some basic properties of farthest points. Also we study the farthest point mapping on \(X\) by virtue of the Gateaux derivative in 2-normed spaces. We show in the case that 2-normed space is strictly convex there exists a unique farthest points of the closed convex set from each point. In the end, we delineate some applications of farthest points in 2-inner product spaces.

2. Preliminaries

Definition 2.1. Let \(X\) be a linear space of dimension greater than 1. Suppose \(\|.,.\|\) is a real-valued function on \(X \times X\) satisfying the following conditions:

\(^1\)Department of mathematical sciences, Shahrood university, Iran, e-mail: m.iranmanesh2012@gmail.com
\(^2\)Department of mathematical sciences, Shahrood university, Iran, e-mail: enfazh.bmaam@gmail.com
a) \( \|x, z\| \geq 0 \) and \( \|x, z\| = 0 \) if and only if \( x \) and \( z \) are linearly dependent.

b) \( \|x, z\| = \|z, x\| \),

d) \( \|\alpha x, z\| = \alpha \|x, z\| \) for any scalar \( \alpha \in \mathbb{R} \),

e) \( \|x + x', z\| \leq \|x, z\| + \|x', z\| \).

Then \( \|\cdot, \cdot\| \) is called a 2-norm on \( X \) and \( (X, \|\cdot, \cdot\|) \) is called a linear 2-normed space.

**Example 2.2.** Let \( X = \mathbb{R}^3 \), and consider the following 2-norm on \( X \):

\[
\|x, y\| = |xy| = |\text{det} \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}|.
\]

where \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \). Then \( X \) is a 2-normed space.

**Example 2.3.** Let \( X \) be a real linear space having two seminorms \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \). Then \( (X, \|\cdot\|) \) is a generalized 2-normed space with the 2-norm defined by

\[
\|x, y\| = \|x\|_1 \|y\|_2, \text{ for } x, y \in X.
\]

Every 2-normed space is a locally convex topological vector space. In fact for a fixed \( b \in X \), \( p_b = \|x, b\| ; x \in X \) is a semi-norm on \( X \) and the family \( P = \{p_b : b \in X\} \) of semi-norms generates a locally convex topology.

**Definition 2.4.** Let \( (X, \|\cdot, \cdot\|) \) be a 2-normed linear space, \( E \) be a nonempty subset of \( X \). The set \( E \) is called b-open if and only if for each \( a_0 \in E \), there exists \( \varepsilon_{a_0} > 0 \) such that for each \( c \in E \) with \( \|a_0 - c, b\| < \varepsilon_{a_0} \) implies \( a_0 - c \in E \). The b-interior of \( E \) is denoted \( \text{int}_b(E) \), is the largest b-open set contained in \( E \).

A sequence \( \{x_n\} \) in 2-normed linear space \( X \) is said to be b-convergent if there exists an element \( a \in X \) such that \( \lim_{n \to \infty} \|x_n - a, b\| = 0 \). A set is b-closed if and only if it contains all of its limit points.

**Definition 2.5.** Let \( (X, \|\cdot, \cdot\|) \) be a 2-normed space, \( b \in X \) be fixed, then a map \( T : X \times \langle b \rangle \to \mathbb{R} \) is called a b-linear functional on \( X \times \langle b \rangle \) whenever

1) \( T(a + c, b) = T(a, b) + T(c, b) \) for \( a, c, b \in X \) such that

2) \( T(\alpha a, b) = \alpha T(a, b) \) for \( \alpha \in \mathbb{R} \).

A b-linear functional \( T : X \times \langle b \rangle \to \mathbb{R} \) is said to be bounded if there exists a real number \( M > 0 \) such that \( |T(x, b)| < M \|x, b\| \) for every \( x \in X \). The norm of the b-linear functional \( T : X \times \langle b \rangle \to \mathbb{R} \) is defined by

\[
\|T\| = \sup\{\|T(x, b)\| : \|x, b\| \neq 0\}.
\]
3. Farthest points in 2-normed spaces

Let $X$ be a 2-normed vector space. For a nonempty subset $G$ of $X$ and $x \in X$, define

$$f_G(x,b) = \sup_{g \in G} \|x-g,b\|.$$  \hfill (3.1)

Recall that a point $g_0 \in G$ is called a b-farthest point for $x \in X$ if

$$\|x-g_0,b\| = f_G(x,b).$$ \hfill (3.2)

The set of all b-farthest points to $x$ from $G$ is denoted by $F_G(x,b)$. Let

$$R_b(G) = \{x \in X : F_G(x,b) \neq \emptyset\}.$$  

The set $G$ is said to be a b-remotal set if $R_b(G) = X$.

**Corollary 3.1.** Let $X$ be a 2-normed vector space and $G$ be a nonempty bounded subset of $X$. Then for any $x, z$ of $X$

i) $|f_G(x,b) - f_G(z,b)| \leq \|x-z,b\|$.  

ii) $\|x-z,b\| \leq f_G(x,b) + f_G(z,b)$.

**Proof.**  

i) Let $y \in F_G(z,b)$. By the definition of b-farthest points, we have

$$f_G(x,b) \geq \|x-y,b\| = \|x-z+z-y,b\| \geq \|x-z,b\| - \|z-y,b\|$$

$$f_G(x,b) - f_G(z,b) \geq \|x-z,b\|.$$  

Interchanging $x$ and $y$, we get

$$f_G(z,b) - f_G(x,b) \geq \|x-z,b\|.$$  

Hence $|f_G(x,b) - f_G(z,b)| \leq \|x-z,b\|$.  

ii) It’s proof is similar to that of (i).

**Theorem 3.2.** Let $G$ is a closed bounded b-remotal set in a 2-normed space $X$. Then $F_G(x,b) \cap \text{int}_b(G) = \emptyset$.

**Proof.** Suppose $e \in G$ such that $e \in F_G(x,b) \cap \text{int}_b(G)$. There exists a number $r > 0$ such that $\{y \in X : \|y-e,b\| < r\} \subseteq G$. Put $u = e - \frac{r}{2\|x-e,b\|}(x-e)$. Then $\|u-e,b\| = \frac{r}{2} \leq r$, and hence $u \in G$ and

$$\|x-u,b\| = \|x-e + \frac{r}{2\|x-e,b\|}(x-e),b\|$$

$$= \|(1 + \frac{r}{2\|x-e,b\|})(x-e),b\|$$

$$= (1 + \frac{r}{2\|x-e,b\|})\|x-e,b\| > \|(x-e),b\|.$$ 

This is a contradiction. \qed
Theorem 3.3. A nonvoid bounded set $G$ in a 2-normed space $X$ is $b$-remotal if and only if the following associated set

$$K_d = G + CB_d^b (0)$$

is closed for $d > 0$, where $CB_d^b (0) = \{ y \in X : \| x, b \| \geq d \}$.

Proof. Let $x$ be an adherent element of $G + CB_d^b (0)$, i.e. there exist a sequence $(x_n)_{n \in N}$ which converges to $x$ and a sequence $(u_n)_{n \in N} \subset G$ such that for all $n \in N \| x_n - u_n, b \| \geq d$. Thus, for every $\varepsilon > 0$ there exists $n_\varepsilon \in N$ such that $\| x_n - u_n, b \| > d - \varepsilon$ for all $n \geq n_\varepsilon$. Now, if $G$ is $b$-remotal, taking an element $g' \in F_G(x, b)$ we obtain that $\| x - g', b \| \geq \| x_n - u_n, b \|$ for all $n \geq n_\varepsilon$ and so $\| x - g', b \| \geq d - \varepsilon$, for every $\varepsilon > 0$. Consequently $\| x - g', b \| \geq d$ i.e. $x \in G + CB_d^b (0)$. Conversely, for an arbitrary element $x \in X$ we take $d = f_G(x, b)$. We can suppose $d > 0$ since $f_G(x, b) = 0$ if and only if $G = \{ x \}$. When $G$ is $b$-remotal obviously for every $n \in N$ exist $u_n \in G$ such that $\| x - u_n, b \| \geq d - \frac{1}{n}$. But, we have

$$\frac{1}{n} (d - \frac{1}{n})^{-1} (x - u_n) + x \in u_n + CB_d^b (0) \subset G + CB_d^b (0),$$

for all $n \in N$ such that $n > 1$. Since $(u_n)_{n \in N}$ is bounded, by passing to the limit we get $x \in G + CB_d^b (0)$). Therefore, if $G + CB_d^b (0)$ is closed there exists $g' \in G$ such that $\| x - g', b \| \geq d$ i.e. $g' \in F_G(x, b)$. Hence the set $G$ is $b$-remotal. \qed

Some characterizations of farthest points in 2-normed spaces are provided in following theorems.

Theorem 3.4. Let $G$ be a subset of a 2-norm space $X$ and $x \in X \setminus M + < b >$, then $g_0 \in F_G(x, b)$, if and only if there exists a $b$-bilinear function $p$ such that

$$p(x - g_0, b) = \sup_{g \in G} \| x - g, b \| \text{ and } \| p \| = 1.$$ \hspace{1cm} (3.3)

Proof. Suppose that there is a $b$-bilinear function $p$ which satisfies (3.3), then

$$\| x - g_0, b \| = \| x - g_0, b \| \| p \| \geq | p(x - g_0, b) | = \sup_{g \in G} \| x - g, b \| \geq \| x - g, b \| .$$

Conversely, let $g_0 \in F_G(x, b)$, by Hahn-Banach theorem in the context of 2-normed spaces (see Theorem 2.2 [11]) there exists a $b$-bilinear function $p$ such that $\| p \| = 1$, $p(x - g_0, b) = \| x - g_0, b \| = \sup_{g \in G} \| x - g, b \|$. \qed

Theorem 3.5. Let $G$ be a subset of a 2-norm space $X$ and $x \in X \setminus M + < b >$. Then the following statements are equivalent.

i) $g_0 \in F_G(x, b)$. 

\hspace{1cm} (i) $g_0 \in F_G(x, b)$.
ii) There is a b-bilinear function \( p \) on \( X \) which satisfies

\[
(3.4) \quad |p(x - g_0, b)| = \sup_{g \in G} \|x - g, b\| \text{ and } \|p\| = 1,
\]

\[
(3.5) \quad |p(x - g_0, b)| \geq |p(x - g, b)|.
\]

iii) There is a b-bilinear function \( p \) on \( X \) which satisfies (3.4) and

\[
(3.6) \quad p(g_0 - g, b)p(g_0 - x, b) \geq 0.
\]

**Proof.** Let \( g_0 \in F_G(x, b) \). Then by Theorem 3.4 we have (3.4) and

\[
|p(x - g_0, b)| = \sup_{g \in G} \|x - g, b\| \geq \|x - g, b\| \geq |p(x - g, b)|,
\]

which proves (3.3). Thus, \((i) \Rightarrow (ii)\).

\((ii) \Rightarrow (iii)\). Suppose that there is a b-bilinear function \( p \) on \( X \) satisfying (3.4), (3.5) then

\[
|p(x - g_0, b)|^2 \geq |p(x - g, b)|^2 = |p(x - g_0, b)|^2 + |p(g - g_0, b)|^2 + 2p(g_0 - g, b)p(g_0 - x, b)
\]

\[
\geq |p(x - g_0, b)|^2 + 2p(g_0 - g, b)p(g_0 - x, b),
\]

whence it follows that \( p(g_0 - g, b)p(g_0 - x, b) \geq 0 \).

\((iii) \Rightarrow (i)\) It is a consequence of Theorem 3.4. \qed

**Definition 3.6.** A linear 2-normed space \((X, \|., .\|)\) is said to be strictly convex if \(\|x + y, c\| = \|x, c\| + \|y, c\|\) and \(c \notin \text{Span}\{x, y\}\) imply that \(x = \alpha y\) for some \(\alpha > 0\).

**Definition 3.7.** A real-valued function \(f\) on \(X \times < b >\) is said to be b-Gateaux differentiable at a point \(x\) of \(X\) if there is a b-linear functional \(df_x\) such that, for each \(y \in X\),

\[
df_x(y, b) = \lim_{t \to 0} \frac{f(x+ty, b) - f(x, b)}{t},
\]

and we call \(df_x\) the b-Gateaux derivative of \(f\) at \(x\).

**Theorem 3.8.** Let \(G\) be a subset of a 2-norm space \(X, x \in X\) and \(y \in F_G(x, b)\). Suppose that the functional \(df_{x,b}\) is the Gateaux derivative of the function \(f_G(., .)\) at the point \(x\). Then

\[
df_x(x - y, b) = \|x - y, b\| \text{ and } \|df_x\| = 1.
\]

**Proof.** If \(G\) is a single point this is clear. Otherwise \(x \neq y\) and \(\|x - y, b\| = f_G(x, b)\), for \(0 < t < 1\),

\[
f_G(x, b) + t\|x - y, b\| = (1 + t)\|x - y, b\| = \|x + t(x - y) - y, b\|
\]

\[
\leq f_G(x + t(x - y), b) \leq f_G(x, b) + t\|x - y, b\|.
\]
As above and Corollary 3.1 so omitted holds throughout, and
\[ df_x(x - y, b) = \lim_{t \to 0} \frac{f_G(x + t(y - x), b) - f_G(x, b)}{t} = \|x - y, b\|. \]

Corollary 3.1 implies that \( \|df_x, b\| \leq 1 \), so this also show that \( \|df_x\| = 1 \).

**Theorem 3.9.** Let \( G \) be a convex subset of a strictly convex 2-normed space \( X, x \in X \setminus G \) and \( b \notin \text{Span}\{x, G\} \). Suppose that the functional \( df_x, b \) is the Gateaux derivative of the function \( f_G(., b) \) at the point \( x \). Then there is at most one \( b \)-farthest point in \( G \) to \( x \).

**Proof.** Suppose that \( y, z \) of \( F_G(x, b) \). Theorem 3.8 shows that
\[ df_x(x - y, b) = \|x - y, b\| = \|x - z, b\| = df_x(x - z, b). \]

\[ f_G(x, b) = \frac{1}{2}((\|x - y, b\| + \|x - z, b\|) = \frac{1}{2}(df_x(x - y, b) + df_x(x - z, b)) \]
\[ = df_x(x - \frac{y + z}{2}, b) \leq \|x - \frac{y + z}{2}, b\| \]
\[ \leq f_G(x, b). \]

Hence equality must hold throughout these inequalities. Since \( X \) is strictly convex 2-normed space and \( b \notin \text{Span}\{x, G\} \), it follows that \( F_G(x, b) \) has at most one element.

The properties of linear 2-normed spaces have been extensively studied by many authors. The same properties also hold in 2-inner product spaces, which were introduced by Diminnie et al [4].

**Definition 3.10.** Let \( X \) be a linear space. Suppose that \( \langle ., . \rangle \) is a \( R \) valued function defined on \( X \times X \times X \) satisfying the following conditions:

a) \( \langle x, x|z \rangle \geq 0 \) and \( \langle x, x|z \rangle = 0 \) if and only if \( x \) and \( z \) are linearly dependent.

b) \( \langle x, x|z \rangle = z, z|x \),

c) \( \langle x, y|z \rangle = \langle y, x|z \rangle , \)

d) \( \langle ax, x|z \rangle = a\langle x, x|z \rangle \) for any scalar \( a \in R \),

e) \( \langle x + x', y|z \rangle = \langle x, y|z \rangle + \langle x', y|z \rangle \).

\( \langle ., .|., . \rangle \) is called a 2-inner product and \( (X, \langle ., .|., . \rangle) \) is called a 2-inner product space (or a 2-perHilbert space).

In any given 2-inner product space \( (X, \langle ., .|., . \rangle) \), we can define a function \( \|., .\| \) on \( X \times X \) by
\[ \|x, z\| = \langle x, x|z \rangle^{\frac{1}{2}}. \]

Using the above properties, we can prove the Cauchy-Schwarz inequality
\[ \|\langle x, y|z \rangle\|^{\frac{1}{2}} \leq \langle x, x|z \rangle \langle y, y|z \rangle. \]
**Theorem 3.11.** Let $G$ be a bounded subset of 2-inner product space $X$, $x \in X$, and $y_0 \in G$. If $\langle x - y, y_0 - y \rangle b \leq 0$ for all $y \in G$, then $y_0 \in F_G(x, b)$.

**Proof.** Suppose that $\langle x - y, y_0 - y \rangle b \leq 0$ for all $y \in G$, then

$$\|x - y, b\| = \langle x - y, x - y \rangle b = \langle x - y, x - y_0 + y_0 - y \rangle b$$

$$= \langle x - y, x - y_0 \rangle b + \langle x - y, y_0 - y \rangle b$$

$$\leq \langle x - y, x - y_0 \rangle b \leq \|x - y, b\| \|x - y_0, b\|.$$ 

Hence $\|x - y, b\|^2 \leq \|x - y_0, b\|$ i.e. $y_0 \in F_G(x, b)$. □

**Definition 3.12.** A set $A$ in a 2-normed space $X$ is said to be b-strongly convex with constant $r > 0$ if there exists a set $A_1 \subseteq E$ such that

$$A = \cap_{a \in A_1} B_r^b(a),$$

where $B_r^b(a) = \{y \in X : \|x - a, b\| \leq r\}.$

A set $A$ is called a b-strongly convex set of radius $R > 0$ if this set is the intersection of balls of radius $R$.

In the following, we study uniqueness problem for a point of closed bounded set that is the farthest point from a given point in 2-inner product spaces.

**Lemma 3.13.** Let $G$ be a b-strongly convex set of radius $r > 0$ in the 2-inner product space $X$. Then the inequality

$$\|a_1 - a_2, b\|^2 \leq R\|a_1 - a_2, p_2 - p_1 | b\|,$$

holds for vectors $p_1, p_2$ such that $\|p_1, b\|, \|p_2, b\| \geq 1.$

**Proof.** We fix vectors $p_1, p_2$. According to the definition of strongly convex sets, we have

$$G \subseteq B_r^b(a_1 - R \frac{p_1}{\|p_1, b\|}) \cap B_r^b(a_2 - R \frac{p_2}{\|p_2, b\|}),$$

which implies the inequalities

$$\|a_2 - a_1 + R \frac{p_1}{\|p_1, b\|}, b\|^2 \leq R^2, \quad \|a_1 - a_2 + R \frac{p_2}{\|p_2, b\|}, b\|^2 \leq R^2$$

and hence

$$\|a_2 - a_1 + R \frac{p_1}{\|p_1, b\|}, b\|^2 = \langle a_2 - a_1 + R \frac{p_1}{\|p_1, b\|}, a_2 - a_1 + R \frac{p_1}{\|p_1, b\|} \rangle b,$$

$$= \langle a_2 - a_1, a_2 - a_1 | b \rangle + R \frac{p_1}{\|p_1, b\|} | R \frac{p_1}{\|p_1, b\|} | b \rangle + 2 \langle a_2 - a_1, R \frac{p_1}{\|p_1, b\|} | b \rangle \leq R^2,$$

and hence

$$\|a_1 - a_2, b\|^2 \leq 2R \langle a_1 - a_2, -p_1 | b \rangle$$

$$\|a_1 - a_2, b\|^2 \leq 2R \langle a_1 - a_2, p_2 | b \rangle.$$

We sum the last two inequalities and obtain the desired inequality. □
For a set $G$ in a 2-normed space $X$ and a number $r > 0$, we define the set
\[ T^b_r(G) = \{ x \in X : f_G(x, b) > r \}. \]

**Theorem 3.14.** Let $G$ be a $b$-strongly convex set of radius $r > 0$ in the 2-inner product space $X$. Then for $x_1, x_2 \in T^b_r(G)$ the inequality
\[ \| f_b(x_1) - f_b(x_2), b \|^2 \leq \frac{r}{R - r} \| x_1 - x_2, b \|, \]
holds for any $R > r$ and $f_b(x_i) \in F_G(x_i, b)$, $i = 1, 2$.

**Proof.** We choose a number $R > r$, and introduce the vectors
\[ p_i = \frac{1}{R} (f_b(x_i) - x_i), i = 1, 2. \]
From Lemma 3.13, we obtain
\[
\| f_b(x_1) - f_b(x_2), b \|^2 \\
\leq \ r (f_b(x_1) - f_b(x_2), p_2 - p_1 | b) \\
= \ r (f_b(x_1) - f_b(x_2), \frac{1}{R} (f_b(x_2) - x_2) - \frac{1}{R} (f_b(x_1) - x_1), | b) \\
= \ \frac{r}{R} \| f_b(x_1) - f_b(x_2), b \|^2 - \frac{r}{R} (f_b(x_1) - f_b(x_2), x_2 - x_1 | b).
\]
Hence by Cauchy-Schwarz inequality we get
\[
(1 - \frac{r}{R}) \| f_b(x_1) - f_b(x_2), b \|^2 \leq \frac{r}{R} \| f_b(x_1) - f_b(x_2), b \| \| x_1 - x_2, b \|.
\]
which implies formula (3.7).

**Corollary 3.15.** Let $G$ be a $b$-strongly convex set of radius $r > 0$ in the 2-inner product space $X$, $x \in T^b_R(G)$ and $b \notin \text{Span}\{G\}$. Then there is at most one $b$-farthest point in $G$ to $x$.

**Proof.** It is a consequence of Theorem 3.14. 

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Some results on farthest points in $2$-normed spaces


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