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TIME AND UNCERTAINTY IN OVERLAPPING GENERATIONS ECONOMIES

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Abstract. This paper shows the general reversibility of every perfect foresight equilibrium of an overlapping generations economy. It then shows and characterizes the existence of reversible sunspot equilibria in these economies as well, which seems to be at odds with our intuition about the irreversibility of a tree of events. Although the paper establishes also that such reversible stochastic equilibria constitute a negligible subset of all the equilibria of their class, their mere existence may be considered somewhat puzzling for this intuition.

1. Introduction

This paper intends to provide some insight about the way in which time and uncertainty get interwoven in the overlapping generations model of the economy ([18]). The conclusion that will be reached at the end of the paper is that, in most of the cases, the introduction of uncertainty changes qualitatively the nature of the model in a way that can give a meaning to the idea of the irreversibility of time: while there is a sense in which every perfect foresight equilibrium can be "read backwards", in general, stochastic equilibria cannot, which fits well our intuition that a tree of events cannot be reversed. Such an intuitive fact might be hardly noteworthy were not true as well, maybe surprisingly this time, that there are actually reversible stochastic equilibria in the overlapping generations economies. I characterize in what follows a family of them and show how they relate to specific perfect foresight equilibria, the cycles of period 2. Fortunately for our intuition, these reversible stochastic equilibria will prove to be really very few in a sensible way. Nevertheless, their mere existence challenges interestingly our intuition on the irreversibility of a tree of events.

What the next sections develop in detail can be expressed casually as follows. If we consider any perfect foresight equilibrium of a simple overlapping generations

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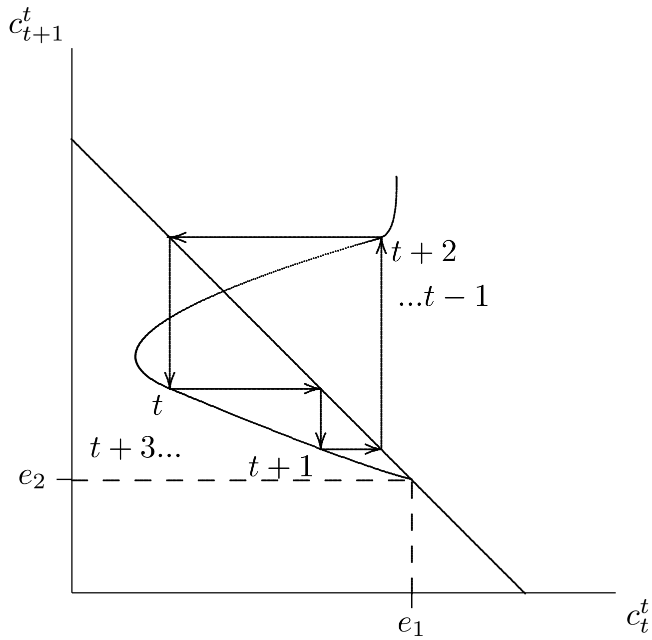
economy,[1] say a cycle of period 3 to fix ideas, its allocation of resources can be readily identified to a set of points laying on the offer curve of the representative agent connected by arrows going, first, horizontally from any given generation's intertemporal profile of consumption to the line of slope -1 and going through the endowments point of the representative agent, and then vertically from this line to the next generation's intertemporal profile of consumption[2] (see Figure 1).

Figure 1



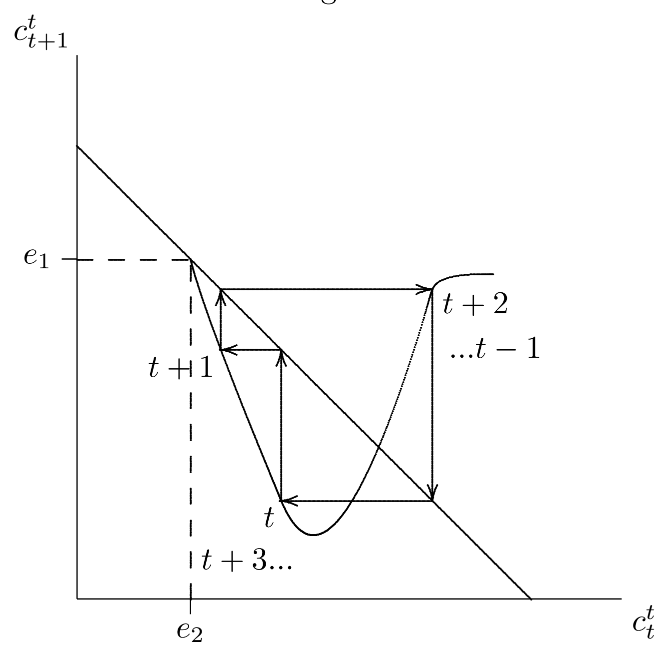
If we now consider the economy that is a sort of mirror image of the previous one, that is to say, the overlapping generations economy whose representative agent mirrors the previous representative agent (more specifically, his endowments and offer curve are the symmetric image across the 45 degrees line of those of the original economy, see Figure 2), then the correspondingly mirrored allocation of resources is not an equilibrium allocation of the new economy. In effect, although the disposition of the intertemporal profiles of consumption on the offer curve is inherited, the way in which they should be connected is not: the arrows just point in the wrong direction. This means that this allocation is not even feasible in the new economy and, so, it could hardly be an equilibrium one. Having said this, we could just think of reversing the arrows in order to make this an equilibrium allocation of the new economy. Now, the consequence would be that the generation following generation t is not $t+1$, but $t-1$. Put in other words, there is a one-to-one correspondence between the perfect foresight equilibria of one of the economies and the equilibria of the other economy with the direction in which time flows reversed.[3]

[1] An economy consisting of a never-ending sequence of agents, indexed by the (negative and positive) integers, identical (up to a time shift in their preferences and endowments) to a representative agent living for two consecutive periods, and with a single dated commodity and no production.

[2] Making these points lay on the offer curve takes care of the individual rationality of the agents' decisions, while their connection in the way just described accounts for the feasibility of the allocation of resources.

[3] See Proposition 2 in Section 2.

Figure 2



The situation changes quite radically, as we might expect, when uncertainty enters the stage. To see why consider the sunspot equilibrium allocation depicted in Figure 3.[4] In this equilibrium, the partition of each period resources between the young and the old fluctuates between six possible partitions, clustered by pairs around the three states of the previous cycle of period 3. The fluctuations are such that from any of the two states with the highest levels of savings (those labeled 1 and 2 in Figure 3) the economy moves, randomly, to one of the two states with medium level savings (3 and 4) and from there, randomly again, two one of the two states with the lowest savings (5 and 6). After that, a new "cycle" starts at one, randomly chosen, of the two high-savings states. This looks pretty much like a cycle of period 3 but is not quite so, since instead of repeating itself exactly every three periods, the allocation is, at any date, only "not far" from where it was three periods before. Why this can be the result of a sunspot equilibrium is a matter to be explained more at length below.[5]Let me only say now that the crucial fact for it to be so is that the offer curve has a slope smaller that 1 in absolute value at each of the three intertemporal profiles of consumption of the pure cycle close to which the sunspot equilibrium fluctuates. In effect, this makes possible for the offer curve to separate the two intertemporal profiles of consumption, contingent to the realization of the uncertainty when old, that every generation may ever obtain (in Figure 3, this means that for each of the three small boxes that the allocation determines around each point of the pure cycle, the offer curve separates top corners from bottom corners).

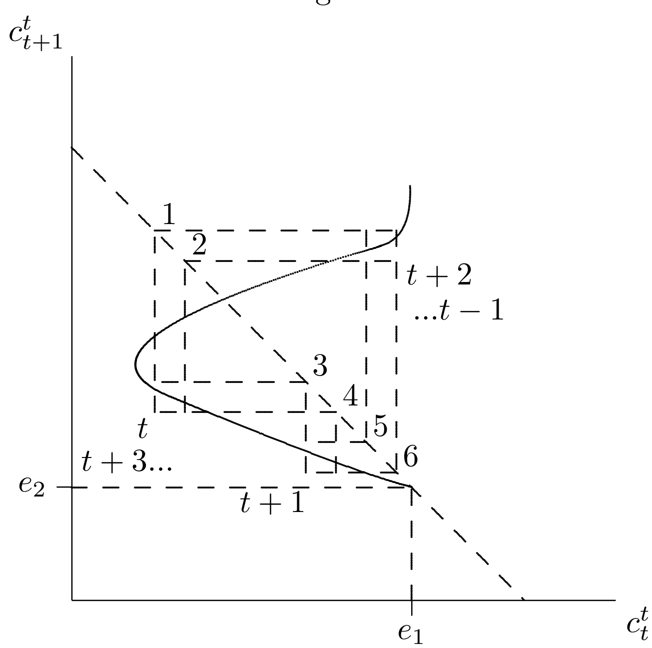
Notice that, on the one hand, the approximately cyclical dynamics described above accounts for the feasibility of the allocation of resources and, on the other hand, the separation property accounts for the individual rationality of every agent's choice. In effect, should any agent choose intertemporal profiles of consumption

[4]For characterizations of the existence on sunspot equilibria ([4]) in an overlapping generations economies see [2], [8], [13], [14], [17], [21], [23].

[5]See Proposition 3 in Section 3 and the subsequent discussion.

contingent to the value taken by the sunspot when old that lay on the same side of the offer curve, i.e. for which the marginal rate of substitution is in any event bigger (respectively smaller) than the relative price of consumption when young and old, then a slightly higher (resp. lower) level of consumption when young would increase his expected utility. Incidentally, notice that in order to exhibit a stationary sunspot equilibrium in which the generation hold positive savings, the samuelsonian case using the terminology of [12], necessarily the income and substitution effects must work in opposite directions for the offer curve to bend backwards and thus the separation condition to be satisfied by a feasible allocation. On the contrary, there is no need for such condition on income and substitution effects or, equivalently, any specific condition on the shape of the offer curve in order to produce stationary sunspot equilibria in the classical case (see [8]).

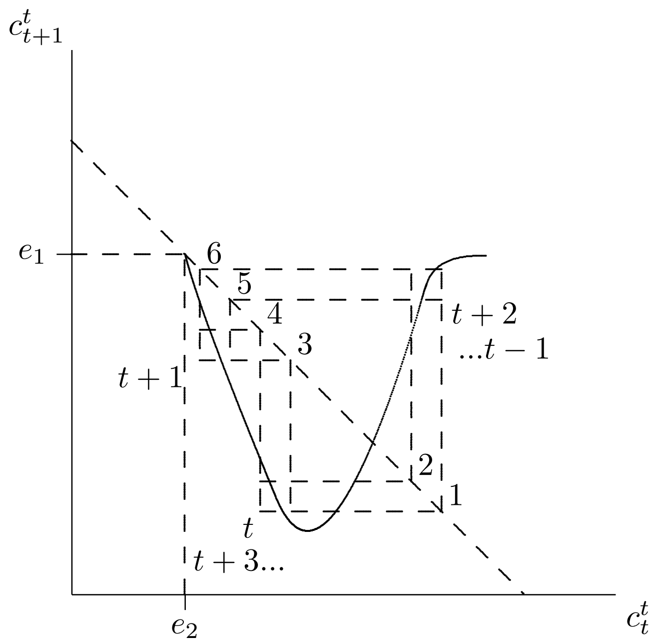
Figure 3



Now, should we flip the whole picture around the 45 degrees line as in Figure 4 in order to see whether the mirror allocation can be an equilibrium one of the mirror economy, then we would get into two problems. First, as in the case without uncertainty, the "approximate cycle" turns in the wrong direction: the clockwise direction. We could hope that reversing the direction of time, letting Mr t be followed by Mr $t-1$ instead of Mr $t+1$ would recover in this case too the feasibility of the allocation of resources, as it actually does. But there is an additional problem now, and it is that during the trip from one economy to the mirror one we not only lost the feasibility, but also the individual rationality of each generation's choice: the offer curve does not separate any longer in Figure 4 the contingent profiles of consumption with the same consumption when young of any generation (it separates now the left and right corners of the small boxes, but unfortunately this does not help much to make the allocation individually rational). While the feasibility could be recovered reversing time, there is no way of making the mirror allocation of this example individually rational in the mirror economy.

The lack of reversibility of sunspot equilibria could actually be guessed intuitively from the fact that a necessary and sufficient condition for the existence of local sunspot equilibria in a stationary overlapping generations economy is the indeterminacy of its steady state. Since the "mirror" transformation of such an economy makes of this steady state a determinate one, no local sunspot equilibrium can survive the transformation. Nevertheless, this "local" argument says nothing about the existence or not of reversible "global" sunspot equilibria sufficiently away from the steady state. The investigation undertaken in this paper addresses precisely this question.

Figure 4



As shown by the previous discussion, no link exists in general between sunspot equilibria of the kind of an overlapping generations economy and the equilibria of its mirror image with time reversed. I say "in general" because, as a matter of fact, there are sunspot equilibria that can be reversed indeed, and the condition characterizing the existence of such sunspot equilibria is that the economy has a cycle of period 2.[6] This does not seem to be too demanding and, hence, the existence of such reversible stochastic equilibria to be a terribly unlikely outcome, which would be at odds with what our intuition tells us about the role of uncertainty in determining a definite direction for time by means of an unfolding tree of events. Nonetheless, our intuition will soon get reassured by the fact that whenever reversible sunspot equilibria appear, irreversible ones do appear too, and they do it in such amount that makes the reversible ones constitute a negligible subset in comparison to the rest and, hence, to lack any likelihood to emerge.

In the rest of the paper, Section 2 develops the model without uncertainty, characterizes its perfect foresight equilibria and establishes the general reversibility of these equilibria. Section 3 introduces extrinsic uncertainty in the model, characterizes the new set of equilibria and shows how to use this characterization to produce allocations of sunspot equilibria following a finite state Markov chain in a stationary overlapping generations economy. Section 4 exhibits a special example of a sunspot equilibrium of this class that is reversible. Section 5 provides a necessary

[6] See Proposition 5 in section 5.

characterization of the finite state Markovian stationary sunspot equilibria of a stationary overlapping generations economy that are reversible, as well as a sufficient characterization of the economies that exhibit them. Section 6 shows that these reversible sunspot equilibria are nonetheless negligible in comparison with all the equilibria of their class. Some concluding remarks can be found in Section 7 and, finally, the appendix collects proofs and lemmas.

2. The Model without Uncertainty

For each $t \in \mathbb{Z}$, let u^t be a utility function of the set[7] U of utility functions that are continuous on $\mathbb{R}^2_+$, twice continuously differentiable on $\mathbb{R}^2_{++}$, monotone,[8] strictly quasi-concave[9] and well-behaved in the boundary,[10] and e^t be in $\mathbb{R}^2_+$ but distinct from 0.

Let $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ denote the overlapping generations economy whose consumers of the generation born at date t are all identical to a representative agent characterized by the preferences represented by u^t and the endowments e^t. Thus the consumers are identical within generations but may differ across generations

An equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ consists of an allocation of resources $\{c^t\}_{t\in\mathbb{Z}}$ and positive prices $\{p_t\}_{t\in\mathbb{Z}}$ such that,

(1) for all $t \in \mathbb{Z}$, c^t solves

$$\begin{aligned} &\max_{c^t \in R^2_+} u^t(c^t) \\ &p^t(c^t - e^t) \leq 0 \end{aligned} \tag{1}$$

where $p^t = (p_t, p_{t+1})$, and

(2) the allocation of resources is feasible, i.e. for all $t \in \mathbb{Z}$,

$$c_2^{t-1} + c_1^t = e_2^{t-1} + e_1^t. \tag{2}$$

The following proposition characterizes completely the equilibrium allocations of resources of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.

Proposition 1.

(1) *If $\{c^t\}_{t\in\mathbb{Z}}$ is an equilibrium allocation of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, then for all $t \in \mathbb{Z}$,*

$$du^t_{c^t}(c^t - e^t) = 0. \tag{3}$$

(2) *If $\{c^t\}_{t\in\mathbb{Z}}$ satisfies* (3) *and*

$$c_2^{t-1} + c_1^t = e_2^{t-1} + e_1^t, \tag{4}$$

for all $t \in \mathbb{Z}$, then it is an equilibrium allocation of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$.

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[7] Endowed with the C^2 uniform convergence on compacta.

[8] In the sense that $Du^t(c^t)$ is always in the strictly positive orthant.

[9] In the sense that $D^2u^t(c^t)$ is always definite negative in the subspace orthogonal to the gradient $Du^t(c^t)$.

[10] In the sense that the limits of $D_1u^t(c^t)$ and $D_2u^t(c^t)$ at any $(c_1^t, 0)$ distinct from the origin are zero and positive respectively, while at any $(0, c_2^t)$ distinct from the origin are positive and zero respectively.

Condition (3) is nothing else than the equalization of the intertemporal marginal rate of substitution and the real rate of interest, expressed as the orthogonality of the gradient of u and the excess demand of every generation ($du^t_{c^t}$ is the differential of u^t at c^t), and condition (4) is the feasibility condition for the allocation of resources. The proof of Proposition 1 is straightforward and is relegated to the appendix.

Now, for each $t \in \mathbb{Z}$, let $\tilde{u}^t$ and $\tilde{e}^t$ be $u^t \circ \rho$ and $\rho(e^t)$ respectively, where ρ is the permutation in $\mathbb{R}^2$ with matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

Notice that, since the effect of the permutation ρ is to exchange the entries of the vector e^t and the arguments of the function u^t, each consumer $(\tilde{u}^t, \tilde{e}^t)$ is, so to speak, the mirror image of the corresponding consumer (u^t, e^t), i.e. its symmetrical counterpart with respect to the diagonal of his consumption space $\mathbb{R}^2_+$.

In the following proposition I establish a one-to-one identification of the equilibria of the overlapping generations $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ to those of the associated overlapping generations $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$.

Proposition 2. *If $\{c^t\}_{t\in\mathbb{Z}}$ is an equilibrium allocation of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, then $\{\rho(c^{-t})\}_{t\in\mathbb{Z}}$ is an equilibrium allocation of the overlapping generations economy $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$.*

Proof. Since, for all $t \in \mathbb{Z}$,

$$\begin{aligned} d\tilde{u}^{-t}_{\rho(c^{-t})}(\rho(c^{-t}) - \tilde{e}^{-t}) &= \\ d(u^{-t} \circ \rho)_{\rho(c^{-t})}(\rho(c^{-t}) - \rho(e^{-t})) &= \\ (du^{-t}_{c^{-t}} \circ \rho)(\rho(c^{-t} - e^{-t})) &= \\ du^{-t}_{c^{-t}}(c^{-t} - e^{-t}) &= 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \rho_2(c^{-(t-1)}) + \rho_1(c^{-t}) &= \\ c_1^{-(t-1)} + c_2^{-t} &= \\ c_2^{-t} + c_1^{-t+1} &= e_2^{-t} + e_1^{-t+1} \\ &= e_1^{-(t-1)} + e_2^{-t} \\ &= \tilde{e}_2^{-(t-1)} + e_1^{-t}, \end{aligned} \tag{7}$$

then the conclusion follows from (2) of Proposition 1. Q.E.D.

A few remarks are due now to try to make clear what the economy $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$ represents. Notice that in considering this economy we are reversing in $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ both the ordering with respect to t and the coordinates and arguments of each e^t and u^t respectively (generation t is $(\tilde{u}^{-t}, \tilde{e}^{-t})$). Thus the overlapping generations economy $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$ is a sort of image of the economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ through a looking-glass. The precise sense in which this is so can be better understood considering the prices $\{\tilde{p}_t\}_{t\in\mathbb{Z}}$ supporting the allocation $\{\rho(c^{-t})\}_{t\in\mathbb{Z}}$ as an equilibrium allocation of the economy $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$. These prices are,[11] for any given $\tilde{p}_0 > 0$,

$$\tilde{p}_t = -\frac{\rho_1(c^0) - \tilde{e}_1^0}{\rho_2(c^{-(t-1)}) - \tilde{e}_2^{-(t-1)}} \cdot \tilde{p}_0 \tag{8}$$

for each $t \in \mathbb{Z}$, where ρ_1 and ρ_2 denote the coordinate functions of the mapping ρ. Thus we have that, on the one hand (rewriting (8) for $-t+1$ and recalling the effect of ρ on any vector of $\mathbb{R}^2$),

$$\tilde{p}_{-t+1} = -\frac{c_2^0 - e_2^0}{c_1^t - e_1^t} \cdot \tilde{p}_0 \tag{9}$$

and, on the other hand (recalling the prices that support $\{c^t\}_{t\in\mathbb{Z}}$ as an equilibrium allocation of the economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$),

$$p_t = -\frac{c_1^0 - e_1^0}{c_2^{t-1} - e_2^{t-1}} \cdot p_0, \tag{10}$$

for any given $p_0 > 0$. Since $c_2^{t-1} - e_2^{t-1} = -(c_1^t - e_1^t)$ holds at any date because of the feasibility of the allocation of resources, the prices $\{\tilde{p}_t\}_{t\in\mathbb{Z}}$ are therefore linked to the prices $\{p_t\}_{t\in\mathbb{Z}}$ supporting the allocation $\{c^t\}_{t\in\mathbb{Z}}$ as an equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, as follows

$$\tilde{p}_t = -\frac{\tilde{p}_0}{p_0}\frac{c_2^0 - e_2^0}{c_1^0 - e_1^0} p_{-t+1}. \tag{11}$$

As a consequence, the problem faced by the generation born at date t of the economy $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$ is actually the same one faced by the generation $-t$ of $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$. In effect,

$$\begin{gathered} \max_{\tilde{c}^t \in R_+^2} \tilde{u}^{-t}(\tilde{c}^t) \\ \tilde{p}_t(\tilde{c}_1^t - \tilde{e}_1^{-t}) + \tilde{p}_{t+1}(\tilde{c}_2^t - \tilde{e}_2^{-t}) = 0 \end{gathered} \tag{12}$$

where $\tilde{c}^t$ stands for $\rho(c^{-t})$ actually, is, as a matter of fact,

$$\begin{gathered} \max_{c^{-t} \in R_+^2} u^{-t}(c^{-t}) \\ \tilde{p}_{t+1}(c_1^{-t} - e_1^{-t}) + \tilde{p}_t(c_2^{-t} - e_2^{-t}) = 0. \end{gathered} \tag{13}$$

Now, we have just seen in (11) that $\tilde{p}_t = -\frac{\tilde{p}_0}{p_0}\frac{c_2^0 - e_2^0}{c_1^0 - e_1^0} p_{-t+1}$ and $\tilde{p}_{t+1} = -\frac{\tilde{p}_0}{p_0}\frac{c_2^0 - e_2^0}{c_1^0 - e_1^0} p_{-t}$, which substituted in this problem gives

$$\begin{gathered} \max_{c^{-t} \in R_+^2} u^{-t}(c^{-t}) \\ p_{-t}(c_1^{-t} - e_1^{-t}) + p_{-t+1}(c_2^{-t} - e_2^{-t}) = 0. \end{gathered} \tag{14}$$

[11] See the proof of Proposition 1 in the Appendix in order to see how to get prices supporting an equilibrium allocation.

This is the problem faced by the generation $-t$ of the economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$. Therefore, the problem of generation $t+1$ in $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$ is that of generation $-t-1$ of $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, and so on. Hence, the effect of ρ on the economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ is that of reversing the direction in which time flows in the model, and what Proposition 2 states is, roughly speaking, that any equilibrium of the economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$ can actually be read backwards in $\{(\tilde{u}^{-t}, \tilde{e}^{-t})\}_{t\in\mathbb{Z}}$.

In the particular case in which the overlapping generations economy is stationary, i.e. for an economy $\{(u, e)\}_{t\in\mathbb{Z}}$ or rather (u, e) from now on, this result can actually be seen more easily looking at the effect of the "mirror" transformation on the dynamics of any equilibrium allocation (see, for instance, Figure 1). Its symmetric image with respect to the diagonal (the result of applying the permutation ρ to utilities and endowments; see Figure 2) delivers an allocation of resources that is not an equilibrium one if the current labeling of time periods is maintained (namely the feasibility condition does not hold). Nonetheless, it becomes an equilibrium allocation once the flowing of time is reversed.

In effect, if we think of the allocation of resources to be implemented by means of a fixed amount of outside money m and, hence, the problem of the representative agent is written as

$$\begin{aligned} &\max_{0\leq c_1^t, c_2^t} u(c_1^t, c_2^t) \\ &p_t c_1^t + m = p_t e_1 \\ &p_{t+1} c_2^t = p_{t+1} e_2 + m \end{aligned} \tag{15}$$

the equilibrium dynamics is completely characterized by the first order condition

$$\theta_t D_1 u(e_1 - \theta_t, e_2 + \theta_{t+1}) = \theta_{t+1} D_2 u(e_1 - \theta_t, e_2 + \theta_{t+1}) \tag{16}$$

where θ_t is the real monetary holdings at date t, i.e. m/p_t. Considering the "mirror" economy of this one the representative agent's problem is

$$\begin{aligned} &\max_{0\leq c_1^t, c_2^t} u(c_2^t, c_1^t) \\ &p_t c_1^t + m = p_t e_2 \\ &p_{t+1} c_2^t = p_{t+1} e_1 + m \end{aligned} \tag{17}$$

and the corresponding equilibrium dynamics

$$\theta_t D_2 u(e_1 + \theta_{t+1}, e_2 - \theta_t) = \theta_{t+1} D_1 u(e_1 + \theta_{t+1}, e_2 - \theta_t). \tag{18}$$

which happens to be the same dynamics (16) as before but for $-\theta_t$, i.e. the intergenerational debt in real terms, with time reversed.

3. The Model with Uncertainty

Consider now this same economy when there is uncertainty about the realization of a sunspot signal, i.e. an extrinsic uncertainty[12] that takes at each period a value

[12] By extrinsic uncertainty it is meant the uncertainty about the realization of states of the world with respect to which the fundamentals of the economy (preferences and endowments here) are constant. On the irrelevancy of the extrinsic uncertainty in the standard Arrow-Debreu framework, see the so-called Ineffectivity Theorem in [4]. On the relevance of such type of uncertainty in equilibria of different departures from the Arrow-Debreu World, see the literature on sunspot equilibria ([20]) with incomplete markets, non-convexities, infinite economies, etc., e.g. in [7], [15] and the citations there.

σ_t in the set $\{1,\ldots,k\}$. Letting $s=(\ldots,s_{t-1},s_t,s_{t+1},\ldots)$ in $S=\{1,\ldots,k\}^{\mathbb{Z}}$ be a realization of the uncertainty, and $s_t=(\ldots,\sigma_{t-1},\sigma_t)$ be a history up to t, let $P(s'_{t+1}|s'_t=s_t)$ denote the probability, conditional to a history up to t, of a particular continuation of this history next period. An equilibrium of the overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ with such a publicly observed sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$, consists of an allocation of resources $\{c^t_{s_{t+1}}\}_{t\in\mathbb{Z},s\in S}$ such that each, for any $t\in\mathbb{Z}$ and $s\in S$, $c^t_{1s_{t+1}}$ does not depend on σ_{t+1} (since generation t makes its decision before σ_{t+1} is observed), and positive prices $\{p_{ts_t}\}_{t\in\mathbb{Z},s\in S}$, verifying

(1) for all $t\in\mathbb{Z}$ and all $s\in S$, $(c^t_{s'_{t+1}})_{s'_{t+1}|s'_t=s_t}$ solves

$$\begin{aligned}&\max_{\substack{c^t_{s'_{t+1}}\in R^2_+\\ s'_{t+1}|s'_t=s_t}} \sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t=s_t)u^t(c^t_{s'_{t+1}})\\ &p^t_{s'_{t+1}}(c^t_{s'_{t+1}}-e^t)\leq 0\ ,\ s'_{t+1}|s'_t=s_t\end{aligned} \tag{19}$$

where $p^t_{s'_{t+1}}=(p_{ts_t},p_{t+1s'_{t+1}})$, and

(2) the allocation of resources is feasible, i.e. for all $t\in\mathbb{Z}$ and all $s\in S$,

$$c^{t-1}_{2s_t}+c^t_{1s_{t+1}}=e^{t-1}_2+e^t_1. \tag{20}$$

The equilibrium allocations of this economy are thus characterized again by, on the one hand, the expected equalization at each date of the intertemporal marginal rate of substitution and the real rate of interest supporting the allocation of resources and, on the other hand, the feasibility condition for any realization of the uncertainty, which can be stated as follows (the proof is relegated to the appendix as well).

Proposition 3.

(1) *If an allocation $\{c^t_{s_{t+1}}\}_{t\in\mathbb{Z},s\in S}$ such that each $c^t_{1s_{t+1}}$ does not depend on σ_{t+1}, and the prices $\{p_{ts_t}\}_{t\in\mathbb{Z},s\in S}$ constitute an equilibrium of the overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$, then for all $s\in S$ and all $t\in\mathbb{Z}$,*

$$\sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t=s_t)du^t_{c^t_{s'_{t+1}}}(c^t_{s'_{t+1}}-e^t)=0. \tag{21}$$

(2) *If an allocation of resources $\{c^t_{s_{t+1}}\}_{t\in\mathbb{Z},s\in S}$ such that each $c^t_{1s_{t+1}}$ does not depend on σ_{t+1}, satisfies* (21) *and the feasibility condition*

$$c^{t-1}_{2s_t}+c^t_{1s_{t+1}}=e^{t-1}_2+e^t_1 \tag{22}$$

for all $s\in S$ and all $t\in\mathbb{Z}$, then it is an equilibrium allocation of the overlapping generations economy $\{(u^t,e^t)\}_{t\in\mathbb{Z}}$ with the sunspot signal $\{\sigma_t\}_{t\in\mathbb{Z}}$.

If an equilibrium allocation of resources happens to depend effectively on s, then the corresponding equilibrium is said to be a sunspot equilibrium;[13] otherwise,

[13]For results on the existence of this type of equilibria in overlapping generations economies see [1], [2], [8], [13], [14], [17], [21] and more generally [5], [9] and [23].

it can be identified with a perfect foresight equilibrium of the economy without uncertainty.

When the overlapping generations economy is stationary, a sunspot equilibrium allocation particularly easy to characterize is that of an equilibrium that is stationary itself, depends only on the current realization of the signal σ_t instead of on the entire history of realizations up to some date, and the stochastic process followed by the signal is Markovian. Then the allocation $\{c^t_{s_{t+1}}\}_{t\in\mathbb{Z},s\in S}$ is characterized by just k possible consumptions when young c^i_1, one for each possible realization of the signal when young $i = 1, \ldots, k$, and similarly k possible consumptions when old c^j_2, such that *i*) they are feasible no matter what is the value of the signal observed at any date, and *ii*) not all the intertemporal profiles of consumption with the same consumption when young are on the same side of the offer curve of the representative agent. This geometrical characterization follows from the requirement in equation (21) that each agent equalizes, in expected value, his marginal rate of substitution between consumption today and consumption tomorrow to the real interest rate: a choice of intertemporal profiles of consumption contingent to tomorrow's sunspot, whose marginal rates of substitution are consistently bigger (respectively, smaller) than the corresponding real interest rate and, hence, on the same side of the offer curve (where the equality holds), could not be individually rational since it would imply that the representative agent consistently values more the consumption when young (respectively, when old), than what the market does, leaving thus room for further exchanges that would improve his expected utility. Therefore, for a stationary economy, the signs of the differentials $du^t_{c^t_{s'_{t+1}}}(c^t_{s'_{t+1}} - e^t)$, i.e. the inner products of the gradients of utility $Du(c_1^{\sigma_t}, c_2^{\sigma_{t+1}})$ and the excess demands $(c_1^{\sigma_t} - e^t_1, c_2^{\sigma_{t+1}} - e_2)$ corresponding to each possible sequence of realizations of the signal (σ_t, σ_{t+1}) for every generation t, cannot be all the same for any given consumption when young c^i_1. Thus zero can be written as a convex linear combination of these inner products, and then the weights be taken as the probabilities of transition between values of the sunspot signal. An example of such a sunspot equilibrium allocation is shown in Figure 3, where the consumptions when young contingent to the sunspot are given by the abscissas of the points 1 through 6, the consumptions when old by their ordinates, and the only non zero probabilities of transition are those from 1 and 2 to 3 or 4, from 3 and 4 to 5 or 6, and from 5 and 6 to 1 or 2. Now an example of a reversible sunspot equilibrium of this kind (although somewhat special) follows.

4. An example of reversible sunspot equilibrium

Consider a stationary overlapping generations economy (u, e) and a sunspot equilibrium that follows a finite state first-order Markov process, i.e. a finite Markovian stationary sunspot equilibrium.[14] If $m^{\sigma_t\sigma_{t+1}}$ denotes the probability of transition from state σ_t to state σ_{t+1}, the equilibrium allocation $\{(c_1^{\sigma_t}, (c_2^{\sigma_{t+1}})^k_{\sigma_{t+1}=1})^k_{\sigma_t=1}\}$

[14] A so-called k-SSE, for Stationary Sunspot Equilibrium of order k, following [6], [14].

must satisfy, for all $i = 1, \ldots, k$

$$\sum_{j=1}^{k} m^{ij} \big(D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^j - e_2) \big) = 0, \tag{23}$$
$$c_1^i + c_2^i = e_1 + e_2$$

Similarly, for the symmetrical allocation $\{(c_2^{\sigma_t}, (c_1^{\sigma_{t+1}})_{\sigma_{t+1}=1}^k)_{\sigma_t=1}^k\}_{t \in \mathbb{Z}}$ to be the allocation of a sunspot equilibrium of its mirror economy $(\tilde{u}, \tilde{e})$, perfectly correlated to the same sunspot signal, it has to satisfy, for all $i = 1, \ldots, k$, the equations

$$\sum_{j=1}^{k} m^{ij} \big(D_1 \tilde{u}(c_2^i, c_1^j)(c_2^i - \tilde{e}_1) + D_2 \tilde{u}(c_2^i, c_1^j)(c_1^j - \tilde{e}_2) \big) = 0, \tag{24}$$
$$c_2^i + c_1^j = \tilde{e}_1 + \tilde{e}_2$$

that is to say, the equations, for all $i = 1, \ldots, k$,

$$\sum_{j=1}^{k} m^{ij} \big(D_1 u(c_1^j, c_2^i)(c_1^j - e_1) + D_2 u(c_1^j, c_2^i)(c_2^j - e_2) \big) = 0, \tag{25}$$
$$c_1^i + c_2^j = e_1 + e_2$$

(notice in (25) the reversed order of the signals in the probabilities of transition and the consumptions when old and young). Therefore, a reversible finite Markovian stationary sunspot equilibrium of a stationary overlapping generations economy (u, e) consists of an allocation $\{(c_1^{\sigma_t}, (c_2^{\sigma_{t+1}})_{\sigma_{t+1}=1}^k)_{\sigma_t=1}^k\}_{t \in \mathbb{Z}}$ and a matrix of probabilities of transition m satisfying (23) and (25) simultaneously. In order to produce such an equilibrium, let e_1 be positive and $c_1^1, \ldots, c_1^k$ be in the open interval $(0, e_1)$, all distinct, c_2^i be $e_1 + e_2 - c_1^i$, for all $i = 1, \ldots, k$, and all such that if $i \neq j$ and $(i', j') \neq (i, j)$, then

$$\frac{c_2^j - e_2}{e_1 - c_1^i} \neq \frac{c_2^{j'} - e_2}{e_1 - c_1^{i'}}. \tag{26}$$

This means that, (i, j) being an arbitrary sequence of consecutive values for the sunspot, there is no such sequence of distinct sunspot signals for which the corresponding real interest rate determined by the allocation of resources, coincides with those of any other sequence of distinct values of the sunspot. Put in other words, the sequence of real interest rates supporting this allocation necessarily changes every period in which it is not zero. Graphically, this means that, any two contingent intertemporal profiles of consumption located off the line $c_1 + c_2 = e_1 + e_2$, lay in distinct rays starting from the endowments. This fact allows to draw an offer curve going through all the profiles of consumption off the line going through the endowment point e with slope -1, a key property for our argument.

In effect, if u is such that the gradient of the utility at the consumption (c_1^i, c_2^j), for all $i \neq j$, is orthogonal to the associated excess of demand, i.e.

$$D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^j - e_2) = 0, \tag{27}$$

(the existence of a utility function u in U with this property and, hence, the existence of an offer curve going through all the consumption points off the line $c_1 + c_2 = e_1 + e_2$, is guaranteed by the fulfillment of (26)) and m is a Markov matrix with a diagonal of zeros, then, whichever are the gradients of u at the consumption points (c_1^i, c_2^i) on the line $c_1 + c_2 = e_1 + e_2$, for all $i = 1, \ldots, k$, it holds that, for all $i, j = 1, \ldots, k$,

$$\begin{aligned} m^{ij}\big(D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^j - e_2)\big) = 0 \\ m^{ji}\big(D_1 u(c_1^i, c_2^j)(c_1^i - e_1) + D_2 u(c_1^i, c_2^j)(c_2^j - e_2)\big) = 0 \end{aligned} \tag{28}$$

either because the inner product in parentheses is null or because the probability of transition is null. Hence, conditions (23) and (25) are simultaneously satisfied. Such an allocation $\{(c_1^{\sigma_t}, (c_2^{\sigma_{t+1}})_{\sigma_{t+1}=1}^k)_{\sigma_t=1}^k\}_{t\in\mathbb{Z}}$ is a sunspot equilibrium allocation of the economy (u, e) while the symmetric counterpart $\{(c_2^{\sigma_t}, (c_1^{\sigma_{t+1}})_{\sigma_{t+1}=1}^k)_{\sigma_t=1}^k\}_{t\in\mathbb{Z}}$ is a sunspot equilibrium allocation of its mirror economy $(\tilde{u}, \tilde{e})$ with the same sunspot signal.

The previous example is clearly an extreme one, since the offer curve is such that every contingent bundle off the feasibility line is on the representative agent's offer curve. This is what allows us not to worry about the impact of such bundles on the fulfillment of the first order conditions characterizing the equilibrium allocation. As a matter of fact, only the contingent bundles on the feasibility line matter for this purpose, but for these we are free to choose probabilities equal to zero, as long as the probability mass is spread over all the rest of the transitions. Nonetheless, the following section characterizes the existence of less trivial reversible sunspot equilibria.

5. Reversible finite Markovian stationary sunspot equilibria

In general, the equilibrium allocation of a finite state Markovian stationary sunspot equilibrium of a stationary overlapping generations economy (u, e) is such that its symmetrical allocation is that of an equilibrium of the mirror economy $(\tilde{u}, \tilde{e})$ too if, as the whole picture of the representative agent flips around the diagonal (i.e. contingent bundles, endowments and offer curve), the set of contingent consumptions when old are still not strictly on the same side of the new offer curve, for each sunspot value that may have been observed when young. This separation property is what actually characterizes the allocation of a sunspot equilibrium. In effect, the offer curve separates the bundles for which the marginal rate of substitution is greater than the real interest rate needed to support them, from those for which it is smaller. Thus, in order to equalize in mathematical expectation the marginal rate of substitution to the real rate of interest, the representative agent has to choose consumptions when old contingent to the sunspot across the offer curve for each of the possible consumptions when young (unless the intertemporal profiles of consumption lay on the offer curve actually, as in the previous example).

The following proposition shows, that a necessary condition for this separation property to hold both for a sunspot-contingent feasible allocation in the economy (u, e) and for its symmetrical counterpart in $(\tilde{u}, \tilde{e})$ is that, if consumption is at any age a normal good, the highest and lowest levels of consumption at equilibrium are those of a cycle of period 2. Moreover, the existence of such a cycle turns out to be sufficient for the existence of reversible sunspot equilibria of this kind.

Proposition 5. *If consumption is a normal good for the representative agent in both periods of life, then*

(1) *for any finite Markovian stationary sunspot equilibrium of a stationary overlapping generations economy (u, e) that is reversible (i.e. its symmetrical allocation across the diagonal is an equilibrium allocation of the symmetrical economy $(\tilde{u}, \tilde{e})$), its highest an lowest states are those of a cycle of period 2, and*

(2) *if an overlapping generations economy (u, e) has a cycle of period 2, then it has reversible finite Markovian stationary sunspot equilibria.*

Proof. The following proof is for the "Samuelsonian" case, following the terminology used in [12], in which the marginal rate of substitution between current and future consumption is, at the endowments point e, smaller than 1, and, hence, the generations want to save when young in order to consume when old more than their endowment. The modifications needed to produce a proof for the classical case, in which this marginal rate of substitution is bigger than 1, are straightforward.

(1) Let $\{(c_1^{\sigma_t}, (c_2^{\sigma_{t+1}})_{\sigma_{t+1}=1}^k)_{\sigma_t=1}^k\}_{t\in\mathbb{Z}}$ be the allocation of a finite Markovian stationary sunspot equilibrium of the stationary overlapping generations economy (u, e), with $c_1^1 < \cdots < c_1^i < \cdots < c_1^k$, without loss of generality, (and hence $c_2^1 > \cdots > c_2^j > \cdots > c_2^k$). Assume moreover that this allocation is such that its symmetrical image across the 45 degrees line $\{(c_2^{\sigma_t}, (c_1^{\sigma_{t+1}})_{\sigma_{t+1}=1}^k)_{\sigma_t=1}^k\}_{t\in\mathbb{Z}}$ is an equilibrium allocation of the mirror economy $(\tilde{u}, \tilde{e})$ as well. If, for the sake of readability, we let $f_{ue}^{ij}(c)$ stand for $du_{c_1^i, c_2^j}(c_1^i - e_1, c_2^j - e_2)$ for any given $c = (c_1, c_2) = (c_1^1, \ldots, c_1^k, c_2^1, \ldots, c_2^k)$ and all $i, j = 1, \ldots, k$, then, according to Proposition 3, zero must be in the convex hull of the set of values $\{f_{ue}^{ij}(c)\}_{j=1}^k$ for all $i = 1, \ldots, k$ as well as in the convex hull of $\{f_{ue}^{ij}(c)\}_{i=1}^k$ for all $j = 1, \ldots, k$.

Assume $f_{ue}^{1k}(c) < 0$, i.e. (c_1^1, c_2^k) is to the left of the offer curve. Then there must be an $h = 1, \ldots, k-1$ such that $f_{ue}^{1h}(c) > 0$, i.e. (c_h^1, c_2^h) is to the right of the offer curve. But since the consumption when old is a normal good and, hence, any excess supply when old is mapped univocally to an excess demand when young, then actually all the points in $\{(c_1^i, c_2^h)\}_{i=1}^k$ are to the right of the offer curve. Therefore zero cannot be contained in the convex hull of $\{f_{ue}^{ih}(c)\}_{i=1}^k$. Assume on the contrary that $f_{ue}^{1k}(c) > 0$, i.e. (c_1^1, c_2^k) is now to the right of the offer curve. Then because the normality of the consumption when old again, the set of points $\{(c_1^i, c_2^k)\}_{i=1}^k$ is entirely to the right of the offer curve, leaving zero out of the convex hull of $\{f_{ue}^{ik}(c)\}_{i=1}^k$. Thus, necessarily, $f_{ue}^{1k}(c) = 0$ holds.

Similarly, assume $f_{ue}^{k1}(c) > 0$, i.e (c_1^k, c_2^1) is to the right of the offer curve. Then, since necessarily $f_{ue}^{kk}(c) > 0$ (otherwise 0 would not be in the convex hull of $\{f_{ue}^{ik}(c)\}_{i=1}^k$), there must be some $h = 1, \ldots, k$ such that $f_{ue}^{kh}(c) < 0$, unless zero is left out of the convex hull of $\{f_{ue}^{kj}(c)\}_{j=1}^k$. But then the set $\{(c_1^i, c_2^h)\}_{i=1}^k$ is entirely to the left of the offer curve and zero would not be in the convex hull of $\{f_{ue}^{ih}(c)\}_{i=1}^k$. Finally, if $f_{ue}^{k1}(c) < 0$, i.e. (c_1^k, c_2^1) is to the left of the offer curve, then the entire set $\{(c_1^i, c_2^1)\}_{i=1}^k$ is to the left of the offer curve and the zero would not be in the convex hull of $\{f_{ue}^{i1}(c)\}_{i=1}^k$. Therefore, $f_{ue}^{k1}(c) = 0$ holds as well.

So, both (c_1^1, c_2^k) and (c_1^k, c_2^1) are necessarily on the offer curve and since moreover $c_1^1 + c_2^1 = e_1 + e_2$ and $c_1^k + c_2^k = e_1 + e_2$, then they constitute a cycle of period 2.

(2) Let (c_1^a, c_2^a) and (c_1^b, c_2^b), with $c_1^a < c_1^b$, be the two consumption bundles of a cycle of period 2 of the stationary overlapping generations economy (u, e), Then, letting c_1^1 and c_1^k be c_1^a and c_1^b respectively (and hence $c_2^k = c_2^a$ and $c_2^1 = c_2^b$), trivially (c_1^1, c_2^k) and (c_1^k, c_2^1) are (c_1^a, c_2^a) and (c_1^b, c_2^b) respectively. On the other hand, since consumption when old is a normal good, c_2 is an increasing function of the real rate of interest if this rate exceeds the representative agents marginal rate of substitution at the endowments point. As a consequence, the steady state consumption when young $\bar{c}_1$ is necessarily between c_1^1 and c_1^k, and the bundles $(\bar{c}_1, c_2^k)$ and $(\bar{c}_1, c_2^1)$ (respectively $(c_1^1, \bar{c}_2)$ and $(c_1^k, \bar{c}_2)$) are separated by the offer curve. Therefore, so are any $k-2$ bundles (c_1^i, c_2^k) and (c_1^i, c_2^1) (respectively (c_1^1, c_2^i) and (c_1^k, c_2^i), with $c_2^i = e_1 + e_2 - c_1^i$), for any c_1^i close enough to $\bar{c}_1$ because of the continuity of the offer curve. Thus, $((c_1^i, c_2^j)_{i=1}^k)_{j=1}^k$ constitute the allocation of a finite Markovian stationary sunspot equilibrium of (u, e), as well as its symmetric counterpart $((c_2^j, c_1^i)_{j=1}^k)_{i=1}^k$ constitute the allocation of a sunspot equilibrium of the same kind for the economy $(\tilde{u}, \tilde{e})$. Q.E.D.

Notice that in the previous characterization only the allocation of resources matters and not the Markov process that governs the stochastic dynamics of the equilibrium. Thus, in principle, one could expect that the Markov matrix that supports the allocation of a reversible finite Markovian stationary sunspot equilibrium of a stationary overlapping generations economy (u, e) can be distinct from the Markov matrix that supports its symmetrical image as an equilibrium allocation of the economy $(\tilde{u}, \tilde{e})$. Nonetheless, even if we require these two matrices to be the same one, it is straightforward to see that for any allocation satisfying the characterization provided in Proposition 5 there is a Markov matrix that makes of both this allocation and its symmetrical counterpart sunspot equilibria of the economies (u, e) and $(\tilde{u}, \tilde{e})$. In effect, for a given pair of k-tuples of consumptions when young and old $c = (c_1, c_2) \in \mathbb{R}_+^k \times \mathbb{R}_+^k$ such that $c_1^i + c_2^i = e_1 + e_2$ for all $i = 1, \ldots, k$ to define the allocation of resources of a reversible finite Markovian stationary sunspot equilibrium supported by the same Markov matrix both in (u, e) and in $(\tilde{u}, \tilde{e})$, this Markov matrix has to be a solution to the system of equations

$$\begin{pmatrix} f_{ue}^{11}(c) & \cdots & f_{ue}^{1k}(c) & & & & \\ & & & \cdots & & & \\ & & & & f_{ue}^{k1}(c) & \cdots & f_{ue}^{kk}(c) \\ f_{ue}^{11}(c) & \cdots & f_{ue}^{k1}(c) & & & & \\ & & & \cdots & & & \\ & & & & f_{ue}^{1k}(c) & \cdots & f_{ue}^{kk}(c) \\ 1 & \cdots & 1 & & & & \\ & & & \cdots & & & \\ & & & & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} m^{11} \\ \vdots \\ m^{1k} \\ \vdots \\ m^{k1} \\ \vdots \\ m^{kk} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{29}$$

(empty entries stand for zeros, dots for obvious sequences). This is an underdetermined linear system of $3k$ equations in the k^2 probabilities of transition as long as $k > 3$, and it is determined for $k = 3$ (notice that the left-hand side matrix has generically full rank since typically the column vector and row vector of the matrix

$(f^{ij}_{ue}(c))$ with the same index, and the vector of k ones $\mathbf{1}$ will be in general position, i.e. linearly independent.

An immediate corollary of the previous proposition is that there is no true 2-state Markovian stationary sunspot equilibrium that is reversible. In effect, it is easy to check that the solution of the system above has only one solution for $k = 2$ and this is precisely the matrix of a pure cycle of period 2. This fact follows easily as well from the characterization provided in Proposition 5: in order to be reversible, the two states of a sunspot equilibrium of this kind fluctuating between just two states have to be those of the cycle of period 2 and, hence, the equilibrium is actually a pure cycle.

In spite of the underdeterminacy showed above, the finite Markovian stationary sunspot equilibria with the symmetry property of being invertible and keeping the same Markov process constitute a negligible subset of the set of all the equilibria of their class, as the following section shows.

6. The reversible finite Markovian stationary sunspot equilibria are negligible

The following proposition states, for the case $k = 3$ but by means of an argument general enough (as explained in the discussion following the proof), that the finite Markovian stationary sunspot equilibria of a stationary overlapping generations economy that are reversible with the same Markov process, constitute typically a negligible subset in comparison with those that are not reversible.

Proposition 6. *For any u in a dense subset of U, the set of 3-state Markovian stationary sunspot equilibria of any stationary overlapping generations economy (u, e) whose symmetric allocation is that of a sunspot equilibrium of the mirror economy $(\tilde{u}, \tilde{e})$ with the same sunspot signal, is null measure in a neighborhood of any regular*[15] *sunspot equilibrium of this kind.*

In order to simplify the notation in the proof of Proposition 6 below, for any given stationary overlapping generations economy (u, e), for any given $k \times k$ Markov matrix m and $c = (c_1, c_2) \in \mathbb{R}^k_{++} \times \mathbb{R}^k_{++}$, let $A^m_{ue}(c)$ and $B^m_{ue}(c)$ be the $k \times k$ matrices whose typical (i, j) entry is, respectively, $m^{ij} f^{ij}_{ue}(c)$ and $m^{ij} f^{ji}_{ue}(c)$ (notice the reverse order in the superscripts), where as previously $f^{ij}_{ue}(c)$ stands for the scalar product of the gradient of utility and the excess demand at (c^i_1, c^j_2).

It then follows readily from the definitions that

(1) $(c_1, c_2) \in \mathbb{R}^k_{++} \times \mathbb{R}^k_{++}$ is a k-state Markovian stationary sunspot equilibrium allocation of a stationary overlapping generations economy (u, e) following a Markov chain with matrix of probabilities of transitions m if, and only if, $c^i_1 + c^i_2 = e_1 + e_2$ and $(1, \ldots, 1) \in \mathbb{R}^k$ is in the null space of $A^m_{ue}(c_1, c_2)$ (this is actually the equilibrium conditions (23));

(2) moreover, a necessary condition for $(c_1, c_2) \in \mathbb{R}^k_{++} \times \mathbb{R}^k_{++}$ such that $c^i_1 + c^i_2 = e_1 + e_2$ for all $i = 1, \ldots, k$, to be the allocation of a k-state Markovian stationary sunspot equilibrium, with a matrix of probabilities of transitions m, for the stationary overlapping generations economy (u, e), is that

[15] In the sense that $f^{ii}_{ue}(c) \neq 0$, for all $i = 1, 2, 3$, i.e. the steady state is not in the support of the equilibrium.

$(1,\ldots,1) \in \mathbb{R}^k$ is in the null space of the matrix $B^m_{\tilde{u}\tilde{e}}(c_2, c_1)$ of its mirror economy $(\tilde{u}, \tilde{e})$ too (see the Lemma 1 in the Appendix).

Bearing this in mind, the proof of Proposition 6 follows.

Proof of Proposition 6. Let (u, e) be a stationary overlapping generations economy and let $(\bar{c}_1, \bar{c}_2) \in \mathbb{R}^3_{++} \times \mathbb{R}^3_{++}$ such that $\bar{c}^i_1 + \bar{c}^i_2 = e_1 + e_2$ for all $i = 1, 2, 3$, and the 3×3 Markov matrix[16] $\bar{m}$ be a regular sunspot equilibrium of (u, e). The set of sunspot equilibria of this class of the economy (u, e) is a smooth manifold in a small enough neighborhood $N_{(\bar{c}_1, \bar{c}_2, \bar{m})}$ of the point $(\bar{c}_1, \bar{c}_2, \bar{m})$ (see Lemma 2 in the Appendix).

Let the point (c_1, c_2, m) in $N_{(\bar{c}_1, \bar{c}_2, \bar{m})}$ be a sunspot equilibrium of the overlapping generations economy (u, e) such that (c_2, c_1, m) is a sunspot equilibrium of its mirror economy $(\tilde{u}, \tilde{e})$. Then, on the one hand, since (c_1, c_2, m) is a sunspot equilibrium of the economy (u, e), the vector $(1, 1, 1)$ is in the null space of $A^m_{ue}(c)$, and on the other hand, since (c_2, c_1, m) is a sunspot equilibrium of $(\tilde{u}, \tilde{e})$, $(1, 1, 1)$ is in the null space of $B^m_{ue}(c)$ too. Therefore, both $|A^m_{ue}(c)|$ and $|B^m_{ue}(c)|$ are zero and hence equal. Thus, from the straightforward computation of these determinants, necessarily either

$$m^{13}m^{32}m^{21} = m^{12}m^{23}m^{31} \tag{30}$$

or

$$f^{13}_{ue}(c)f^{32}_{ue}(c)f^{21}_{ue}(c) = f^{12}_{ue}(c)f^{23}_{ue}(c)f^{31}_{ue}(c). \tag{31}$$

Should none of these equations be transversal at (c_1, c_2, m) to the manifold of 3-state Markovian stationary sunspot equilibria of the economy (u, e) in $N_{(\bar{c}_1, \bar{c}_2, \bar{m})}$, there would be arbitrarily close to u in U, with respect to C^2 topology, another utility function v such that

(1) (c_1, c_2, m) is a sunspot equilibrium of (v, e) as well,
(2) (c_2, c_1, m) is a sunspot equilibrium of the mirror economy $(\tilde{v}, \tilde{e})$ of (v, e) and
(3) the transversality holds.[17]

Hence the null measure of the subset in $N_{(\bar{c}_1, \bar{c}_2, \bar{m})}$ of sunspot equilibria $(c_1, c_2, m) \in \mathbb{R}^3_{++} \times \mathbb{R}^3_{++} \times (0, 1)^6$ of the economy (u, e) such that (c_2, c_1, m) is a sunspot equilibrium of its mirror economy $(\tilde{u}, \tilde{e})$. Q.E.D.

A comment should be made on the fact that the claim is constrained to hold only around equilibria which are regular in the sense of not containing the steady state in their support. This fact does not limit the scope of the statement in any serious way since non-regular equilibria constitute clearly a negligible subset among all the equilibria,[18] and thus non-regular equilibria can sensibly be considered quite unlikely.

The proof of proposition 6 deserves a few comments about how to extend it to any finite state Markovian stationary sunspot equilibrium. The strategy followed in that proof consists of noticing first that, necessarily, both determinants $|A^m_{ue}(c)|$ and $|B^m_{ue}(c)|$ have to take the same value for $(c, m) \in \mathbb{R}^k_{++} \times \mathbb{R}^k_{++} \times (0, 1)^{k(k-1)}$

[16] Considered as a point $(m_{12}, m_{13}, m_{21}, m_{23}, m_{31}, m_{32})$ of the open 6-dimensional cube $(0, 1)^6$.

[17] See Lemmas 3 and 4 in the Appendix.

[18] Specifically the set of non-regular equilibria is, for a dense subset of economies, contained in a manifold of smaller dimension than the manifold of regular equilibria (see Lemma 5 in the Appendix).

such that $c_1^i + c_2^i = e_1 + e_2$ to be a reversible k-state Markovian stationary sunspot equilibrium of (u, e). As a matter of fact, these determinants share a very peculiar structure. In effect, if we let $*$ denote an operation on $k \times k$ matrices such that, for any given P, Q, $P * Q$ has $p_{ij}q_{ij}$ as (i, j) entry, then $|A^m_{ue}(c)|$ and $|B^m_{ue}(c)|$ are of the form $|P * Q|$ and $|P * Q^t|$ respectively, for $P = (m^{ij})$ and $Q = (f^{ij}_{ue}(c))$. Thus the necessary condition for the reversibility of the equilibrium is $|P * Q| - |P * Q^t| = 0$, i.e.

$$\sum_{\rho \in P} (-1)^{\mathrm{sign}(\rho)} \prod_{i=1}^{k} p_{i\rho(i)} q_{i\rho(i)} - \sum_{\rho \in P} (-1)^{\mathrm{sign}(\rho)} \prod_{i=1}^{k} p_{i\rho(i)} q_{\rho(i)i} = 0, \tag{32}$$

where P stands for the set of permutations of $\{1, \ldots, k\}$, or, equivalently,

$$\sum_{\rho \in \tilde{P}} (-1)^{\mathrm{sign}(\rho)} \prod_{i=1}^{k} p_{i\rho(i)} \Big(\prod_{h=1}^{k} q_{h\rho(h)} - \prod_{h=1}^{k} q_{\rho(h)h} \Big) = 0, \tag{33}$$

where $\tilde{P}$ stands for the subset of asymmetric permutations, i.e. those which are not their own inverse, since the terms corresponding to the symmetric ones cancel out in the equation (32).

For $k = 3$, four of the six permutations of $\{1, 2, 3\}$ are symmetric and the necessary condition for the reversibility of the equilibrium takes the form

$$\begin{aligned}(m_{12}m_{23}m_{31} - m_{13}m_{32}m_{21})(f^{12}_{ue}(c) f^{23}_{ue}(c) f^{31}_{ue}(c) - \\ f^{13}_{ue}(c) f^{32}_{ue}(c) f^{21}_{ue}(c)) = 0.\end{aligned} \tag{34}$$

Hence the equations whose transversality to the manifold of equilibria is studied in the proof of Proposition 6. For values of k bigger than 3, no such simple factorization exists[19] and necessarily the transversality of (32) would have to be checked directly, which is far from being a simple task. Still, the conjecture of (32) being transversal at any regular sunspot equilibrium for some v arbitrarily close to a given u in the space of utility functions U is likely to hold in general, since this transversality depends on the values of the second order partial derivatives of u at the allocation, on which there are no restrictions (even if there are on their signs).

Interestingly enough, if $k = 2$, the only two permutations of $\{1, 2\}$ are symmetric. This means that in this case the condition (32) is an identity, and thus it is surely satisfied for sunspot equilibria fluctuating randomly between two states. Nevertheless, this does not mean that such equilibria can be reversed, but just that this proof cannot be extended to that case (actually the 2-states Markovian stationary sunspot equilibria cannot be reversed, as it has already been shown to follow easily from Proposition 5).

7. Concluding remarks

The differences shown in the previous sections between the certainty and the uncertainty cases for the reversibility of the equilibria of an overlapping generations economy were actually hinted at by previous characterizations of the existence of sunspot equilibria in these economies. In effect, [8] shows that a necessary and

[19] For instance, if $k = 4$, then just 9 out of the 24 permutations of $\{1, 2, 3, 4\}$ are symmetric and thus 15 asymmetric ones are left to be taken into account.

sufficient condition for the existence of finite state Markovian stationary sunspot equilibria translate into a condition on the representative agent's offer curve much weaker in the so called "classical" case than in the "Samuelsonian" case (following the terminology used in [12]). Accordingly many more classical overlapping generations economies exhibit such sunspot equilibria than the Samuelsonian ones do. Bearing this in mind, since the mirror economy of any Samuelsonian overlapping generations economy is a classical economy and viceversa, that result pointed already at a missing one-to-one correspondence between the sunspot equilibria of any two such mirrored economies like the one existing for the perfect foresight equilibria.

Incidentally, at the heart of the proof of Proposition 6 there is an argument used in [10] to show the impossibility of a robust equivalence between the finite Markovian stationary sunspot equilibria of a simple overlapping generations economy and the correlated equilibria of the finite economy with asymmetric information considered in [16]. The idea of a connection between the equilibria of an open-ended economy and those of a related one-shot economy has already been explored to some extent, starting from the link between the cycles of an overlapping generations economy and the multiple equilibria of finite economies with a symmetric structure, as shown in [3] and [22]. I show in [11] how to circumvent the impossibility to extend the connection to sunspot equilibria in the way conjectured in [16] and thus make it hold in general.

As for the reversibility issue itself, the only reference I am aware of about the extent to which the overlapping generations models determine a definite direction of time is [19].[20]That paper considers only the case of certainty and concludes, as it follows from the analysis presented here as well, that the overlapping structure is not enough to determine unequivocally the direction in which time flows for perfect foresight equilibria. Stochastic equilibria are not considered in that paper.

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Appendix

Proof of Proposition 1.

(1) If $\{c^t\}_{t\in\mathbb{Z}}$ and $\{p_t\}_{t\in\mathbb{Z}}$ constitute an equilibrium of the overlapping generations economy $\{(u^t, e^t)\}_{t\in\mathbb{Z}}$, then for every $t \in \mathbb{Z}$, c^t is a solution to (1) i.e. there exists a positive multiplier λ^t such that

$$\begin{aligned} D_1u^t(c^t) - \lambda^t p_t &= 0 \\ D_2u^t(c^t) - \lambda^t p_{t+1} &= 0 \\ p_t(c_1^t - e_1^t) + p_{t+1}(c_2^t - e_2^t) &= 0. \end{aligned} \tag{35}$$

Multiplying the first equation by $(c_1^t - e_1^t)$, the second by $(c_2^t - e_2^t)$ and adding them up taking into account the budget constraint in the third equation, it follows

$$D_1u^t(c^t)(c_1^t - e_1^t) + D_2u^t(c^t)(c_2^t - e_2^t) = 0 \tag{36}$$

that is to say

$$du^t_{c^t}(c^t - e^t) = 0. \tag{37}$$

[20]I thank Frank Heinemann for drawing my attention upon the existence of this paper.

(2) In order to produce prices $\{p_t\}_{t\in\mathbb{Z}}$ supporting $\{c^t\}_{t\in\mathbb{Z}}$ as an equilibrium allocation, let p_0 be any positive price and define,[21] for each $t \in \mathbb{Z}$, the price

$$p_t = -\frac{c_1^0 - e_1^0}{c_2^{t-1} - e_2^{t-1}} p_0 \tag{38}$$

and the multiplier

$$\lambda^t = -D_2 u^t(c^t) \frac{c_2^t - e_2^t}{c_1^0 - e_1^0} \frac{1}{p_0}. \tag{39}$$

Then, for all $t \in \mathbb{Z}$, the allocation of resources is feasible by assumption and, moreover, the first order conditions (35) are satisfied since, firstly, the budget constraint holds

$$\frac{p_t}{p_{t+1}} = \frac{-\frac{c_1^0 - e_1^0}{c_2^{t-1} - e_2^{t-1}} p_0}{-\frac{c_1^0 - e_1^0}{c_2^t - e_2^t} p_0} = \frac{c_2^t - e_2^t}{c_2^{t-1} - e_2^{t-1}} = -\frac{c_2^t - e_2^t}{c_1^t - e_1^t}; \tag{40}$$

secondly, the partial derivative of the lagrangian with respect to c_2^t is zero because of the very definition of λ^t; and, finally, the partial derivative with respect to c_1^t is also zero since

$$\begin{aligned} & D_1 u^t(c^t) - \lambda^t p_t = \\ & D_1 u^t(c^t) + D_2 u^t(c^t) \frac{c_2^t - e_2^t}{c_1^0 - e_1^0} \frac{1}{p_0} \cdot - \frac{c_1^0 - e_1^0}{c_2^{t-1} - e_2^{t-1}} p_0 = \\ & \frac{1}{c_1^t - e_1^t} \left(D_1 u^t(c^t)(c_1^t - e_1^t) + D_2 u^t(c^t)(c_2^t - e_2^t) \right) = \\ & \frac{1}{c_1^t - e_1^t} du^t_{c^t}(c_1^t - e_1^t) = 0 \end{aligned} \tag{41}$$

by mere substitutions, recalling (3) and (4).

Q.E.D.

Proof of Proposition 3.

(1) If $\{c^t_{s_{t+1}}\}_{t\in\mathbb{Z},s\in S}$ such that each $c^t_{1s_{t+1}}$ does not depend on σ_{t+1}, and $\{p_{ts_t}\}_{t\in\mathbb{Z},s\in S}$ constitute an equilibrium, then, for any $s \in S$ and any $t \in \mathbb{Z}$, $(c^t_{s'_{t+1}})_{s'_{t+1}|s'_t=s_t}$ is the solution to (19), i.e. there exist positive multipliers $\lambda^t_{s'_{t+1}}$ for all s'_{t+1} such that $s'_t = s_t$, for which

$$\begin{aligned} \sum_{s'_{t+1}|s'_t=s_t} P(s'_{t+1}|s'_t = s_t) D_1 u^t(c^t_{s'_{t+1}}) - \sum_{s'_{t+1}|s'_t=s_t} \lambda^t_{s'_{t+1}} p_{ts_t} = 0 \\ P(s'_{t+1}|s'_t = s_t) D_2 u^t(c^t_{s'_{t+1}}) - \lambda^t_{s'_{t+1}} p_{t+1s'_{t+1}} = 0 \ , \ s'_{t+1}|s'_t = s_t \\ p_{ts_t}(c^t_{1s'_{t+1}} - e_1^t) + p_{t+1s'_{t+1}}(c^t_{2s'_{t+1}} - e_2^t) = 0 \ , \ s'_{t+1}|s'_t = s_t. \end{aligned} \tag{42}$$

[21] Notice that, $Du^t(c^t) \in R^2_{++}$ and the feasibility condition guarantee that, at equilibrium, if $c_1^0 - e_1^0 < 0$ (> 0), then $c_1^t - e_1^t < 0$ (> 0), i.e. $c_2^{t-1} - e_2^{t-1} > 0$ (< 0), for all $t \in \mathbb{Z}$ and thus p_t and λ^t are always positive. If the allocation of resources is the autarky, then the first order conditions will be satisfied by any positive price p_0, with $p_t = \Pi_{i=1}^t (D_2 u^i(e^i)/D_1 u^i(e^i)) p_0$ if $t > 0$, $p_t = \Pi_{i=t}^{-1} (D_1 u^i(e^i)/D_2 u^i(e^i)) p_0$ if $t < 0$, and λ^t being $-D_2 u^t(e^t) \Pi_{i=1}^{t+1} (D_1 u^i(e^i)/D_2 u^i(e^i)) \cdot 1/p_0$ if $t > -1$, $-D_2 u^t(e^t) \cdot 1/p_0$ if $t = -1$ and $-D_2 u^t(e^t) \Pi_{i=t+1}^{-1} (D_2 u^i(e^i)/D_1 u^i(e^i)) \cdot 1/p_0$ if $t < -1$.

Multiplying the first equation by $(c^t_{1s'_{t+1}} - e^t_1)$, each of the equations in the second line by the corresponding $(c^t_{2s'_{t+1}} - e^t_2)$ and adding all them up taking into account the budget constraints in the third line, then the condition (21) follows.

(2) In order to produce prices supporting $\{c^t_{s_{t+1}}\}_{t\in\mathbb{Z},s\in S}$ as an equilibrium allocation, let p_{0s_0}, for every[22] $s \in S$, be any positive price such that, for any other $s' \in S$,

$$\frac{p_{0s_0}}{p_{0s'_0}} = \frac{c^0_{1s'_1} - e^0_1}{c^0_{1s_1} - e^0_1}. \tag{43}$$

Then define, for all $s \in S$ and all $t \in \mathbb{Z}$, the prices

$$p_{ts_t} = -\frac{c^0_{1s_1} - e^0_1}{c^{t-1}_{2s_t} - e^{t-1}_2} p_{0s_0}, \tag{44}$$

and for all s'_{t+1} such that $s'_t = s_t$, the multipliers

$$\lambda^t_{s'_{t+1}} = -P(s'_{t+1}|s'_t = s_t) D_2 u^t(c^t_{s'_{t+1}}) \frac{c^t_{2s'_{t+1}} - e^t_2}{c^0_{1s'_1} - e^0_1} \frac{1}{p_{0s'_0}}. \tag{45}$$

Then, for all $t \in \mathbb{Z}$ and all $s \in S$, the feasibility constraint is satisfied by assumption and the first order conditions (42) are satisfied too: firstly, the budget constraints are satisfied since, for all s'_{t+1} such that $s'_t = s_t$,

$$\frac{p_{ts_t}}{p_{t+1s'_{t+1}}} = \frac{-\frac{c^0_{1s_1} - e^0_1}{c^{t-1}_{2s_t} - e^{t-1}_2} p_{0s_0}}{-\frac{c^0_{1s'_1} - e^0_1}{c^t_{2s'_{t+1}} - e^t_2} p_{0s'_0}} = \frac{c^t_{2s'_{t+1}} - e^t_2}{c^{t-1}_{2s_t} - e^{t-1}_2} = -\frac{c^t_{2s'_{t+1}} - e^t_2}{c^t_{1s'_{t+1}} - e^t_1}, \tag{46}$$

where the second equality results from the normalization (43) adopted; secondly, the partial derivatives of the lagrangian with respect to $c^t_{2s'_{t+1}}$ are satisfied by the very definition of the multipliers $\lambda^t_{s'_{t+1}}$, and finally, the partial derivative with respect to $c^t_{1s'_{t+1}}$ is again satisfied by mere substitutions, recalling (21) and (22).

Q.E.D.

Lemma 1. *If $(c^i_1, (c^j_2)^k_{j=1})^k_{i=1}$ is the allocation of resources of a finite Markovian stationary sunspot equilibrium of (u, e) governed by the Markov matrix m, then the $(1, \dots, 1) \in \mathbb{R}^k$ is in the null space of the matrix $B^m_{\tilde{u}\tilde{e}}(c_2, c_1)$ of its mirror economy $(\tilde{u}, \tilde{e})$.*

Proof. If $(c^i_1, (c^j_2)^k_{j=1})^k_{i=1}$ is the allocation of resources of a finite markovian stationary sunspot equilibrium of (u, e) governed by the Markov matrix m, then for all $i = 1, \dots, k$,

$$\sum_{j=1}^k m^{ij} du_{(c^i_1, c^j_2)}(c^i_1 - e_1, c^j_2 - e_2) = 0, \tag{47}$$

[22] Actually, just one p_{0s_0} needs to be fixed arbitrarily, all the other prices at date 0 and at every state of the world s being then determined by the normalization (43).

that is to say, for all $i = 1, \dots, k$,

$$\sum_{j=1}^{k} m^{ij} d\tilde{u}_{(c_2^j, c_1^i)}(c_2^j - \tilde{e}_1, c_1^i - \tilde{e}_2) = 0, \tag{48}$$

or, equivalently, for all $i \in \{1, \dots, k\}$,

$$(\, m^{ij} f^{ji}_{\tilde{u}\tilde{e}}(c_2, c_1)\,)\, \mathbf{1} = 0 \tag{49}$$

that is to say, $(1, \dots, 1) \in \mathbb{R}^k$ is in the null space of the matrix $B^m_{\tilde{u}\tilde{e}}(c_2, c_1)$ of its mirror economy $(\tilde{u}, \tilde{e})$. Q.E.D.

Lemma 2. *The set of 3-state Markovian stationary sunspot equilibria of an overlapping generations economy (u, e) is locally a smooth manifold around any of them that is regular.*

Proof. Let (u, e) be a stationary overlapping generations economy such that the set of its 3-state Markovian stationary sunspot equilibria is non-empty and let a $(\bar{c}_1, \bar{c}_2, \bar{m})$ be such an equilibrium that is regular, i.e. such that for all $i = 1, 2, 3$, $f^{ii}_{ue}(\bar{c}) \neq 0$. Let $F_{ue} \colon \mathbb{R}^3_{++} \times \mathbb{R}^3_{++} \times (0,1)^6 \to \mathbb{R}^3$ be such that

$$F_{ue}(c, m) = \begin{pmatrix} \sum_{j=1}^3 m^{1j} f^{1j}_{ue}(c) \\ \sum_{j=1}^3 m^{2j} f^{2j}_{ue}(c) \\ \sum_{j=1}^3 m^{3j} f^{3j}_{ue}(c) \\ c_1^1 + c_2^1 - e_1 - e_2 \\ c_1^2 + c_2^2 - e_1 - e_2 \\ c_1^3 + c_2^3 - e_1 - e_2 \end{pmatrix}. \tag{50}$$

Then ${F_{ue}}^{-1}(0)$ is the set of 3-state Markovian stationary sunspot equilibria and thus $(\bar{c}_1, \bar{c}_2, \bar{m}) \in {F_{ue}}^{-1}(0)$. The jacobian of F_{ue} at $(\bar{c}_1, \bar{c}_2, \bar{m})$, $DF_{ue}(\bar{c}_1, \bar{c}_2, \bar{m})$, is of the form

$$\begin{pmatrix} C & C' & C'' \\ I_3 & I_3 & 0 \end{pmatrix} \tag{51}$$

where

$$C = \begin{pmatrix} \sum_{j=1}^3 \bar{m}^{1j} D_1 f^{1j}_{ue}(\bar{c}) & 0 & 0 \\ 0 & \sum_{j=1}^3 \bar{m}^{2j} D_2 f^{2j}_{ue}(\bar{c}) & 0 \\ 0 & 0 & \sum_{j=1}^3 \bar{m}^{3j} D_3 f^{3j}_{ue}(\bar{c}) \end{pmatrix}, \tag{52}$$

$$C' = \begin{pmatrix} \bar{m}^{1j} D_4 f^{11}_{ue}(\bar{c}) & \bar{m}^{12} D_5 f^{12}_{ue}(\bar{c}) & \bar{m}^{13} D_6 f^{13}_{ue}(\bar{c}) \\ \bar{m}^{21} D_4 f^{21}_{ue}(\bar{c}) & \bar{m}^{2j} D_5 f^{2j}_{ue}(\bar{c}) & \bar{m}^{13} D_6 f^{13}_{ue}(\bar{c}) \\ \bar{m}^{31} D_4 f^{31}_{ue}(\bar{c}) & \bar{m}^{32} D_5 f^{32}_{ue}(\bar{c}) & \bar{m}^{3j} D_6 f^{3j}_{ue}(\bar{c}) \end{pmatrix} \tag{53}$$

and

$$C'' = \begin{pmatrix} f^{12}_{ue}(\bar{c}) - f^{11}_{ue}(\bar{c}) & f^{13}_{ue}(\bar{c}) - f^{11}_{ue}(\bar{c}) \\ 0 & 0 & \cdots \\ 0 & 0 \end{pmatrix}$$

$$\begin{matrix} & 0 & 0 & \\ \cdots & f^{21}_{ue}(\bar{c}) - f^{22}_{ue}(\bar{c}) & f^{23}_{ue}(\bar{c}) - f^{22}_{ue}(\bar{c}) & \cdots \\ & 0 & 0 & \end{matrix} \tag{54}$$

$$\left. \begin{matrix} \cdots & 0 & 0 \\ & 0 & 0 \\ & f_{ue}^{31}(\bar{c}) - f_{ue}^{33}(\bar{c}) & f_{ue}^{32}(\bar{c}) - f_{ue}^{33}(\bar{c}) \end{matrix} \right),$$

with D_i denoting the i-th partial derivative.

Should $DF_{ue}(\bar{c}, \bar{m})$ not be full rank, then necessarily $f_{ue}^{i1}(\bar{c}) = f_{ue}^{i2}(\bar{c}) = f_{ue}^{i3}(\bar{c})$ for some $i \in \{1, 2, 3\}$, but then $(\bar{c}, \bar{m}) \in F_{ue}^{-1}(0)$ would imply $f_{ue}^{ii}(\bar{c}) = 0$. Thus, if for all $i = 1, 2, 3$, $f_{ue}^{ii}(\bar{c}) \neq 0$, then $DF_{ue}(\bar{c}, \bar{m})$ is full rank and therefore ${F_{ue}}^{-1}(0)$ is a manifold (of dimension 6) in a neighborhood of $(\bar{c}, \bar{m})$. Q.E.D.

Lemma 3. *If a stationary overlapping generations economy (u, e) has a regular 3-state Markovian stationary sunspot equilibrium $(\bar{c}, \bar{m})$ such that*

$$f_{ue}^{12}(\bar{c}) f_{ue}^{23}(\bar{c}) f_{ue}^{31}(\bar{c}) - f_{ue}^{13}(\bar{c}) f_{ue}^{32}(\bar{c}) f_{ue}^{21}(\bar{c}) = 0, \tag{55}$$

then arbitrarily close to this economy there is another economy (v, e) for which $(\bar{c}, \bar{m})$ is an equilibrium as well and whose equilibria of the same kind that satisfy (55) *constitute a null measure subset of the set of such equilibria in a neighborhood of $(\bar{c}, \bar{m})$.*

Proof. Let us see that the subset of utility functions in U such that the equation (55) is transversal to the manifold S of 3-state Markovian stationary sunspot equilibria at $(\bar{c}, \bar{m})$ is dense in U. In effect, let (u, e) be a stationary overlapping generations economy that has a regular 3-state Markovian stationary sunspot equilibrium $(\bar{c}, \bar{m})$ satisfying (55). Let F_{ue} be as in Lemma 2 and $G_{ue} \colon \mathbb{R}_{++}^3 \times \mathbb{R}_{++}^3 \times (0, 1)^6 \to \mathbb{R}$ be such that

$$G_{ue}(c, m) = f_{ue}^{12}(c) f_{ue}^{23}(c) f_{ue}^{31}(c) - f_{ue}^{13}(c) f_{ue}^{32}(c) f_{ue}^{21}(c). \tag{56}$$

Then $(\bar{c}, \bar{m}) \in F_{ue}^{-1}(0) \cap G_{ue}^{-1}(0)$. Thus $G_{ue}^{-1}(0)$ is transversal to ${F_{ue}}^{-1}(0)$ at $(\bar{c}, \bar{m})$ if $DG_{ue}(\bar{c}, \bar{m})$ is not in the span of the rows of $DF_{ue}(\bar{c}, \bar{m})$. Since $DG_{ue}(\bar{c}, \bar{m})$ is a 12-tuple whose six first entries are expressions in terms of the second order partial derivatives of u at the points $(\bar{c}_1^i, \bar{c}_2^j)$, for all $i, j = 1, 2, 3$, its six last entries are zero and $f_{ue}^{ii}(\bar{c}) \neq 0$, for all $i = 1, 2, 3$, then $DG_{ue}(\bar{c}, \bar{m})$ could only be the trivial linear combination of the rows of $DF_{ue}(\bar{c}, \bar{m})$ with zero scalars. Thus the six first entries of $DG_{ue}(\bar{c}, \bar{m})$ would have to be zero too, which requires the second order partial derivatives of u at the points $(\bar{c}_1^i, \bar{c}_2^j)$ to satisfy six equations. But there is a function v in U arbitrarily close to u with the same gradients as u at the points $(\bar{c}_1^i, \bar{c}_2^j)$ —and hence such that $(\bar{c}, \bar{m}) \in F_{ve}^{-1}(0) \cap G_{ve}^{-1}(0)$ as well— but such that $DG_{ve}(\bar{c}, \bar{m})$ is not in the span of the rows of $DF_{ve}(\bar{c}, \bar{m})$, i.e. such that $F_{ve}^{-1}(0)$ is transversal to $G_{ve}^{-1}(0)$ at $(\bar{c}_1^i, \bar{c}_2^j)$. Q.E.D.

Lemma 4. *If a stationary overlapping generations economy (u, e) has a regular 3-state Markovian stationary sunspot equilibrium $(\bar{c}, \bar{m})$ satisfying*

$$m^{12} m^{23} m^{31} - m^{13} m^{32} m^{21} = 0, \tag{57}$$

then arbitrarily close to this economy there is another economy (v, e) for which $(\bar{c}, \bar{m})$ is an equilibrium as well and whose equilibria of the same kind that satisfy (57) *constitute a null measure subset of the set of such equilibria in a neighborhood of $(\bar{c}, \bar{m})$.*

Proof. Let us see that the subset of functions in U such that the equation (57) is transversal to the manifold of these equilibria at $(\bar{c}, \bar{m})$ is dense in U. In effect, let

(u, e) be a stationary overlapping generations economy that has a regular 3-state Markovian stationary sunspot equilibrium $(\bar{c}, \bar{m})$ satisfying (57). Let F_{ue} be as in Lemma 2 and $H\colon \mathbb{R}^3_{++} \times \mathbb{R}^3_{++} \times (0,1)^6 \to \mathbb{R}$ be such that

$$H(c, m) = m^{12}m^{23}m^{31} - m^{13}m^{32}m^{21}. \tag{58}$$

Then $(\bar{c}, \bar{m}) \in {F_{ue}}^{-1}(0) \cap H^{-1}(0)$. Thus $H^{-1}(0)$ is transversal to ${F_{ue}}^{-1}(0)$ at $(\bar{c}, \bar{m})$ only if $DH(\bar{c}, \bar{m})$ is not in the span of the rows of $DF_{ue}(\bar{c}, \bar{m})$. Since $DH(\bar{c}, \bar{m})$ is a 12-tuple whose six first entries are zeros and its six last entries are

$$(\bar{m}^{23}\bar{m}^{31}, -\bar{m}^{32}\bar{m}^{21}, -\bar{m}^{13}\bar{m}^{32}, \bar{m}^{12}\bar{m}^{31}, \bar{m}^{12}\bar{m}^{23}, -\bar{m}^{32}\bar{m}^{21}), \tag{59}$$

should $D_c F_{ue}(\bar{c}, \bar{m})$ not be full rank, then there is a function v in U arbitrarily close to u with the same gradients at the points $(\bar{c}^i_1, \bar{c}^j_2)$ —and hence such that $(\bar{c}, \bar{m}) \in {F_{ve}}^{-1}(0) \cap H^{-1}(0)$ as well— such that $D_c F_{ue}(\bar{c}, \bar{m})$ is full rank indeed. Thus if moreover $DH(\bar{c}, \bar{m})$ were in the span of the rows of $DF_{ve}(\bar{c}, \bar{m})$, then it would necessarily be the trivial linear combination with scalars 0, which cannot be since $\bar{m} \in (0,1)^6$. Therefore, $H^{-1}(0)$ is transversal to ${F_{ue}}^{-1}(0)$ at $(\bar{c}, \bar{m})$. Q.E.D.

Lemma 5. *For any given endowments e and any utility function u from a dense subset of U, the set of nonregular 3-state Markovian stationary sunspot equilibria of the stationary overlapping generations economy (u, e) is, in a neighbourhood of any of them, contained in a manifold of a dimension smaller than that of the local manifold of regular equilibria.*

Proof. Let (u, e) be a stationary overlapping generations economy that has a non-regular 3-state Markovian stationary sunspot equilibrium $(\bar{c}, \bar{m})$, i.e. such that for some $i = 1, 2, 3$, $f^{ii}_{ue}(c) = 0$. Let F_{ue} be as in Lemma 2 and $\Psi^i_{ue}\colon \mathbb{R}^3_{++} \times \mathbb{R}^3_{++} \times (0,1)^6 \to \mathbb{R}$ be such that $\Psi^i_{ue}(c, m) = f^{ii}_{ue}(c)$.

Then $(\bar{c}, \bar{m}) \in {F_{ue}}^{-1}(0) \cap {\Psi^i_{ue}}^{-1}(0)$. Thus all the entries of $D\Psi^i_{ue}(\bar{c}, \bar{m})$ other than $D_{c^i_1}\Psi^i_{ue}(\bar{c}_1, \bar{m})$ and $D_{c^i_2}\Psi^i_{ue}(\bar{c}_1, \bar{m})$ are zero. Should both $D_{c^i_1}\Psi^i_{ue}(\bar{c}_1, \bar{m})$ and $D_{c^i_2}\Psi^i_{ue}(\bar{c}_1, \bar{m})$ —in terms of the second order partial derivatives of u at $(\bar{c}^i_1, \bar{c}^i_2)$— be null too, then there is v in U arbitrarily close to u with the same gradient at the points $(\bar{c}^i_1, \bar{c}^j_2)$ —and hence such that $(\bar{c}, \bar{m}) \in {F_{ve}}^{-1}(0) \cap {\Psi^i_{ve}}^{-1}(0)$ as well— such that $D\Psi^i_{ve}(\bar{c}, \bar{m})$ is non-null. Hence $D\Psi^i_{ve}(\bar{c}, \bar{m})$ is not in the span of the rows of $DF_{ve}(\bar{c}, \bar{m})$ (otherwise, necessarily for all $h = 1, 2, 3$, $f^{h1}_{ve}(\bar{c}) = f^{h2}_{ve}(\bar{c}) = f^{h3}_{ve}(\bar{c})$, which from $(\bar{c}, \bar{m}) \in {F_{ve}}^{-1}(0)$ implies $f^{hh}_{ve}(\bar{c}) = 0$ for all $h = 1, 2, 3$, i.e.

$$D_1 v(\bar{c}^h_1, e_1 + e_2 - \bar{c}^h_1)(\bar{c}^h_1 - e_1) + D_2 v(\bar{c}^h_1, e_1 + e_2 - \bar{c}^h_1)(e_1 - \bar{c}^h_1) = 0. \tag{60}$$

But the strict quasi-concavity and monotonicity of v guarantees that there is a unique c satisfying this condition. Thus, necessarily, all $\bar{c}^h_1 = c$ and hence $\bar{c}^i_1 = \bar{c}^1_1$ for all $i = 1, 2, 3$, contradicting the assumption). The conclusion follows immediately. Q.E.D.

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