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Numerical Solution for Nonlinear MHD Jeffery-Hamel Blood Flow Problem through Neural Networks Optimized Techniques

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ABSTRACT

The purpose of study is to develop numerical techniques for nonlinear magnetohydrodynamics (MHD) Jeffery-Hamel blood flow problem to analyze the behavior of blood flow and its contribution in high blood pressure through artificial neural networks trained with Active Set and Interior Point Algorithm. First we transform three-dimensional flow problem into two-dimensional MHD Jeffery-Hamel flow problem, which is converted into an equivalent third order nonlinear ordinary differential equation. These neural network models using log-sigmoid activation function are developed for new transformed equation. Detailed statistical analysis is also included to ensure the reliability and accuracy of the proposed methods through large number of independent runs. Further, comparative studies of the proposed solutions with standard numerical results are presented.
KEYWORDS: Blood flow, Jeffery-Hamel Problem, Neural Networks, Nonlinear ODEs, Boundary value problems,

1 INTRODUCTION

Jeffery-Hamel problems are considered as incompressible viscous fluid flows between non-parallel sheets. Study of Jeffery-Hamel flows have been commonly used in various fields of applied science and engineering like mechanical and bio-mechanical engineering, fluid mechanics, environmental science. Jeffery [1] and Hamel [2] have been proposed the mathematical formulation of the problem in detail. Jeffery-Hamel flows actually provide an exact similarity solution of the Navier-Stokes equations in the special case of a two-dimensional blood flow through a tube with inclined plane sheets converges at a source or sink at the single point. Further for historical background, its applications and importance in various fields reader can go through the references [3-8]. The classical view of Jeffery-Hamel problems with use of an external magnetic field on a conducting fluid were studied in [9] by taking the magnetic field as a control parameter. The MHD Jeffery-Hamel flow problems do not exist any exact solution due to their highly non-linearity in the literature. However, their analytical and numerical solutions have been frequently reported in the literature, like Homotopy Perturbation method (HPM) [10-13], Homotopy analysis methods (HAM) [14-15], the Adomian decomposition method (ADM) [16-17], the Differential transform method (DTM) [18-19], Variational iteration methods (VIM) [20-21], and so on.

Recently studied on the Jeffery- Hamel flow equations are presented in [22-27]. Therefore there is a need to find stochastic numerical methods based on computational intelligence techniques to solve these problems.

Today the most serious physiological problem was stenosis (narrowing) of the arteries because they develop and causes many harmful vascular diseases which have very close relationship with the nature of blood flow and deformation of vascular walls. The stenosis of artery causes by the decomposition of fibrous tissue and fats in artery lumen which restricts the normal movement of blood where reduces the transport of blood in a whole body. Furthermore, the transport of blood entirely depends on the heart pumping action in the circulatory system of human being and produces a pressure gradient. Due to the stenosis of artery pressure increases that causes the heart to work hard.

Stochastic algorithm based on artificial intelligence techniques using neural networks have been applied extensively by the many researcher to solve a variety of initial and boundary value problems of linear and non-linear differential equations [28-31]. Recently, uses of these algorithms are non-linear Van-der Pol oscillators [32], Troesch's problems arising in plasma physics [33], solution of thin plate bending problem [34], tracking problems of a spherical inverted pendulum [35-36], the first Painlevé transcendent [37], surrogate modeling for the solution of integral equations [38], Bratu's problem in fuel ignition modeling [39], etc.

In our proposed model blood is consider as Newtonian fluid, due to which problem become simple and still valid for blood in large artery. The purpose of this study is to develop the relationship between the rate of the blood

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flow and cross-sectional area of artery. To understand the conditions that contribute in hypertension this increases the risk of heart diseases.

This mathematical model containing two partial differential equations, which are used to find the cross sectional area, blood flow rate and pressure. By using the transformation to convert cylindrical system into another system for sack of simplicity of problem. The non-linear system of equations are governed and converted into linear equations by linearization method. In section 2, we formulate the problem and next section proposed a mathematical model for this equation with the help of log-sigmoid function. In section 4 we presented numerical and graphical results. Finally, we put a comparative analysis through Active set Algorithm (AST) and Interior Point technique (INT) through MATLAB. We concluded the paper in the last section 5.

1.2**Mathematical Formulation of Problem**

Consider cylindrical coordinates (r,θ,z) and a steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink. Where r is radial, z is axial component and "θ" is angular coordinate. Consider a fluid in the problem is human blood and visco-elastic effect is neglected, therefore behave like water. We consider pipe like behavior of human artery in this problem and construct a cylindrical problem in three dimensional partial differential equations (PDES). The governing mathematical relations are given as.

$$\frac{\rho}{r}\frac{\partial}{\partial r}[ru(r,\theta)]=0 \quad . \qquad (1)$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+v[\frac{\partial^2 u(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}-\frac{u(r,\theta)}{r^2}] \quad .(2)$$

$$\frac{1}{\rho r}\frac{\partial P}{\partial \theta}-\frac{2v}{r}\rho\frac{\partial u(r,\theta)}{\partial \theta}=0 \quad . \qquad (3)$$

Where ρ is density, v is kinematic viscosity, $u(r,\theta)$ is the component of velocity in radial direction and P denotes fluid pressure.

Integrating the Eq. (1) with respect to r. We get the following equation.

$$ru(r,\theta)=f(\theta) \quad . \qquad (4)$$

Now we introduce the new function $f(\eta)$ as following,

$$f(\eta)=\frac{f(\theta)}{A} \quad . \qquad (5)$$

Where by using dimensionless parameters,

$$\eta=\frac{\theta}{\alpha} \;,\; A=f_{\max} \quad .$$

From Eq. (4) and Eq. (5), we conclude that

$$f(\eta)=\frac{ru(r,\theta)}{A} \quad . \qquad (6)$$

$$u(r,\theta)=\frac{Af(\eta)}{r} \quad . \qquad (7)$$

Differentiate Eq. (7) w.r.t "r" and "θ".
First we take derivative w. r. t "r",

$$\frac{\partial u(r,\theta)}{\partial r}=-\frac{Af(\eta)}{r^2} \quad . \qquad (8)$$

By taking second derivative of $u(r,\theta)$ w. r. t "r",

$$\frac{\partial^2 u(r,\theta)}{\partial r^2} = \frac{2Af(\eta)}{r^3} \quad . \tag{9}$$

Again differentiating Eq. (7) w.r.t "θ", we get

$$\frac{\partial u(r,\theta)}{\partial \theta} = \frac{A}{r}\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial \theta} = \frac{A}{r\alpha} f'(\eta) \quad . \tag{10}$$

Second derivative of $u(r,\theta)$ w.r.t "θ".

$$\frac{\partial^2 u(r,\theta)}{\partial \theta^2} = \frac{A}{r\alpha}\frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial \theta} = \frac{A}{r\alpha^2} f''(\eta) \ . \tag{11}$$

Integrating Eq. (3) and, after simplification we get the following result;

$$P = \frac{2v}{r}\rho u(r,\theta) \ . \tag{12}$$

Differentiate pressure "P" in Eq. (12) w.r.t "r", we obtain the result,

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{2v}{r^2} u(r,\theta) + \frac{2v}{r}\frac{\partial u(r,\theta)}{\partial r} \ . \tag{13}$$

Now putting the values from above equations into Eq. (13) we get,

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{4v}{r^3} Af(\eta) \ . \tag{14}$$

Substituting the values of Eqs. (7 -11) into Eq. (3), we obtain

$$-\frac{A^2}{r^3} f^2(\eta) = \frac{4v}{r^3} Af(\eta) + v\left[\frac{2Af(\eta)}{r^3} - \frac{1}{r}\frac{Af(\eta)}{r^2} + \frac{1}{r^3}\frac{Af''(\eta)}{\alpha^2} - \frac{Af(\eta)}{r^3}\right] \ . \tag{15}$$

Now dividing both sides by "$\frac{2A}{r^3}$", we obtain

$$-Af^2(\eta) = 4vf(\eta) + v\left[2f(\eta) - f(\eta) + \frac{f''(\eta)}{\alpha^2} - f(\eta)\right] \ .$$

After simplify the above equation takes the final form as,

$$Af^2(\eta) + 4vf(\eta) + v\frac{f''(\eta)}{\alpha^2} = 0 \ . \tag{16}$$

Differentiate the Eq. (16) w.r.t "η" we can get,

$$2Af'(\eta)f(\eta) + 4vf'(\eta) + v\frac{f'''(\eta)}{\alpha^2} = 0 \ . \tag{17}$$

Multiplying "$\frac{\alpha^2}{v}$" with the Eq. (17) both sides, we get

$$2\frac{\alpha^2}{v}Af'(\eta)f(\eta) + 4\alpha^2 f'(\eta) + f'''(\eta) = 0 \ . \tag{18}$$

Put $\mathrm{Re} = \frac{\alpha^2}{v}A$ (19)

Substitute the Eq. (19) into Eq. (18), we obtain boundary value problem of a third order ordinary differential equation for the normalized function profile $f(\eta)$,

$$f'''(\eta)+2\alpha\,\mathrm{Re}\,f'(\eta)f(\eta)+4\alpha^2 f'(\eta)=0\ . \qquad (20)$$

With boundary conditions

$$f(0)=1,\ f'(0)=0,\ f(1)=0, \qquad (21)$$

Here "Re" is the Reynolds numbers, which is defined as:

$$\mathrm{Re}=\frac{f_{\max}\alpha}{v}=\frac{U_{\max}r\alpha}{v}\begin{cases} divergent-channel,\alpha>0,f_{\max}>0\\ convergent-channel,\alpha<0,f_{\max}<0\end{cases} \qquad (22)$$

1.3 **Neural Networks Modeling**

The solution of the Jeffery-Hamel problems $f(\eta)$ through neural networks which are well known approximates and its "n[th]"order derivatives $f^{(n)}(\eta)$ can be approximated by the following continuous mapping in this methodology. We construct the mathematical model based on active set (AST), Interior Point technique (INT) with fitness function. The following activation functions called log-sigmoid based on logarithmic functions was used in the mapping [40-44].

$$LS=\frac{1}{1+e^{-(C+Bx)}}\ . \qquad (22)$$

The solution $f(\eta)$ of the differential equation (20) along with its third order derivative $f^{(3)}(\eta)$ can be approximated and $\hat{f}(\eta)$ is defined as

$$\hat{f}(\eta)=\sum_{i=1}^{N} A_i\phi(B_i\eta+C_i)\ . \qquad (23)$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are real-valued bounded adaptive parameters, can be combined in vector form as written as:

$$\mathbf{W}=(A_1,A_2,...,A_N,B_1,B_2,...,B_N,C_1,C_2,...,C_N),$$

where N is the number of neurons. This proposed mathematical model is using f_{LS}, for the approximation of the solution of Eq. (23) along with first and 3[rd] order derivative.

$$\hat{f}_{LS}(\eta)=\sum_{i=1}^{N}\frac{A_i}{1+e^{-(B_i\eta+C_i)}} \qquad (24)$$

$$\hat{f}'_{LS}(\eta) = \sum_{i=1}^{N} A_i B_i \left[\frac{e^{-B_i\eta - c_i}}{(1+e^{-B_i\eta - C_i})^2} \right] \quad (25)$$

$$\hat{f}'''_{LS}(\eta) = \sum_{i=1}^{N} A_i B_i^3 \left[\frac{6e^{-3(B_i\eta + c_i)}}{(1+e^{-B_i\eta - C_i})^4} - \frac{6e^{-2(B_i\eta + c_i)}}{(1+e^{-B_i\eta - C_i})^3} + \frac{e^{-(B_i\eta + c_i)}}{(1+e^{-B_i\eta - C_i})^2} \right]. \quad (26)$$

where (') mean derivative with respect to η. The mathematical model for Eq. (20) can be formulated by a linear combination of networks, Eq. (24), (25) and (26), called a differential equation neural network (DENN).

The fitness function for proposed model "E" has been formulated for the Eqs.(20)-(22) using Mathematical model by defining the unsupervised error as the sum of mean squared errors: $E = E_1 + E_2$.

The error term E_1 is connected with the physical problem (20) is given as:

$$E_1 = AVERAGE\left[\hat{f}_i''' + 2\alpha \operatorname{Re} \hat{f}_i \hat{f}_i' + 4\alpha^2 \hat{f}_i' \right]. \quad (27)$$

for $i = 1, 2, 3, ..., N$.

where $\hat{f}_i = \hat{f}(\eta_i)$, $\eta_i = Nh$ with increment 'h' i.e., value at N subintervals in [0,1], $[\eta_1, \eta_2]$, $[\eta_2, \eta_3]$..., $[\eta_{N-1}, \eta_N]$.

Also E_2 for initial values can be defined as

$$E_2 = AVERAGE\left[(\hat{f}_0 - 1)^2 + (\hat{f}_0')^2 + (\hat{f}_N)^2 \right] \quad (28)$$

Optimization procedure for numerical result. Furthermore, we give in detail about the procedural steps for the optimization in MATLAB Optimtool, is given below.

- Step 1:Initialization: A vector is generated bounded real values of length equal to the number of weights in given Mathematical model randomly plays as the starting point for each solver: $\mathbf{W} = (A_1, A_2, ..., A_N, B_1, B_2, ..., B_N, C_1, C_2, ..., C_N),$ Here N represents the number of neurons.
- Step 2: Fitness Evaluation: The MATLAB Optimtool for constrained optimization problems is invoked for each model.
- Step 3: Termination Criteria: Terminate the execution of the solver, if any of the following criteria is satisfied:
  - required level of predefined fitness obtained, i.e., $\mathbf{E \le 10^{-12}}$.
  - total number of iterations executed.
- Step 4: Storage: Save the final optimal weights (variables) along with fitness values and total computational time taken by the algorithm.
- Step 5: Statistical Analysis: Repeat the process from steps 1 to 4 for sufficiently large number of runs to perform an effective and reliable statistical analysis.

1.4Numerical solutions

In this section we consider the case of Jeffery-Hamel flows with Reynolds number $\operatorname{Re} = 110$ and channel angles $\alpha = 3^0$, we show that the solutions of Jeffery-Hamel flow problems with the uses of proposed method along with two optimizer like AST and INT techniques. Further, the exact solution for this equation is not available, therefore we calculate the values of f_{RF} used as a reference solution with technique of MATHEMATICA in this case and we now apply the neural network models with 10 neurons each to solve the problem. In each model there are a total of 30 unknown adjustable parameters or weights, its numerical solution $\hat{f}_{LS}$ with the help of proposed method. Moreover, we calculate the value of absolute error $\left|f_{RS} - \hat{f}_{LS}\right|$ with AST and INT. Through these solvers we showed that the present solution is highly accurate as compared to others methods present in literature.

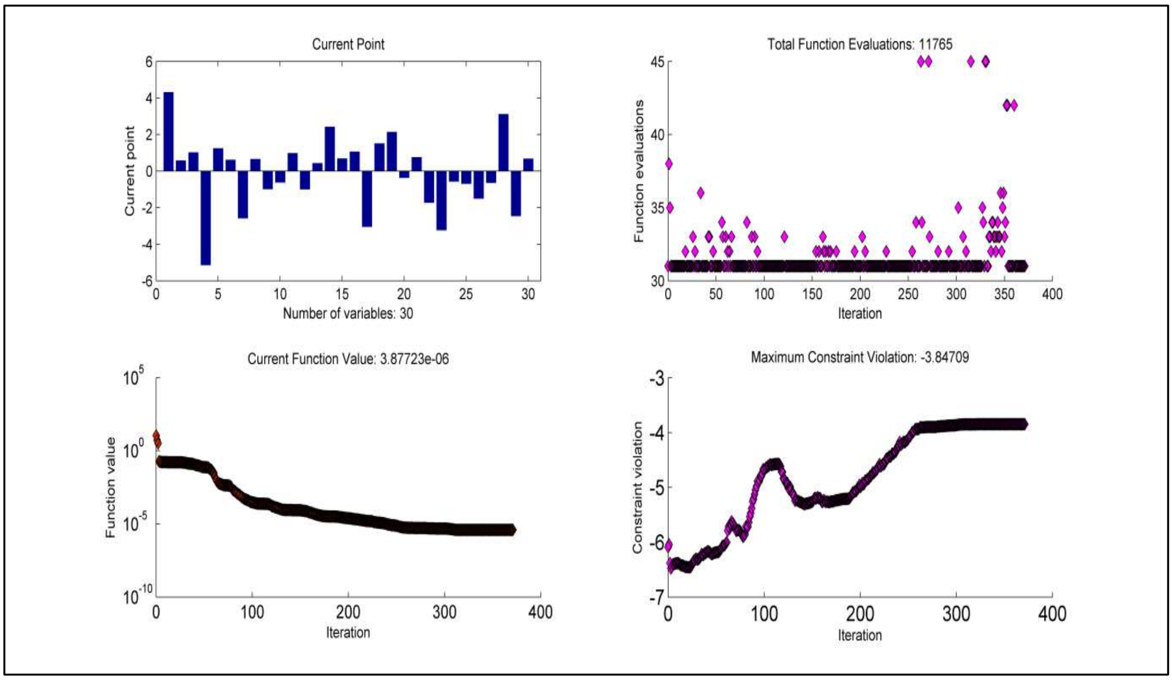

Figure1. AST technique result for proposed MHD problem

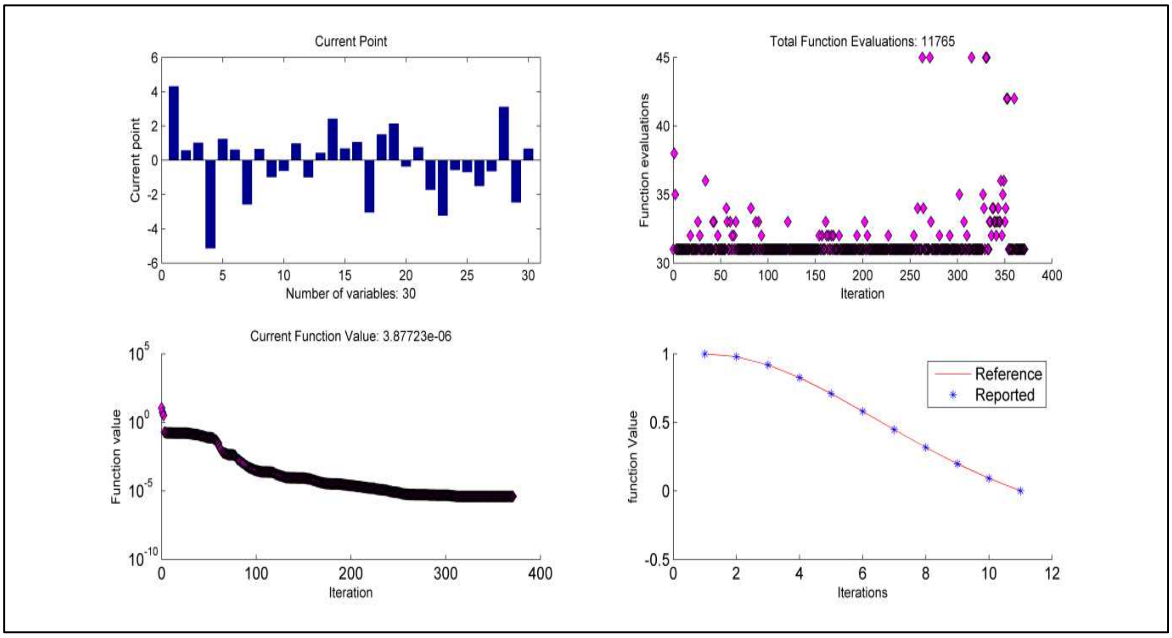

Figure 2. Comparison of numerical result of AST with Reference solution

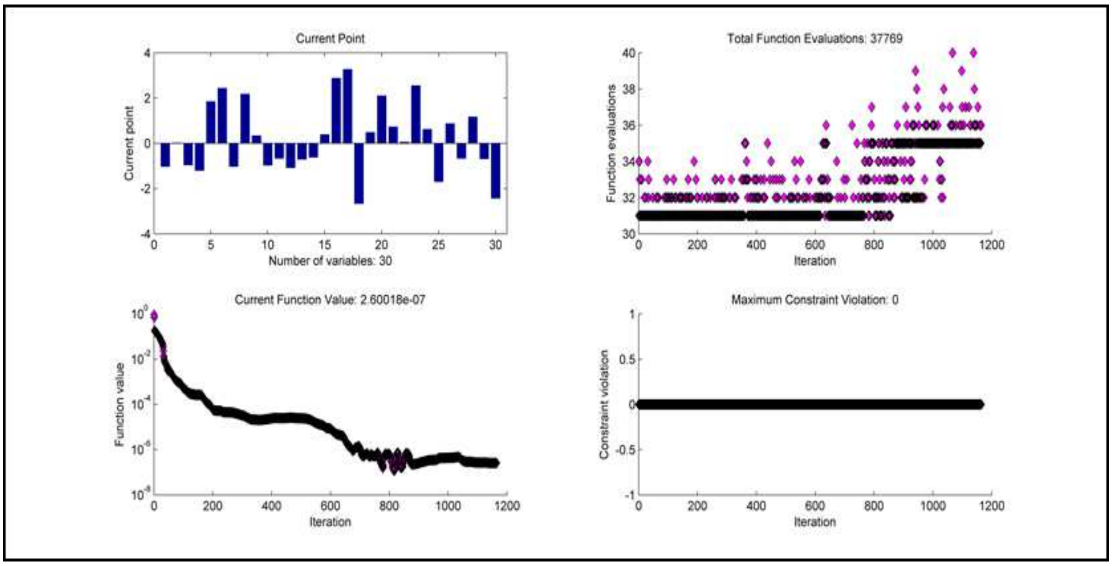

Figure3. Numerical result of INT for Proposed problem.

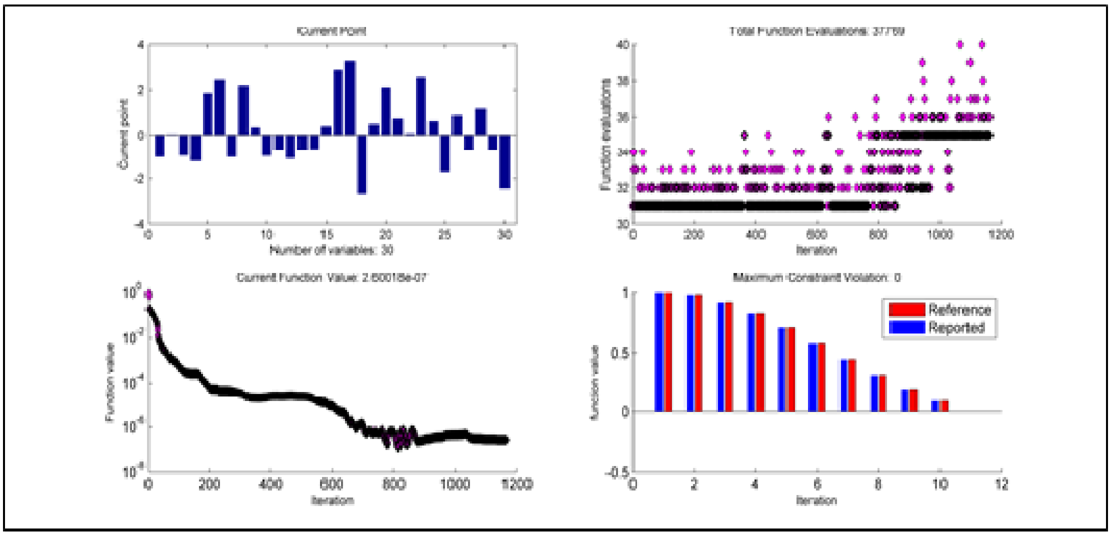

Figure 4. Comparison of numerical result of INT with Reference solution

Fig. 1 and Fig. 2 are shown the behavior of flow through AST and its comparison with reference solution and similarly Fig. 3 and Fig. 4 are represented the behavior of Jeffery-Hamel flow through INT technique and provide its comparison with reference solution. The numerical solutions obtained by the neural network models consistently overlap the reference solution, as shown in both figures. In order to elaborate small differences, values of absolute error (AE) are calculated, and results reported in Table I and Table II of AST and INT techniques respectively and their graphical representation is shown in Fig. 5 and Fig. 6 in this case.

Table1: Absolute Error (AE) for multi-runs of AST technique.

Result/ η	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
AE1	2.34E-05	2.20E-05	1.97E-05	1.68E-05	1.38E-05	1.11E-05	9.17E06	8.20E-06	8.43E-06	9.96E-06	1.29E-05
AE2	4.72E-05	4.48E-05	4.03E-05	3.45E-05	2.82E-05	2.27E-05	1.87E-05	1.63E-05	1.62E-05	1.90E-05	2.46E-05
AE3	4.56E-05	4.34E-05	3.97E-05	3.48E-05	2.90E-05	2.41E-05	2.08E-05	1.92E-05	1.94E-05	2.25E-05	2.86E-05
AE4	4.85E-05	4.61E-05	4.11E-05	3.36E-05	2.44E-05	1.58E-05	8.78E-06	2.62E-06	2.19E-06	4.15E-06	3.63E-06
AE5	5.35E-05	5.04E-05	4.56E-05	3.97E-05	3.34E-05	2.83E-05	2.52E-05	2.44E-05	2.61E-05	3.12E-05	4.00E-05
AE6	9.94E-06	9.47E-06	8.68E-06	7.49E-06	5.76E-06	4.22E-06	3.23E-06	2.38E-06	1.64E-06	1.60E-06	2.23E-06
AE7	8.27E-05	7.80E-05	7.00E-05	5.96E-05	4.83E-05	3.87E-05	3.19E-05	2.80E-05	2.78E-05	3.29E-05	4.31E-05
AE8	4.70E-05	4.46E-05	4.01E-05	3.42E-05	2.74E-05	2.14E-05	1.69E-05	1.40E-05	1.32E-05	1.51E-05	1.98E-05
AE9	1.34E-05	1.32E-05	1.22E-05	1.03E-05	7.76E-06	5.19E-06	2.94E-06	8.46E-07	9.77E-07	2.09E-06	2.58E-06
AE10	3.43E-05	3.23E-05	2.93E-05	2.58E-05	2.23E-05	1.97E-05	1.84E-05	1.87E-05	2.07E-05	2.51E-05	3.18E-05
AE11	5.24E-05	4.95E-05	4.49E-05	3.91E-05	3.26E-05	2.72E-05	2.41E-05	2.29E-05	2.39E-05	2.83E-05	3.63E-05
AE12	2.66E-05	2.55E-05	2.30E-05	1.92E-05	1.51E-05	1.12E-05	7.87E-06	5.33E-06	3.97E-06	4.05E-06	5.47E-06
AE13	4.58E-05	4.38E-05	3.91E-05	3.21E-05	2.35E-05	1.55E-05	8.84E-06	3.11E-06	1.33E-06	3.23E-06	2.79E-06
AE14	8.27E-05	7.80E-05	7.00E-05	5.96E-05	4.83E-05	3.87E-05	3.19E-05	2.80E-05	2.78E-05	3.29E-05	4.31E-05
AE15	8.69E-05	8.12E-05	7.28E-05	6.29E-05	5.30E-05	4.51E-05	4.06E-05	4.01E-05	4.39E-05	5.34E-05	6.88E-05

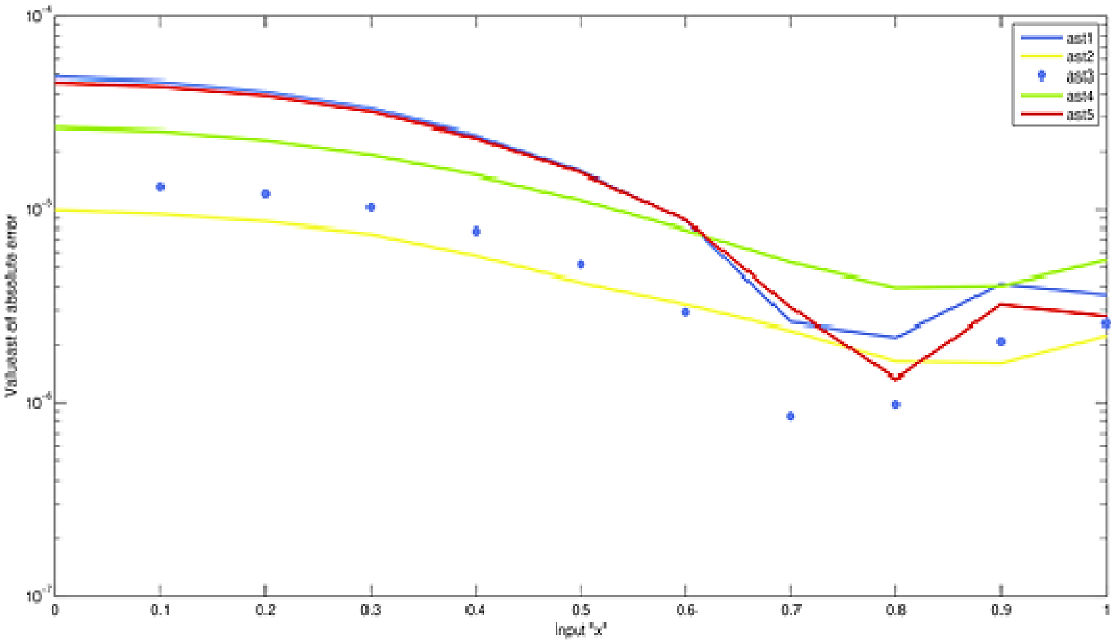

Figure5. Absolute Error of AST technique with Reference solution.

Table2: Absolute Error (AE) for multi-runs of INT technique.

Result/ η	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
INT1	9.25E-05	8.76E-05	7.82E-05	6.55E-05	5.12E-05	3.84E-05	2.84E-05	2.12E-05	1.78E-05	2.00E-05	2.74E-05
INT2	3.49E-05	3.37E-05	3.06E-05	2.63E-05	2.14E-05	1.65E-05	1.28E-05	1.03E-05	9.00E-06	9.72E-06	1.25E-05
INT3	7.77E-05	7.37E-05	6.64E-05	5.73E-05	4.76E-05	3.89E-05	3.30E-05	3.03E-05	3.10E-05	3.65E-05	4.69E-05
INT4	6.23E-05	5.95E-05	5.36E-05	4.61E-05	3.83E-05	3.09E-05	2.55E-05	2.28E-05	2.30E-05	2.65E-05	3.40E-05
INT5	7.89E-05	7.52E-05	6.79E-05	5.84E-05	4.81E-05	3.87E-05	3.19E-05	2.82E-05	2.79E-05	3.22E-05	4.15E-05
INT6	3.83E-05	3.64E-05	3.28E-05	2.82E-05	2.35E-05	1.91E-05	1.60E-05	1.45E-05	1.48E-05	1.73E-05	2.22E-05
INT7	9.80E-06	9.61E-06	8.74E-06	7.41E-06	5.81E-06	4.12E-06	2.64E-06	1.47E-06	5.97E-07	1.68E-07	2.32E-07
INT8	2.26E-05	2.19E-05	1.98E-05	1.69E-05	1.35E-05	1.00E-05	7.21E-06	5.13E-06	3.76E-06	3.61E-06	4.72E-06
INT9	2.50E-05	2.40E-05	2.15E-05	1.83E-05	1.48E-05	1.13E-05	8.57E-06	6.95E-06	6.28E-06	6.90E-06	9.04E-06
INT10	3.49E-05	3.37E-05	3.06E-05	2.63E-05	2.14E-05	1.65E-05	1.28E-05	1.03E-05	9.00E-06	9.72E-06	1.25E-05
INT11	6.44E-05	6.10E-05	5.50E-05	4.75E-05	3.98E-05	3.31E-05	2.85E-05	2.67E-05	2.82E-05	3.34E-05	4.27E-05
INT12	9.68E-05	9.17E-05	8.25E-05	7.10E-05	5.89E-05	4.83E-05	4.11E-05	3.79E-05	3.91E-05	4.63E-05	5.95E-05
INT13	6.92E-05	6.54E-05	5.90E-05	5.12E-05	4.32E-05	3.63E-05	3.18E-05	3.04E-05	3.25E-05	3.87E-05	4.94E-05
INT14	7.77E-05	7.37E-05	6.64E-05	5.73E-05	4.76E-05	3.89E-05	3.30E-05	3.03E-05	3.10E-05	3.65E-05	4.69E-05
INT15	6.23E-05	5.95E-05	5.36E-05	4.61E-05	3.83E-05	3.09E-05	2.55E-05	2.28E-05	2.30E-05	2.65E-05	3.40E-05

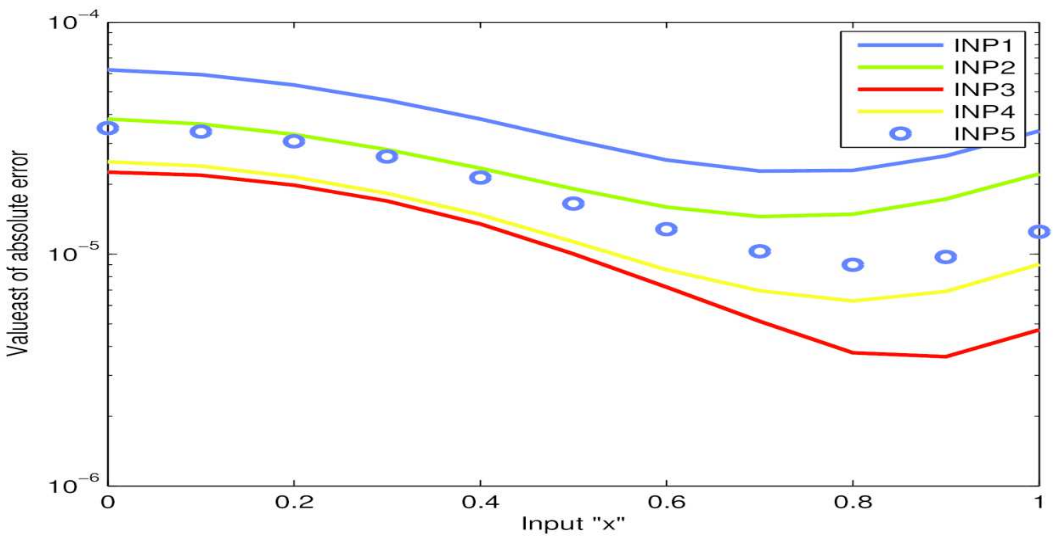

Figure6. Absolute Error of INT technique with Reference solution.

1.5 Conclusion.

- These solvers depend on neural network models using log-sigmoid function, optimized with active set and an interior point method can provide reliable solutions for the nonlinear two time transformed problem of the MHD Jeffery-Hamel flow equations.

- Comparative study of the results of the proposed models shows that solutions in case of log-sigmoid-INT and log-sigmoid-AST match upto 5 to 6 decimal places of accuracy. The results reported here are better in accuracy.
- The proposed solvers have some advantages over other numerical techniques.
- The beauty of proposed method is its simplicity.
- In future, one may work other computational intelligence techniques based on neural network models, optimized with global and local search algorithms.
- Moreover, one may explore to extend these methodologies to solve stiff, highly nonlinear differential equations with singularities and requiring convergent solutions on larger scale for better application of these problems.

REFERENCES

1. Jeffery, G. B. "The two-dimensional steady motion of a viscous fluid", Philoso-phical Magazine, (1915); 6 (29) 455-465.
2. Hamel, G. "Spiral förmige Bewgungen Zäher Flüssigkeiten, Jahresbericht der DMV-Deutsche Mathematiker-Vereinigung", (1916); 25 34-60.
3. Batchelor, K, "An Introduction to Fluid Dynamics", Cambridge University Press, (1967).
4. Sadri, R. M. "Channel Entrance Flow", Ph.D. Thesis, Department of Mechanical Engineering, The University of Western Ontario, (1997).
5. Hamadiche, M., Scott, J., &Jeandel, D. "Temporal stability of Jeffery-Hamel flow", Journal of Fluid Mechanics, (1994); 268 71-88.
6. LE, F. "Laminar flow in symmetrical channels with slightly curved walls I: on the Jeffery-amel solutions for flow between plane walls", Proceedings of Royal Society London, (1962). A 267 119-138.
7. Makinde, O. D., & Mhone, P. Y. "Hermite-Padé approximation approach to MHD Jeffery-Hamel flows", Applied Mathematics and Computation, (2006). 181 966-972.
8. Schlichting, H. "Boundary-layer Theory, McGraw-Hill Press", (2000). New York.
9. Axford, W. I. "The Magnetohydrodynamic Jeffrey-Hamel problem for a weakly conducting fluid", Quarterly Journal of Mechanics and Applied Mathematics, (1961), 14 335-351.
10. Jalaal, M., Ganji, D. D., & Ahmadi, G. "Analytical investigation on acceleration motion of a vertically falling spherical particle in incompressible Newtonian media", Advanced Powder, (2010). Technology, 21 (3) 298-304.
11. Jalaal, M., &Ganji, D. D, "On unsteady rolling motion of spheres in inclined tubes filled with in compressible Newtonian fluids", Advanced Powder Technology, (2011). (1) 58-67.
12. Jalaal, M., &Ganji, D. D,"An analytical study on motion of a sphere rolling down an inclined plane submerged in a Newtonian fluid", Power Technology, (2010); 198 (1) 82-92.
13. Ganji, D. D., & Sadighi, A, "Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations", International Journal of Nonlinear Sciences and Numerical Simulation, (2011). 7 (4) 411-418.
14. Domairry, G., A Mohsenzadeh, A., &Famouri, M, "The application of Homotopy analysis method to solve nonlinear differential equation governing Jeffery-Hamel flow", Communications in Nonlinear Science and Numerical Simulation, (2008);14 85–95.
15. Liao, S. J, "The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems", PhD thesis, Shanghai Jiao Tong University, (1992).
16. Esmaili, Q., Ramiar, A., Alizadeh, E., &Ganji, D. D, "An approximation of the analytical solution of the Jeffery-Hamel flow by decomposition method", Physics Letters A, (2008); 372 3434-3439.
17. Makinde, O. D, Effect of arbitrary magnetic Reynolds number on MHD flows inconvergent-divergent channels, International Journal of Numerical Methods Heat Fluid Flow, (2008); 18 (6) 697-707.
18. Hosseini, R., Poozesh, S., and Dinarvand, S, "MHD flow of an incompressible v through convergent or divergent channels in presence of a high magnetic field", Journal of Applied Mathematics, (2012); 2012/157067. (Article ID 157067).
19. Dinarvand, S, "Reliable treatments of differential transform method for two-dimensional incompressible viscous flow through slowly expanding or contracting porous walls with small-to-oder at epermeability", International Journal of Physical Sciences, (2012); 7 (8) 1166-1174.
20. He, J. H, "Coupling method of a homotopy technique and a perturbation technique for non-linear problems", International Journal of Non-linear Mechanics, (2000); 35 37-43.

21. Ganji, D. D, “A semi-Analytical technique for non-linear settling particle equation ofmotion”, Journal of Hydro-Environment Research, (2012); 6 (4) 323–327.
22. Jalaal, M., Ganji, D. D., and Ahmadi, G, “An analytical study on settling of non-spherical particles”, Asia-pacific Journal of Chemical Engineering, (2012); 7 (1) 63–72.
23. Sheikholeslami, M., Ganji, D. D., Ashorynejad, H. R., &Rokni, H. B. Analytical investigation of Jeffery-Hamel flow with high magnetic field and nano particle by Adomian decomposition method, Applied Mathematics and Mechanics-Engl. Ed, 33 25–36.
24. Bararnia, H., Ganji, Z. Z, Ganji, D. D., &Moghimi, S. M, “Numerical and analytical approaches to MHD Jeffery-Hamel flow in a porous channel”, International Journal of Numerical Methods for Heat and Fluid Flow, (2012); 22 (4) 491–502.
25. Abbasbandy, S., & Shivanian, E, “Exact an alytical solution of the MHD Jeffery-Hamel flow problem”, Meccanica, (2012); 47 (6) 1379-1389.
26. Mustafa, I., Akgül, A., Kýlýçman, A., &Ganji, D. D, “A new application of their producing Kernel Hilbert Space Method to solve MHD Jeffery-Hamel flows problem in non-parallel walls”, Abstract and Applied Analysis, Hindawi Publishing Corporation , (2013).
27. Marinca, V., &Heriþanu, N, “An optimal homotopy asymptotic approach applied to nonlinear MHD Jeffery-Hamel flow”, Mathematical Problems in Engineering, (2011); (Article ID169056,16pages).
28. McFall, K. S, “Automated design parameter selection for neural networks solving coupled partial differential equations with discontinuities”, Journal of the Franklin Institute, (2013); 350 (2) 300–317.
29. Garaluz, E. G., Atencia, M., Joya,G., Lagos, F., G., &Sandova, F, “Hopfield networks for identification of delay differential equations with an application to dengue fever epidemics in Cuba”, Neurocomputing, (2011); 74 (16) 2691-2697.
30. Tsoulos, I. G., Gavrilis, D., &Glavas, E, “Solving differential equations with constructed neural networks”, Neurocomputing, (2009); 72 (10–12) 2385-2391.
31. Choi, B., & Lee, J. H, “Comparison of generalization ability on solving differential equations using back propagation and reformulated radial basis function networks”, Neurocomputing, (2009); 73(1–3) 115-118.
32. Khan, J. A., Raja M. A. Z, & Qureshi, I. M, “Novel approach for vander Pol Oscillatoron the continuous Time Domain”, Chinese Physics Letters, (2011); 28 (11) 110205.
33. Raja M. A. Z, “Stochastic numerical techniques for solving Troesch's Problem”, Information Sciences, (2014); 279 860-873.
34. Li, X., & Ouyang, j, “Integration modified wavelet neural networks for solving thin plate bending problem”,Applied Mathematical Modelling, (2013);37 (5) 2983–2994.
35. Khan, J. A., Raja M. A. Z, & Qureshi, I. M, “Numerical treatment of nonlinear Emden- Fowler equation using stochastic technique”, Annals of Mathematics and Artificial Intelligence, (2011); 63 (02) 185-207.
36. Ping, Z, “Tracking problems of a spherical inverted pendulum via neural network enhanced design”, Neurocomputing, (2013); 106 137-147.
37. Raja M. A. Z, Khan, j. A., Ahmad, S.I., & Qureshi, I. M, “Numerical Treatment of Pain leve equation I using neural networks and stochastic solvers", Springer, book series, Studies in Computational intelligence, 442, Innovations in Intelligent Machines-3, chapter 7, (2013).
38. Kurková, v, “Surrogate modelling of solutions of integral equations by neural networks”, Artificial Intelligence Applications and Innovations, Springer, BerlinHeidelberg, (2012); 88-96.
39. Raja M. A. Z., & Ahmad, S. I, “Numerical treatment for solving one-dimensional Bratu Problem using Neural Networks”, Neural Computing and Application, online first, (2012).
40. Khan, J. A., & Raja, M. A. Z, “Artificial Intelligence based Solver for Governing Model of Radioactivity Cooling”, Self-gravitating Clouds and Clusters of Galaxies, Research Journal of Applied Sciences, Engineering and Technology, (2013); 6 (3) 450-456.
41. Raja , M.A. Z., Khan, J. A., & Qureshi, I. M, “A new stochastic approach for solution of Riccati differential equation of fractional order”, Ann Math ArtifIntell, (2010); 60 229-250.
42. M.A. Zahoor Raja, M. A., & Samar, R, “Numerical Treatment for nonlinear MHD Jeffery-Hamel problem using Neural Networks Optimized with Interior Point Algorithm”, Neurocomputing, (2014); 124, 178-193.
43. Saini, I., P. Singh, & Malik, V, “Genetic algorithem approach for solving the Falkner-Skan equation", World Academy of Science, Engineering and Technology, IJC, Information Science and Engineering, (2013); 7 3.
44. Ahmad, I. & Bilal, M, “Numerical Solution of Blasius Equation through Neural Networks Algorithm”, American Journal of Computational Mathematics, (2014);4 223-232.