The method of projections as applied to the numerical solution of two point boundary value problems using cubic splines

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Introduction; Notation

The discretization problem is rather central to the field of Numerical Analysis. In most general terms, it arises in the numerical solution of an equation

$$Mx = y^* \,, \tag{1}$$

where y^* is a given element of some set Y, a solution x^* being sought in some set X which is mapped by M to Y. The problem consists in choosing finitely many functionals on X and means for determining (approximately) the value of these functionals on the unknown solution x^* from the information available, i.e., from the map M and the element y^*.

The term "discretization" originated with the finite difference method for the solution of differential equations, where the functionals are chosen to be point functionals and the above process amounts to replacing the equation

$$(Mx)\,(t) = y^*\,(t)\,, \quad \text{all} \quad t \in S\,,$$

by an equation on a finite point set.

The method of projections is a rather general means to discretize (1) in case X and Y are linear spaces: the given equation (1) is replaced by

$$P_n M x = P_n y^* \,, \tag{2}$$

for which a solution, x_n^*, is sought in some n–dimensional subspace X_n of X; here P_n is a projection on Y of finite rank, i.e., $P_n^2 = P_n$ and $P_n[Y]$ is finite dimensional.

For the case of a differential equation, special instances of this method have been in use for some time; thus Galerkin's method, the Collocation method, the Least-Squares method and the method of moments all are projection methods, and even finite difference methods can be interpreted to be of this type.

Nevertheless, literature on the method of projections as such is scarce; the only book that contains a treatment of it is Kantorovich' and Akilov's "Functional analysis in normed spaces" [10, Ch.XIV]; and it seems justified (and not merely a matter of national pride) that the references given there are almost exclusively Russian.

But in [10], the method of projections appears only as a special case (referred to as having "the special structure described in XIV, 2.3") of an even more general setup. This seems to make it worthwile to give a more direct treatment of it, as is done in Sections 2 and 4 below.

From the outset, the following assumptions are made. M is taken to be a linear operator, mapping X in a 1-1 manner onto Y so that M^{-1} exists (and is linear); both X and Y are assumed to be normed linear spaces, and both M and M^{-1} are assumed to be bounded; P_n is taken to be a bounded linear projection operator. Under these assumptions, equation (1) admits of exactly one solution, x^*, and equation (2) is equivalent to

$$\lambda M x = \lambda y^* \quad \text{all} \quad \lambda \in \Lambda_n \,, \tag{2'}$$

where Λ_n is some n–dimensional space of linear functionals on Y. Hence, equation (2) has one and only one solution in X_n for arbitrary y^* if and only if the only element x in X_n for which $\lambda M x = 0$, all $\lambda \in \Lambda_n$, is $x = 0$.

These and other facts about projectors are discussed in Section 1. In Section 2, the method of projections is introduced, and some examples are given in Section 3, which also contains a short discussion of the relationship between the projection method and the method of finite differences.

A projection scheme for the solution of $Mx = y^*$ is given by a sequence $\{X_n\}$ of finite dimensional subspaces of X and a corresponding sequence $\{\Lambda_n\}$ of finite dimensional spaces of linear functionals on Y; and it is called convergent if, for all $y^* \in Y$, equations (2′) define an approximant $x_n^* \in X_n$ for all sufficiently large n and $\lim_{n\to\infty} \|x^* - x_n^*\| = 0$. Section 4 gives rather simple conditions under which the convergence of a projection scheme for the equation $Mx = y^*$ follows from its convergence for a, presumably simpler, equation $M_1 x = y^*$.

As an illustration, these results are applied to the numerical solution of the differential equation

$$x^{(2)}(t) + p_1(t)x^{(1)}(t) + p_0(t)x(t) = y^*(t), \quad t \in [0,1], \quad x(0) = x(1) = 0, \tag{3}$$

with the assumption that $p_0, p_1 \in C[0,1]$. X is taken to be the linear space $\{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$, and, correspondingly, $Y = C[0,1]$. The rôle of M_1 is played by

$$(M_1 x)(t) = x^{(2)}(t)\,.$$

In order to test the hypothesis that advantages can be gained by using for X_n a set of piecewise polynomial functions rather than the customary algebraic or trigonometric polynomials, X_n is chosen to consist of cubic splines. The necessary information about spline functions is contained in Section 5. In Section 6, the boundedness of M and M^{-1} with respect to certain norms on X and Y is established.

Section 7 contains a proof of the convergence of the Collocation method, followed by the construction of a higher order method: For both methods, order of convergence estimates are derived; but, whereas the Collocation method gives

$$\|x^* - x_n^*\|_\infty = O(n^{-j})\,, \quad \text{if } x^* \in C^{(2+j)}[0,1]\,, \quad j = 1,2, \tag{4}$$

which is commensurate with the rate achieved by finite difference approximations, the second method gives

$$\|x^* - x_n^*\|_\infty = O(n^{-2-j})\,, \quad \text{if } x^* \in C^{(2+j)}[0,1]\,, \quad j = 0,1,2. \tag{5}$$

The question as to the existence of a finite difference scheme giving such a rate of convergence is raised but, regrettably, not answered.

The estimate (5) can also be established for Galerkin's method, as is shown in Section 8. Section 9 contains a brief discussion of the Least-Squares method and, to encourage more work on it, of the Golomb-Weinberger method.

A few remarks concerning the computation of x_n^* make up the last Section. But this thesis is not supported by numerical evidence. Justification on practical grounds for studying the methods discussed in the following pages cannot be based on the few examples that the scope of a doctoral dissertation permits, but must come from extensive numerical experiments. It is the goal of this thesis to encourage such activity.

Notation

Throughout the text, the following conventions are used without further explanation.

"I" always denotes the identity map on the linear space given in the context.

For a map T from X to Y and a subset S of X, "$T|S$" denotes the restriction of T to S, while "$T[S]$" denotes the image of S under T.

"$\{x_i\}_1^n$" is shorthand for "$\{x_i : i = 1, \ldots, n\}$"; "$\{x_i\}$" denotes an infinite sequence.

If X is a linear space and $\{x_i\}_1^n \subset X$, then "$\langle \{x_i\}_1^n \rangle$" denotes the linear subspace of X spanned by the elements $x_1, \ldots, x_n$.

"$C^{(j)}[a, b]$" stands for the linear space of all real valued functions on $[a, b]$ which possess continuous derivatives up to and including the j–th; for $x \in C^{(j)}[a, b]$, "$x^{(j)}$" denotes the j–th derivative. As is usual, we write $C[a, b]$ for $C^{(0)}[a, b]$.

"$\mathrm{Lip}^{(j)}[a, b]$" denotes the set of all $x \in C^{(j)}[a, b]$ with the property that there exists a constant K so that

$$|x^{(j)}(s) - x^{(j)}(t)| \leq K|s - t|, \quad \text{all } s, t \in [a, b].$$

For $x \in C[a, b]$, "ω_x" denotes the modulus of continuity of x, i.e.,

$$\omega_x(h) = \sup\{ |x(s) - x(t)| : \ |s - t| \leq h, \quad s, t \in [a, b] \}, \quad h \geq 0,$$

while "$\|x\|_\infty$" is defined as

$$\|x\|_\infty = \max\{ |x(t)| : t \in [a, b] \}.$$

For both, the interval $[a, b]$ will be clear from the context and will usually be the interval $[0, 1]$.

"The Banach space $C[0, 1]$" is the linear space $C[0, 1]$ with the norm $\|x\| = \|x\|_\infty$.

Finally, "$\delta_c^{(j)}$" is shorthand for the rule which assigns to a function $x \in C^{(j)}[a, b]$ the number $x^{(j)}(c)$; therefore, with less precision, $\delta_c^{(j)}$ stands for the linear functional given by that rule on whatever space of functions the context may indicate. We write δ_c for $\delta_c^{(0)}$.

CHAPTER 1: THE METHOD OF PROJECTIONS

1. Linear projection operators.

Let X be a linear space; a linear operator $P : X \to X$ is called a **linear projection operator**, or, for short, a **projector**, if $P^2 = P$. In this section, various properties of projectors are listed. Most of the material can be found in such standard texts as [8] and [18].

If P is a projector, then so is $I - P$; each projector P determines a decomposition of X into an algebraic direct sum, $X = P[X] \dot{+} (I - P)[X]$, and each algebraic direct decomposition $X = X_1 \dot{+} X_2$ of X determines a projector P on X such that $P[X] = X_1$ and $(I - P)\,[X] = X_2$.

If P is a projector on X, and X^a is the algebraic dual of X (or, more generally, $\langle X, X^a \rangle$ is some pairing), then the **dual** of P is the linear map $P^a : X^a \to X^a$, given by the rule

$$P^a \lambda = \lambda \circ P, \quad \text{all } \lambda \in X^a,$$

so that $(P^a\lambda)\ (x) = \lambda(Px)$, all $\lambda \in X^a$ and all $x \in X$. P^a is also a projector, and the corresponding decomposition of X^a can be obtained from that of X for P: for $S \subset X$, define

$$S^- = \big\{\lambda \in X^a : \lambda\,[S] = \{0\}\big\};$$

then $P^a[X^a] = \{(I - P)\,[X]\}^-$ and $(I - P^a)\,[X^a] = \{P[X]\}^-$.

If P is of rank n, i.e., if dim $P[X] = n$, then so is P^a. Hence each projector P on X of rank n determines an n–dimensional subspace $X_1 = P[X]$ of X and an n–dimensional subspace $\Lambda_1 = P^a[X^a]$ of X^a satisfying $X_1 \cap \Lambda_1^- = \{0\}$ and $X = X_1 + \Lambda_1^-$, where without danger of confusion the set

$$\{x \in X : \quad \text{for all} \quad s \in S, \quad sx = 0\}$$

with $S \subset X^a$ is also denoted by S^-. Conversely, each n–dimensional subspace X_1 of X and each n–dimensional subspace Λ_1 of X^a satisfying $X_1 \cap \Lambda_1^- = \{0\}$ determine a projector P on X of rank n by the rule

$$Px \in X_1, \quad (x - Px) \in \Lambda_1^-, \quad \text{all } x \in X.$$

Therefore, each projector P on X of finite rank is an **interpolation** operator in the sense that P associates with each element $x \in X$ the unique element $Px \in X_1$ which interpolates x with respect to the linear functionals $\{\lambda_i\}_1^n$, i.e., satisfies

$$\lambda_i(Px) = \lambda_i x, \quad i = 1, \ldots, n,$$

where $\{\lambda_i\}_1^n$ is some basis of Λ_1.

For the remainder of this section, let X be a normed linear space. For a projector P on X to be continuous, it is necessary that both $P[X]$ and $(I - P)\,[X]$ be closed; this condition is also sufficient in case X is a Banach space. If P is of finite rank, then P is continuous if and only if Λ_1 consists of continuous linear functionals, i.e., $\Lambda_1 \subset X'$, where X' is the linear space of all continuous linear functionals on X, or, the topological dual of X.

The smaller the norm of P or $(I - P)$, the more apt is Px to be a good approximation to x. This is so because of

Lemma 1.1. *Let P be a continuous projector on the normed linear space X, with $X_1 = P[X]$; then*

$$\|x - Px\| \le \|I - P\|\,\|x - z\| \le \big(1 + \|P\|\big)\,\|x - z\|, \quad \text{all } x \in X, \quad \text{all } z \in X_1\,.$$

In particular, if $E(x,X_1) = \inf\{\|x-z\| :\ z \in X_1\}$ *is the distance of* x *from* X_1*, then*

$$E(x,X_1) \le \|x-Px\| \le \|I-P\| E(x,X_1) \le \big(1+\|P\|\big) E(x,X_1)\,.$$

We will make extensive use of this simple lemma later on.

Unless P is trivial, we have $\|P\| \ge 1$. If X is a Hilbert space, then $\|I-P\| = 1$ if and only if P is orthogonal projection. This fact generalizes in the following way to normed linear spaces: By Lemma 1.1, $\|I-P\| = 1$ if and only if P is a **metric** projection, i.e., if and only if

$$E(x,X_1) = \|x-Px\|, \quad \text{all } x \in X\,.$$

This serves to illustrate the fact that in normed linear spaces projectors of norm 1 are not very frequent, for even in a uniformly convex Banach space in which every closed linear subspace possesses a unique metric projection, it is rare for a metric projection to be linear.

In fact, even in a uniformly convex Banach space X, not every closed linear subspace X_1 need possess a projector at all (in the sense that there exists a continuous projector P on X with $X_1 = P[X]$);cf.,e.g.,[16]. Of course, by the Hahn-Banach Theorem, every finite dimensional subspace of a normed linear space possesses plenty of continuous projectors; and even though it may not possess a projector of norm 1, it possesses projectors of minimal norm, or **minimal** projectors as well as **optimal** projectors, i.e., projectors for which $\|I-P\|$ is minimal. Neither minimal nor optimal projectors need be unique, and it is not known how to construct either one even in rather simple cases.

We will have to deal later with a sequence $\{X_n\}$ of subspaces of a normed linear space X and a corresponding sequence $\{P_n\}$ of projectors and it will be important to know when $P_n \to I$, i.e., when P_n converges strongly to the identity on X, or, what is the same, when, for all $x \in X$, $\lim_{n\to\infty} \|P_n x - x\| = 0$. For a sequence $\{X_n\}$ of subspaces of X, define

$$\overline{\lim_{n\to\infty}}\ X_n = \big\{x \in X :\ \lim_{n\to\infty} E(x,X_n) = 0\big\}\,.$$

Lemma 1.2. *Let* X *be a normed linear space and* $\{P_n\}$ *a sequence of continuous projectors on* X*. Then the conditions that* $\overline{\lim}_{n\to\infty}\ P_n[X] = X$ *and* $\{P_n\}$ *is uniformly bounded are sufficient for* $P_n \to I$*. If* X *is a Banach space, then these conditions are also necessary.*

Proof. If $\{P_n\}$ is uniformly bounded, then there exists, by definition, $c > 0$ such that $\|P_n\| \le c$ for all n. If also $\overline{\lim}\ P_n[X] = X$, then, by Lemma 1.1,

$$\lim_{n\to\infty} \|x - P_n x\| \ \le\ \lim_{n\to\infty} (1+c)\, E(x, P_n[X]) = 0, \quad \text{all } x \in X,$$

showing that $P_n \to I$.

Conversely, assume that $P_n \to I$. Then obviously, $\overline{\lim}\ P_n[X] = X$, while $\{P_n\}$ is not necessarily uniformly bounded. But if X is a Banach space, then any strongly convergent sequence must be uniformly bounded, by the Banach-Steinhaus Theorem, cf.,e.g., [10, Thm.3,p.252], Q.E.D.

It is not sufficient for a sequence $\{X_n\}$ of finite dimensional subspaces of a Banach space X to satisfy $\overline{\lim}\ X_n = X$ in order to guarantee the existence of a sequence $\{P_n\}$

of projectors with $X_n = P_n[X]$ so that $P_n \to I$. A well-known example is the sequence $\{X_n\}$ in $X = C[-1,1]$ with $X_n = \{\sum_0^n a_i t^i\}$; any projector P_n on X to X_n satisfies $\|P_n\| \geq 2(\ln n)/\pi^2 + O(1)$ as is shown in [9].

That such examples exist in "almost all" Banach spaces, is partially the content of the following interesting conjecture [16]: If, for a Banach space X, there exists a $c > 0$ such that every finite dimensional linear subspace possesses a projector with norm $\leq c$, then X is (topologically and algebraically) isomorphic to a Hilbert space.

Finally, we note that although a sequence $\{P_n\}$ may not converge strongly to the identity, it may converge pointwise on (i.e., for each x in) some set which may be quite "large" and even dense even though it must be of the first category. Thus, for the example given, one can find a sequence $\{P_n\}$ of projectors which converges to the identity on the dense subset $C^{(1)}[-1,1]$ of $C[-1,1]$, e.g., interpolation at the zeroes of the appropriate Tschebycheff polynomial.

Lemma 1.3. *If $\{P_n\}$ is a sequence of projectors on the normed linear space X, then* $\lim \|x - P_n x\| = 0$ *for all $x \in X$ such that* $\lim \|P_n\| \, E(x, P_n[X]) = 0$.

2. The method of projections

Assume given a linear map $M : X \to Y$ from a linear space X onto a linear space Y, and assume that M is 1-1, so that M^{-1} exists. The problem is to find, given $y^* \in Y$, the element $x^* \in X$ such that $Mx^* = y^*$.

This problem is, in general, not solvable numerically; it is possible only to compute (approximately) the values of finitely many functionals on the unknown solution x^*. This fact can be reconciled with the desire to find an element of X (rather than finitely many numbers) by seeking an approximation x_n^* to x^* in some finite dimensional subspace X_n of X, and the method of projections (or, generalized Galerkin's method) is one way to accomplish this.

One picks an n–dimensional subspace X_n of X and an n–dimensional subspace Λ_n of Y^a so that $M[X_n] \cap \Lambda_n^- = \{0\}$. As $M[X_n]$ is then n–dimensional, $M[X_n]$ and Λ_n determine a projector, P_n. The corresponding approximant, x_n^*, to x^*, is then defined as

$$x_n^* = M^{-1} P_n y^* . \tag{2.1}$$

Since $P_n M x = Mx$ for $x \in X_n$, x_n^* is the solution in X_n to the "projected" equation

$$P_n M x = P_n y^* . \tag{2.2}$$

For the numerical determination of x_n^*, one picks a convenient basis $\{x_i\}_1^n$ of X_n and a basis $\{\lambda_i\}_1^n$ of Λ_n and determines the coefficients $a_1^*, \ldots, a_n^*$ of x_n^* with respect to this basis as the solution to the system of n linear equations

$$\sum_{j=1}^{n} \lambda_i \left(M x_j\right) a_j^* = \lambda_i y^* , \quad i = 1, \ldots, n . \tag{2.3}$$

The error in approximating x^* by x_n^* is

$$x^* - x_n^* = M^{-1}(I - P_n)\, y^* \,; \tag{2.4}$$

the accuracy of the approximation depends therefore on how well the "right side", y^*, can be approximated by elements in $M[X_n]$ using the projector P_n. As this is so, one would be tempted, in case Y is a metric space, to use for P_n a metric projection onto $M[X_n]$. But, unless Y is an inner product space, this would entail solving the nonlinear problem of finding the best approximation to y^* by elements in $M[X_n]$; and, as the elements of $M[X_n]$ can have quite a complicated description in practice, this will often be a very difficult task. In view of Lemma 1.1, it seems to be more advisable to attempt instead to find a projector P_n of small norm.

The mapping $Q_n : x^* \to x_n^*$ is given by

$$Q_n = M^{-1} P_n M = \big(M \mid X_n\big)^{-1} P_n M \,; \tag{2.5}$$

hence, Q_n is a projector. In fact, Q_n is given by X_n and $M^a[\Lambda_n]$, where M^a is the dual of M, i.e., M^a is the linear map from Y^a to X^a given by the rule

$$M^a \lambda = \lambda \circ M, \quad \text{all } \lambda \in Y^a \,.$$

The projection method is, therefore, "merely" an interpolation method and consists in determining the element x_n^* in X_n which interpolates x^* with respect to the linear functionals $\{M^a \lambda_i\}_1^n$, where $\{\lambda_i\}_1^n$ is a basis for Λ_n.

We note that the method of projections can be dualized. We approximate the dual equation $M^a \lambda = \mu$ by $Q_n^a M^a \lambda = Q_n^a \mu$, $\lambda \in \Lambda_n$, where Q_n^a is the projector given by $M^a[\Lambda_n]$ and X_n. The notation is correct, Q_n^a is indeed the dual of Q_n. Thus, linear methods such as the method of finite differences, which compute approximately the value of some linear functionals on the unknown solution x^*, can be interpreted to be just the projection method applied to the dual of M (cf. Section 3).

The sad fact that the original problem cannot be solved numerically is usually made up for by a demonstration showing that one can come arbitrarily close to the solution by numerical methods. Accordingly, one has in the projection method a sequence $\{X_n\}$ of finite dimensional subspaces of X and a sequence $\{\Lambda_n\}$ of corresponding finite dimensional subspaces of Y^a, and can then hopefully show that the elements of the sequence $\{x_n^*\}$ of corresponding approximants are defined and that the sequence converges to x^* in some sense.

Let both X and Y be normed linear spaces, and assume that both M and M^{-1} are bounded. We will call a projection scheme for M given by the sequences $\{X_n\}$ and $\{\Lambda_n\}$ **convergent** if $M[X_n]$ and Λ_n define a projector, P_n, for all sufficiently large n and if the resulting sequence of approximants converges in norm to x^* for all $y^* \in Y$. Hence a projection scheme is convergent if and only if P_n (or Q_n) is defined for all $n \geq n_0$ and $P_n \to I$ (or $Q_n \to I$). In case one (and therefore the other) of X or Y is a Banach space, this implies that $\{P_n\}$ and $\{Q_n\}$ are uniformly bounded. But if Y is not complete, then it is possible for a sequence $\{P_n\}$ of projectors on Y to converge strongly to the identity even though none of the P_n are even bounded (cf., e.g., Section 7). We will call a projection scheme **boundedly convergent** if it is convergent and if the sequence $\{P_n\}$, and therefore the sequence $\{Q_n\}$, is uniformly bounded.

Lemma 2.1. *Let X and Y be normed linear spaces, and $M : X \to Y$ a bounded linear map with bounded inverse; let $\{X_n\}$ be a sequence of finite dimensional subspaces of X. Then it is possible to construct a boundedly convergent projection scheme for M using the sequence $\{X_n\}$ if and only if $\overline{\lim}\, X_n = X$ and, for some $c > 0$, each X_n possesses a projector with norm no greater than c. In case one (and therefore the other) of X or Y is a Banach space, these conditions are necessary to guarantee the existence of a convergent projection scheme using $\{X_n\}$.*

As an illustration, choose $X = \{x \in C^{(1)}[0,1] : x(0) = 0\}$, $Y = C[0,1]$, $Mx(t) = x^{(1)}(t)$, norm Y in the usual way and set

$$\|x\|_X = \|x^{(1)}\|_\infty, \quad \text{all } x \in X\,.$$

Then both X and Y are Banach spaces and M is an isometry. But we cannot establish a convergent projection scheme for M using polynomials, i.e., using the sequence $\{X_n\}$ with $X_n = \{\sum_0^n a_i t^i\}$, even though $\overline{\lim}\, X_n = X$, since by the result cited in Section 1 the sequence $\{M[X_n]\}$ fails to have a bounded sequence of projectors. In contrast, if we choose for X_n the set of all piecewise parabolic functions in X with interior knots (cf. Section 5) i/n, $i = 1, \ldots, n-1$, then X_n possesses a projector P_n of norm 1, given by X_n and $\langle \{\delta_{i/n}^{(1)}\}_{i=1}^n \rangle$.

This also illustrates the point implicit in this thesis that there is theoretically, if not practically, no difficulty in constructing (boundedly) convergent projection schemes for the solution of ordinary linear differential equations using piecewise polynomial functions. But lest I be accused of polemic I hasten to point out that by Lemma 1.3 projection schemes which fail to be convergent still may give a sequence of approximants which converge to x^*, provided y^* is "nice" enough. Kantorovich and Akilov [10] give several examples of this nature.

3. Examples

The method of projections has been used in various concrete forms for some time (cf., e.g., [5, Kap.I, par.4] and [10, ch.XIV] and the references given there). These forms have been honored with different names according to specific ways of choosing Λ_n. In the German literature, they all fall under the heading of "Fehlerabgleichungsmethoden", as the approximation x_n^* is determined by the condition that the defect $(Mx_n^* - y^*)$ be "small" in some sense, viz., so that $P_n(Mx_n^* - y^*) = 0$, or, what is the same, so that

$$\lambda_i(Mx_n^* - y^*) = 0, \quad i = 1, \ldots, n,$$

where $\Lambda_n = \langle \{\lambda_i\}_1^n \rangle$.

As this thesis will deal with second order ordinary linear differential equations only, the examples are brought in this setting. Accordingly, let M be a second order ordinary linear differential operator in normal form,

$$(Mx)(t) = x^{(2)}(t) + p_1(t)x^{(1)}(t) + p_0(t)x(t), \quad t \in [0,1], \tag{3.1}$$

with $p_0, p_1 \in C[0,1]$; set $X = \{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$, and assume that M is 1-1 on X. Then M maps X onto $Y = C[0,1]$. We will write $X_n = \langle \{x_i\}_1^n \rangle$, and $\Lambda_n = \langle \{\lambda_i\}_1^n \rangle$.

Computationally perhaps the simplest method is the **Collocation** method, for which

$$\lambda_i y = y(t_i), \quad i = 1, \dots, n,$$

with $0 \le t_1 < t_2 < \cdots < t_n \le 1$, so that P_n becomes interpolation by elements in $M[X_n]$ at the points t_i, $i = 1, \dots, n$.

Less simple is the **Least-squares** method, in which P_n becomes orthogonal projection, i.e.,

$$\lambda_i y = \int_0^1 y(t)\,(Mx_i)\,(t)\,dt, \quad i = 1, \dots, n.$$

Galerkin's method requires the defect to be orthogonal to X_n rather than $M[X_n]$, so that

$$\lambda_i y = \int_0^1 y(t) x_i(t)\,dt, \quad i = 1, \dots, n,$$

and coincides more or less with **Ritz**' method whenever the latter is applicable.

In the "**Sub-domain**" method, the λ_i are given by

$$\lambda_i y = \int_{t_{i-1}}^{t_i} y(t)\,dt, \quad i = 1, \dots, n,$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$, while the "**orthogonality**" method or method of **moments** comes closest to the generality of the previous section: the λ_i are given by

$$\lambda_i y = \int_0^1 y(t)\phi_i(t)\,dt, \quad i = 1, \dots, n,$$

where $\{\phi_i\}_1^n$ are some functions, usually the first n of a complete orthonormal set on $[0,1]$.

To this list, I would like to add the **Golomb-Weinberger** method to be discussed in Section 9, which consists in choosing, given Λ_n, the subspace X_n of X in such a way that Q_n (given by X_n and $M^a[\Lambda_n]$) becomes orthogonal projection with respect to some convenient inner product; and also the method of finite differences or nets.

The method of finite differences consists of picking points $0 \le t_0 < t_1 < \cdots < t_n \le 1$ and finding approximations to the numbers

$$\mu_i x^* = x^*(t_i), \quad i = 1, \dots, n$$

in the following way. One chooses numbers a_{ij} such that

$$\sum_j a_{ij}\mu_j - M^a \lambda_i, \quad i = 1, \dots, n, \tag{3.2}$$

can be expected to be small on x^*, where $\lambda_i y = y(t_i)$, $i = 1, \ldots, n$; usually, one sets $a_{ij} = 0$ for $|i-j|$ greater than some given integer. Then one solves, if possible,

$$\sum_j a_{ij} b_j = (M^a \lambda_i) x^*, \quad i = 1, \ldots, n,$$

for the b_i's, the right side being known since

$$(M^a \lambda_i) x^* = \lambda_i (M x^*) = \lambda_i y^* = y^*(t_i), \quad i = 1, \ldots, n.$$

Because the linear functionals (3.2) are supposedly small on x^*, the number b_i is then hopefully a good approximation to the number $\mu_i x^*$, $i = 1, \ldots, n$.

In other words, one establishes an easily invertible map Q_0^a from $S = \langle \{\mu_i\}_1^n \rangle$ to $M^a[\Lambda_n]$, where $\Lambda_n = \langle \{\lambda_i\}_1^n \rangle$, and assumes that x^* is annihilated by $\mu - Q_0^a \mu$, for all $\mu \in S$. To establish the connection with the projection method, extend Q_0^a in any of the many possible ways to a projector Q_n^a on X^a to $M^a[\Lambda_n]$; for this, we can pick any n–dimensional subspace X_n of X satisfying $X_n^- \cap M^a[\Lambda_n] = \{0\}$ and $X_n^- \cap \{\mu - Q_0^a \mu : \mu \in S\} = \{0\}$, and then define Q_n^a as the projector given by $M^a[\Lambda_n]$ and X_n. Then $Q_n^a|S = Q_0^a$ and the projection method with X_n and Λ_n gives an approximant x_n^* satisfying $\mu_i x_n^* = b_i$, $i = 1, \ldots, n$.

In conclusion, any such finite difference scheme can be considered to be a special instance of the Collocation method; and any concrete example of the Collocation method for which the resulting projector Q_n^a is 1-1 on S gives rise to a finite difference scheme.

4. How to construct convergent projection schemes

Except in special cases, e.g., in the Least-squares method and its dual, the Golomb-Weinberger method, or in the Galerkin method when M is positive definite, it is not clear, given X_n and Λ_n, that $M[X_n]$ and Λ_n define a projector; and even if this can be ascertained for a given sequence $\{X_n\}$ and a sequence $\{\Lambda_n\}$ at least for large enough n, it is not obvious under what conditions on $\{X_n\}$ and $\{\Lambda_n\}$ the projection scheme for M is convergent.

One difficulty arises from the fact that even though X_n may be a set of functions about which we know quite a bit, such as polynomials or piecewise polynomial functions, the subspace $M[X_n]$ of Y can be made as complicated as we wish with the "proper" choice of p_0, p_1. Of course, if X_n consists of piecewise polynomial functions, and $M = M_1$, where

$$(M_1 x)(t) = x^{(2)}(t), \quad \text{all } t \in [0, 1],$$

then $M[X_n]$ consists also of piecewise polynomial functions, so that the existence of P_n is more easily ascertained in this case. In this section, we prove a theorem which permits us to conclude the convergence of a projection scheme for M from its convergence for M_1.

We return to the generality of Section 2, and assume given a bounded linear map $M : X \to Y$, where X and Y are normed linear spaces. We need the notion of total boundedness. A subset S of a normed linear space X is **totally bounded** if, given $\in > 0$, there exists a finite subset $\{x_i\}_1^n$ of X such that

$$S \subset \bigcup_{i=1}^{n} \{x \in X : \|x - x_i\| \leq \in \}.$$

A linear operator $T : X \to Y$ is **totally bounded** if T maps bounded sets into totally bounded sets. Hence T is totally bounded if the set

$$\{Tx : x \in X, \quad \|x\| \leq 1\}$$

is totally bounded. Thus, any compact (or completely continuous) linear operator is totally bounded, but the converse is not true.

Lemma 4.1. *Let X and Y be normed linear spaces and $T : X \to Y$ a totally bounded linear operator; let $\{R_n\}_1^\infty$ be a uniformly bounded sequence of linear operators on Y, which converges strongly to some bounded linear operator R on Y. Then* $\lim \|R_n T - RT\| = 0$, *i.e., the sequence $\{R_n T\}$ converges uniformly to RT.*

Proof. This lemma is a consequence of the well-known fact (c.f., e.g., [11, Thm.8.17]) that for an equicontinuous family of linear maps from one topological vector space to another the topologies of pointwise convergence and of uniform convergence coincide on totally bounded sets. But, for completeness, we prove the lemma directly.

It is sufficient to give the proof for the case $R = 0$. We have

$$\|R_n T\| = \sup\big\{\|R_n Tx\| : x \in X, \quad \|x\| \leq 1\big\} = \sup\big\{\|R_n y\| : y \in B\big\}, \tag{4.1}$$

where $B = \{Tx : x \in X, \|x\| \leq 1\}$, so that B is a totally bounded set. By assumption, there is a $c > 0$ so that, for all n, $\|R_n\| \leq c$. Let $\varepsilon > 0$ be given; then there exists a finite set $\{y_i\}_1^r \subset Y$ such that

$$B \subset \bigcup_{i=1}^{r} \big\{y \in Y : \|y - y_i\| \leq \varepsilon/2c\big\}.$$

Since $R_n \to 0$, there exists n_0 so that for $n \geq n_0$ and $i = 1, \ldots, r$, $\|R_n y_i\| < \varepsilon/2$. But then, for all $n \geq n_0$ and $i = 1, \ldots, r$, we have

$$\|R_n y\| \leq \|R_n y_i\| + \|R_n(y - y_i)\| < \varepsilon/2 + c\|y - y_i\| \leq \varepsilon,$$

whenever $\|y - y_i\| < \varepsilon/2c$, so that for $n \geq n_0$,

$$\sup\big\{\|R_n y\| : y \in B\big\} \leq \max_{i=1,\ldots,r} \sup\big\{\|R_n y\| : \|y - y_i\| \leq \varepsilon/2c\big\} < \varepsilon.$$

As ε was arbitrary, we have, with (4.1), that $\lim \|R_n T\| = 0$, Q.E.D.

Theorem 4.1. *Let X and Y be normed linear spaces, let M and M_1 be bounded linear maps form X to Y with bounded inverse and such that*

$$M_2 = M - M_1$$

is totally bounded. Let $\{Y_n\}$ be a sequence of finite dimensional subspaces of Y such that $\overline{\lim}\ Y_n = Y$, and let $\{\Lambda_n\}$ be a corresponding sequence of finite dimensional subspaces of Y' (the topological dual of Y) such that Y_n and Λ_n define a projector, $\bar{P}_n$, for all n. If the sequence $\{\bar{P}_n\}$ is uniformly bounded, then the projection scheme for M given by $\{M_1^{-1}[Y_n]\}$ and $\{\Lambda_n\}$ is boundedly convergent.

Proof. Observe that since $\overline{\lim}\ Y_n = Y$ and $\{\bar{P}_n\}$ is uniformly bounded, we have from Lemma 1.2 that $\bar{P}_n \to I$. Hence, by Lemma 4.1,

$$\lim \|M_2 - \bar{P}_n M_2\| = 0. \tag{4.2}$$

Set $X_n = M_1^{-1}[Y_n]$, all n. Since $\dim M[X_n] = \dim Y_n$, and Y_n and Λ_n define a projector $\bar{P}_n$, $M[X_n]$ and Λ_n define a projector, P_n, if and only if $M[X_n] \cap \Lambda_n^- = \{0\}$, or, as $\Lambda_n^- = \ker\ \bar{P}_n$, if and only if the map

$$R = \bar{P}_n \mid M[X_n] \tag{4.3}$$

is 1-1. Hence the existence of P_n is proven, once R is shown to be bounded below.

Set $M_0 = M \mid X_n$.We have, for $x \in X_n$, $(I - R)M_0 x = (I - \bar{P}_n)M_2 x$, hence

$$\|I - R\| = \sup\{\|y - \bar{P}_n y\| : y \in M[X_n], \|y\| \leq 1\} \leq \|M_2 - \bar{P}_n M_2\|\ \|M_0^{-1}\|. \tag{4.4}$$

Therefore,

$$\|Ry\| \geq \|y\| - \|(I - R)y\| \geq (1 - \|M_2 - \bar{P}_n M_2\|\ \|M_0^{-1}\|)\ \|y\|, \quad \text{all} \quad y \in M[X_n]. \tag{4.5}$$

Because of (4.2) and since $\|M_0^{-1}\| \leq \|M^{-1}\|$, (4.5) implies the existence of n_0 such that, for $n \geq n_0$, R is bounded below; thus P_n is defined.

But as R is of finite rank, R is then also onto Y_n, so that R^{-1} is defined and, by (4.5),

$$\|R^{-1}\| \leq (1 - \|M_2 - \bar{P}_n M_2\|\ \|M_0^{-1}\|)^{-1}. \tag{4.6}$$

Since $P_n = R^{-1}\bar{P}_n$, we have with this

$$\|P_n\| \leq \|R^{-1}\|\ \|\bar{P}_n\| \leq (1 - \|M_2 - \bar{P}_n M_2\|\ \|M_0^{-1}\|)^{-1}\ \|\bar{P}_n\|, \tag{4.7}$$

showing that $\{P_n\}$ is uniformly bounded. But as $M[X_n] = MM_1^{-1}[Y_n]$, and $(MM_1^{-1})^{-1}$ is bounded and $\overline{\lim}\ Y_n = Y$, we also have $\overline{\lim}\ M[X_n] = Y$, so that $P_n \to I$, Q.E.D.

Corollary 1. *Under the assumptions of the theorem, we have for* $n \geq n_0$

$$\|y^* - P_n y^*\| \leq \|y^* - \bar{P}_n y^*\| + \|R - I\| \, \|R^{-1}\| \, \|\bar{P}_n y^*\|, \tag{4.8}$$

and, therefore,

$$\|x^* - x_n^*\| \leq \|M^{-1}\| \Big(\|y^* - \bar{P}_n y^*\| + \|M_2 - \bar{P}_n M_2\| \, \|M_0^{-1}\| \cdot (1 - \|M_2 - \bar{P}_n M_2\| \, \|M_0^{-1}\|)^{-1} \|\bar{P}_n y^*\| \Big). \tag{4.9}$$

Proof. Observe that

$$I - P_n = I - R^{-1}\bar{P}_n = I - \bar{P}_n + (R - I)\, R^{-1}\bar{P}_n,$$

and use (4.4), (4.6) and (4.2), Q.E.D.

The error estimate (4.9) is not very helpful, as it is usually impossible in practice to compute the number $\|M^{-1}\|$ by any means short of knowing M^{-1}. Should it happen that $\|M_1^{-1} M_2\| < 1$, we get, of course, the estimate

$$\|M^{-1}\| \leq \|M_1^{-1}\| / (1 - \|M_1^{-1}\| \, \|M_2\|), \tag{4.10}$$

which is often simpler than it looks; in the application later on, e.g., we will choose the norms on X and Y always in such a way that M_1 is an isometry. Also, if in our application M is positive definite, there are standard ways of estimating $\|M^{-1}\|$. Otherwise, we must rely on the fact that $\|M_0^{-1}\|$ can be computed (even though that may not be easy) and use an estimate of $\|M^{-1}\|$ which uses only the numbers $\|M_0^{-1}\|$, $\|P_n\|$, $\|M_2 - \bar{P}_n M_2\|$ and $\|M\|$.

To derive such an estimate, recall that $x_n^* = Q_n x^*$, where Q_n is a projector, $Q_n = M_0^{-1} P_n M$, given by X_n and $M^a[\Lambda_n]$, so that by Lemma 1.1,

$$\|x^* - x_n^*\| \leq (\|Q_n\| + 1) \, \|x^* - x\|, \quad \text{all} \quad x \in X_n. \tag{4.11}$$

Now

$$\|x^* + M_1^{-1}\bar{P}_n M_2 x^*\| \leq \|M_1^{-1}\| \, \|M_1 x^* + \bar{P}_n M_2 x^*\| \leq \|M_1^{-1}\| \, (\|Mx^*\| + \|M_2 - \bar{P}_n M_2\| \, \|x^*\|),$$

and $M_1^{-1}\bar{P}_n M_2 x^* \in X_n$, so that with (4.11),

$$\|x^*\| \leq \|x^* - x_n^*\| + \|Q_n x^*\| \leq \|M_1^{-1}\| \, (\|Q_n\| + 1) \, (\|Mx^*\| + \|M_2 - \bar{P}_n M_2\| \, \|x^*\|) + \|Q_n x^*\|.$$

Since $\|Q_n x^*\| \leq \|M_0^{-1} P_n\| \, \|Mx^*\|$, this gives

$$\|x^*\| \Big(1 - \|M_1^{-1}\| \, (\|Q_n\| + 1) \, \|M_2 - \bar{P}_n M_2\|\Big) \leq \|Mx^*\| \, \Big(\|M_0^{-1} P_n\| + (\|Q_n\| + 1)\|M_1^{-1}\|\Big).$$

We have proved

Corollary 2. *For sufficiently large n, we have*

(4.12)

$$\|M^{-1}\| \le \Big(\|M_0^{-1}P_n\| + \big(\|Q_n\|+1\big)\|M_1^{-1}\|\Big)\Big/\Big(1 - \|M_1^{-1}\|\,(\|Q_n\|+1)\,\|M_2 - \bar{P}_n M_2\|\Big).$$

Remark. Theorem 4.1 is a slight extension of Theorem 6 in Ch. XIV of Kantorovich and Akilov [10]. The latter is proved there by referring back to a series of more general results. But because of the importance of Theorem 4.1 for the purposes of this thesis, it seemed advantageous to give a direct proof. The above proof of Theorem 4.1 is only vaguely related to the chain of reasoning employed in [10]. By contrast, the argument for Corollary 2 is essentially that for Theorem 4 in Ch. XIV of [10].

The content of Theorem 4.1 can be phrased somewhat differently.

Corollary 3. *Let M and M_1 be bounded linear maps with bounded inverse from the normed linear space X to the normed linear space Y. If $M_2 = M - M_1$ is totally bounded, then a projection scheme is boundedly convergent for M if and only if it is boundedly convergent for M_1.*

Finally, Theorem 4.1 starts with a uniformly bounded sequence $\{\bar{P}_n\}$ of projectors on Y satisfying $\bar{P}_n \to I$, for which the sequence $\{X_n\}$ has to be constructed via $X_n = M_1^{-1}\bar{P}_n[Y]$. In practice, one is more likely to start with the sequence $\{X_n\}$ and attempt to construct, for each n, a projector $\bar{P}_n$ with $\bar{P}_n[Y] = M_1[X_n]$. But one can also begin with a uniformly bounded sequence $\{\bar{Q}_n\}$ of projectors on X satisfying $\bar{Q}_n \to I$.

Corollary 4. *Let M and M_1 be bounded linear maps with bounded inverse from the normed linear space X to the normed linear space Y and let $\{\bar{Q}_n\}$ be a uniformly bounded sequence of projectors on X such that $\bar{Q}_n \to I$, where $\bar{Q}_n$ is given by X_n and Σ_n, all n. If $M_2 = M - M_1$ is totally bounded, then $\{X_n\}$ and $\{\Lambda_n\}$ give a boundedly convergent projection scheme for M, with $\Lambda_n = (M_1^{-1})^a[\Sigma_n]$, all n.*

Proof. Observe that the sequence $\{\bar{P}_n\}$, with $\bar{P}_n = M_1\bar{Q}_n M_1^{-1}$, all n, satisfies all assumptions of Theorem 4.1.

Note that the practical use of this corollary (as of Theorem 4.1) presupposes that M_1^{-1} is known.

CHAPTER 2: SPLINE FUNCTIONS

5. Spline functions

In this section, spline functions are introduced, and certain of their properties are derived. The main portion of the section is taken up with the investigation of two projectors on $C[0,1]$ to a subspace of cubic splines.

A **spline function** $x(t)$ **on** $[a,b]$ **of degree** $k \ge 0$ **with knots** (or joints) $\{t_i\}_1^n$, where $a < t_1 < t_2 < \cdots < t_n < b$, is, by definition, a function in $C^{(k-1)}[a,b]$ which on each of the intervals (a, t_1), (t_1, t_2), ... , (t_n, b) agrees with some polynomial of degree k (or less). Any such function can be written (in exactly one way) as

$$x(t) = \sum_{i=0}^{k} a_i t^i + \sum_{j=1}^{n} b_j (t - t_j)_+^k, \tag{5.1}$$

where we use the by now customary notation

$$(t)_+^m = \begin{cases} t^m, \ t \geq 0 \\ 0 \ , \ t < 0 \end{cases} \quad , \quad m \geq 0. \tag{5.2}$$

Denote the linear space of all such functions by

$$S_k \, (a, t_1, \ldots, t_n, b).$$

The literature on spline functions, with its earliest entry [13], has grown to some 40 articles. The few that are pertinent for this thesis are listed in the bibliography; for a rather complete list, see Chapter 10 and the bibliography of [12].

Of the many facts known about splines, we will need only the following. For a function $x(t)$ and points $t_0 < t_1 < \cdots < t_k$, denote by $x(t_0, \ldots, t_k)$ the k–th divided difference of x on these points.

Lemma 5.1 [7]. *Let $g_k(s;t) = (s-t)_+^{k-1}/(k-1)\,!$, $k \geq 1$. Then, for all $x \in C^{(k)}[a,b]$ and all points $\{t_i\}_0^k$ with $a \leq t_0 < t_1 < \cdots < t_k \leq b$,*

$$x(t_0, \ldots, t_k) = \int_a^b g_k(t_0, \ldots, t_k; s) x^{(k)}\,(s)\, ds. \tag{5.3}$$

Proof. By Taylor's formula with integral remainder,

$$x(t) = \sum_{i=0}^{k-1} x^{(i)}\,(a)\,(t-a)^i/i\,! + \int_a^b g_k(t;s) x^{(k)}\,(s)\, ds, \tag{5.4}$$

for all $t \in [a,b]$, from which (5.3) follows with the observation that for the $(k-1)$–st degree polynomial

$$y(t) \equiv \sum_{i=0}^{k-1} x^{(i)}\,(a)\,(t-a)^i/i\,!$$

any k–th divided difference is zero, Q.E.D.

We note that (5.3) remains valid even if some of the points coincide as long as $t_0 < t_k$. The function $g_k(t_0, \ldots, t_k; t)$ is called a **basic** or **fundamental** spline [13], [14].

Corollary. *The function* $x(t) = g_k(t_0, \ldots, t_k; t)$ *is nonnegative, and positive if and only if* $t \in (t_0, t_k)$. *Further*

$$\int_{t_0}^{t_k} g_k(t_0, \ldots, t_k; t)\, dt = 1/k\,!. \tag{5.5}$$

Proof. This follows at once from (5.3) and from the fact that

$$x(t_0, \ldots, t_k) = x^{(k)}(\xi)/k\,!, \quad \text{some } \xi \in (t_0, t_k), \tag{5.6}$$

but can, of course, be ascertained directly from the definition of $g_k(t_0, \ldots, t_k; t)$; Q.E.D.

Lemma 5.2. *Let* $n \geq 1$, $k \geq 0$, *and* $0 = t_0 < t_1 < \cdots < t_n = 1$, *and introduce auxiliary points* $t_i = i/n$, $i = -k, \ldots, -1, n+1, \ldots, n+k$. *Then* $\{x_{ik}(t)\}_{i=1}^{n+k}$, *given by*

$$x_{ik}(t) = g_{k+1}(t_{i-k-1}, t_{i-k}, \ldots, t_i; t), \quad t \in [0,1], \quad i = 1, ., n+k, \tag{5.7}$$

is a basis for $S = S_k(0, t, \ldots, t_{n-1}, 1)S$.

Proof. First, $\{x_i\}_1^{n+k} \subset S$, where we omit the subscript k. Also, because of the preceding corollary, the set $\{g_{k+1}(t_{i-k-1}, \ldots, t_i; t)\}_{i=1}^{n+k}$ is linearly independent. We need more of an argument as we must show their linear independence on $[0,1]$. So assume

$$\sum_{i+k}^{n+k} a_i x_i = 0.$$

Then, by (5.3),

$$\sum_{i=1}^{n+k} a_i y(t_{i-k-1}, \ldots, t_i) = 0 \tag{5.8}$$

for all $y \in C^{(k+1)}[t_{-k}, t_{n+k}]$, whose $(k+1)$–st derivative vanishes outside the interval $[0,1]$. But this implies $a_i = 0$, $i = 1, ., n+k$: If $i \leq n/2$, let $p(t)$ be the k–th degree polynomial satisfying

$$p(t_j) = 0, \quad j = i-k, \ldots, i-1, \quad p(t_i) = 1,$$

and let $q(t)$ be the $(2k+3)$rd degree polynomial satisfying

$$q^{(j)}(t_{i-1}) = 0, \quad q^{(j)}(t_i) = p^{(j)}(t_i), \quad j = 0, \ldots, k+1.$$

Then

$$y(t) = \begin{cases} 0 & , \ t < t_{i-1}, \\ q(t) & , \ t \in [t_{i-1}, t_i], \\ p(t) & , \ t > t_i, \end{cases}$$

is in $C^{(k+1)}$ and $y^{(k+1)}(t) = 0$ for $t \notin [0,1]$, while $y(t_{j-k-1}, \ldots, t_j) \neq 0$ if and only if $j = i$, so, by (5.8), $a_i = 0$. If $i \geq n/2$, use the symmetric construction to get $a_i = 0$ for $i \geq n/2$. In conclusion, $\{x_i\}_1^{n+k}$ is a linearly independent set. But, by (5.1), S is of dimension not greater than $k+1+n-1 = k+n$, so that $\{x_i\}_1^{n+k}$ is a maximal linearly independent set in S, hence a basis, Q.E.D.

The practical importance of this basis, which derives from the Corollary to Lemma 5.2, is discussed in Section 10.

The case of uniformly spaced knots is of particular interest.

Lemma 5.3. *Let $n \geq 1$, $k \geq 0$, and $t_i = i/n$, $i = 1, \ldots, n-1$, and let $\{x_{ik}\}_{i=1}^{n+k}$ be the basis of $S_k(0, 1/n, \ldots, (n-1)/n, 1)$ given by (5.7). Then, for $k \geq 1$,*

$$\left(\sum_{i=1}^{n+k} a_i x_{ik}(t)\right)^{(1)} = \frac{n}{k+1} \sum_{i=1}^{n+k-1} (a_{i+1} - a_i)\, x_{i,k-1}(t). \tag{5.9}$$

Also, for $k \geq 0$,

$$x_{ik}(t) \equiv x_{jk}(t + (j-i)/n), \quad \text{all } i, j, \tag{5.10}$$

$$\sum_{i=1}^{n+k} x_{ik}(t) \equiv n/(k+1)\,!\,. \tag{5.11}$$

Proof. We have $(\partial/\partial t) g_{k+1}(s;t) = -g_k(s;t)$, and

$$g_{k+1}(t_{i-k-1}, \ldots, t_i; t) = \Big(g_{k+1}(t_{i-k}, \ldots, t_i; t) - g_{k+1}(t_{i-k-1}, \ldots, t_{i-1}; t)\Big)/(t_i - t_{i-k-1}),$$

from which (5.9) follows. Also

$$g_{k+1}(t_{i-k-1}, \ldots, t_i; t) \equiv g_{k+1}(t_{i-k-1} + c, \ldots, t_i + c; t + c) \tag{5.12}$$

for all c, from which (5.10) follows. Finally, one checks (5.11) directly for $k = 0$. For $k \geq 1$, we have by (5.9) that the left hand side of (5.11) is constant on $[0, 1]$. with (5.10), this constant equals

$$\sum_{i=1}^{n+k} x_{ik}(0) = \sum_{i=0}^{k+1} g_{k+1}(t_{i-k-1}, \ldots t_i; 0) = \phi(t_0, \ldots, t_{k+1}),$$

where

$$\phi(t) = \sum_{i=0}^{k+1} (t - (k+1-i)/n)_+^k / k\,!.$$

As $\phi(t_j, t_{j+1}) = n((j+1)/n)^k / k\,!$, $j = 0, \ldots, k$, a polynomial of degree k in (j/n) with leading coefficient $n/k\,!$, we have $\phi(t_0, \ldots, t_{k+1}) = n/(k+1)\,!$, Q.E.D.

We note that most of these and many other facts are contained in the rather thorough treatment of splines with uniformly spaced knots [13]; missing only is the generating function for the numbers

$$g_{k+1}(0, 1, \ldots, k+1; i), \quad i = 0, \ldots, k+1,$$

which can be found already in [6], and has been rediscovered recently by others (cf.,e.g., [1]).

In the remainder, we will be concerned with linear splines (broken lines) and cubic splines only. We begin with a straightforward lemma on interpolation by broken lines.

Lemma 5.4. *Let* $n \geq 1$, $X_n = S_1(t_{0,n}, \ldots, t_{n,n})$, *with* $0 = t_{0,n} < \cdots < t_{n,n} = 1$; *set* $h_n = \max\{t_{i+1,n} - t_{i,n} : i = 0, \ldots, n-1\}$, *and* $\Lambda_n = \langle\{\delta_{t_{i,n}}\}_{i=0}^n\rangle$, *considered as a subspace of the topological dual of the Banach space* $X = C[0,1]$. *Then* X_n *and* Λ_n *define a projector,* $\bar{P}_n$, *on* X, *and* $\|\bar{P}_n\| = 1$, *and* $\|I - \bar{P}_n\| = 2$, *and*

$$\|x - \bar{P}_n x\|_\infty \leq \omega_x(h_n), \quad \text{all } x \in X, \tag{5.12}$$

$$\|x - \bar{P}_n x\|_\infty \leq h_n \omega_{x^{(1)}}(h_n)/2, \quad \text{all } x \in C^{(1)}[0,1]. \tag{5.13}$$

Remark. The estimate (5.13) gives

$$\|x - \bar{P}_n x\|_\infty = O(h_n^2), \quad \text{all } x \in \mathrm{Lip}^{(1)}[0,1], \tag{5.14}$$

and any further increase in the smoothness assumption on x will not increase the convergence rate above $O(h_n^2)$, as the function $x(t) = t^2$ readily shows.

Theorem 5.1. *For* $n \geq 3$, *let* $X_n = S_3(0, t_{2,n}, \ldots, t_{n-2,n}, 1)$, *with* $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = 1$; *let* $h_n = \max\{\Delta t_{i,n} : i = 0, \ldots, n-1\}$ *and* $q_n = \max\{\Delta t_{i,n}/\Delta t_{j,n} : i, j = 0, \ldots, n-1\}$, *where* $\Delta t_{i,n} = t_{i+1,n} - t_{i,n}$, *and let* $\Lambda_n = \langle\{\delta_{t_{i,n}}\}_0^n\rangle$, *considered as a subspace of the topological dual of the Banach space* $X = C[0,1]$. *Then* X_n *and* Λ_n *define a projector,* $\bar{Q}_n$, *on* X, *and*

$$\|\bar{Q}_n\| \leq 1 + 5q_n^2/2, \quad \text{all } n.$$

Hence, if $\lim h_n = 0$ *while, for some* c *and all* n, $q_n \leq c$, *then* $\bar{Q}_n \to I$.

Proof. Fix n and drop the second subscript in $t_{i,n}$. We use the abbreviation

$$x''_i = x^{(2)}(t_i).$$

If $y \in X_n$, (so that $y^{(2)}$ is a broken line), then one computes from (5.3),

$$\Delta t_{i-1} y''_{i-1} + 2(\Delta t_{i-1} + \Delta t_i) y''_i + \Delta t_i y''_{i+1} = 6(\Delta t_{i-1} + \Delta t_i)\, y\,(t_{i-1}, t_i, t_{i+1}), \quad i = 1, \ldots, n-1, \tag{5.15}$$

while the lack of a knot at t_1 and at t_{n-1} gives

$$\begin{aligned} \Delta t_1 y''_0 &- (\Delta t_1 + \Delta t_0) y''_1 + \Delta t_0 y''_2 = 0, \\ \Delta t_{n-1} y''_{n-2} &- (\Delta t_{n-2} + \Delta t_{n-1}) y''_{n-1} + \Delta t_{n-2} y''_n = 0. \end{aligned} \tag{5.16}$$

Combining these with the first and last equation of (5.15) gives

$$(\Delta t_0 + \Delta t_1)\,(2 + \Delta t_0/\Delta t_1) y''_1 + \Big(\Delta t_1 - (\Delta t_0)^2/\Delta t_1\Big) y''_2 = 6(\Delta t_0 + \Delta t_1)\, y\,(t_0, t_1, t_2), \tag{5.17a}$$

$$(\Delta t_0 + \Delta t_1) y''_0 + (\Delta t_0 + \Delta t_1) y''_1 + (\Delta t_0 + \Delta t_1) y''_2 = 6(\Delta t_0 + \Delta t_1)\, y\,(t_0, t_1, t_2), \tag{5.17b}$$

and two analogous equations involving y''_{n-2}, y''_{n-1}, y''_n. Adjoin (5.17a) and its counterpart to (5.15) to get, for any $y \in X_n$,

(5.18)

$$\frac{2\Delta t_1 + \Delta t_0}{\Delta t_1} y''_1 + \frac{\Delta t_1 - \Delta t_0}{\Delta t_1} y''_2 = 6y(t_0, t_1, t_2),$$

$$\frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i} y''_{i-1} + 2y''_i + \frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i} y''_{i+1} = 6y(t_{i-1}, t_i, t_{i+1}), \quad i = 2, \ldots, n-2,$$

$$\frac{\Delta t_{n-2} - \Delta t_{n-1}}{\Delta t_{n-2}} y''_{n-2} + \frac{2\Delta t_{n-2} + \Delta t_{n-1}}{\Delta t_{n-2}} y''_{n-1} = 6y(t_{n-2}, t_{n-1}, t_n).$$

The matrix of this system of equations is diagonally dominant, giving the lower bound of 1 for the modulus of any of its eigenvalues. Therefore, if $y \in X_n \cap \Lambda_n^-$, then $y(t_i) = y^{(2)}(t_i) = 0$, $i = 0, \ldots, n$, hence $y = 0$ (cf. equation (5.21) below). Therefore, $\bar{Q}_n$ is defined. Also

$$|y''_i| \le 6 \max_j |\, y(t_{j-1}, t_j, t_{j+1}) \,|, \quad i = 1, \ldots, n-1, \tag{5.19a}$$

so that, with (5.17b) and its counterpart

$$|y''_i| \le 18 \max_j |\, y(t_{j-1}, t_j, t_{j+1}) \,|, \quad i = 0, n. \tag{5.19b}$$

Further, if $x \in X$ and $y = \bar{Q}_n x$, then $y(t_{i-1}, t_i, t_{i+1}) = x(t_{i-1}, t_i, t_{i+1})$, so that

$$|y(t_{j-1}, t_j, t_{j+1}) \,| \le 2q_n^2 h_n^{-2} \|x\|, \quad j = 1, \ldots, n-1. \tag{5.20}$$

Hence, as

$$\begin{aligned} y(t) = x(t_i)\frac{t_{i+1} - t}{\Delta t_i} + +x(t_{i+1})\frac{t - t_i}{\Delta t_i} + \\ + \frac{1}{6}(t - t_i)(t - t_{i+1}) \cdot \Big(y''_i(\frac{t_{i+1} - t}{\Delta t_i} + 1) + y''_{i+1}(\frac{t - t_i}{\Delta t_i} + 1) \Big), \\ t \in [t_i, t_{i+1}], \end{aligned} \tag{5.21}$$

for $i = 0, \ldots, n-1$, we have with (5.19a–b) and (5.20),

$$\|y\| \le \|x\| + \frac{1}{6}(h_n/2)^2\, 5 \cdot 12 \cdot q_n^2 h_n^{-2} \|x\|, \tag{5.22}$$

giving $\|\bar{Q}_n\| \le 1 + 5q_n^2/2$.

To prove that $\bar{Q}_n \to I$ if $\lim h_n = 0$ and $\{q_n\}$ is bounded, we can either appeal to Lemma 1.2 using the fact that under these circumstances $\{\bar{Q}_n\}$ is uniformly bounded while $\overline{\lim}\, X_n = X$, or else prove it directly as follows: Since

$$|x(t_{j-1}, t_j, t_{j+1}) \,| \le \omega_x(h_n) q_n^2 h_n^{-2}, \quad j = 1, \ldots, n-1,$$

we have that

$$\begin{aligned} |\frac{1}{6}(t - t_i)(t - t_{i+i}) \Big(y''_i(\frac{t_{i+1} - t}{\Delta t_i} + 1) + y''_{i+1}(\frac{t - t_i}{\Delta t_i} + 1) \Big) \,| \le \\ \le \frac{1}{6}(h_n/2)^2 \cdot 5 \cdot 6 \cdot q_n^2 h_n^{-2} \omega_x(h_n) = \frac{5}{4} q_n^2 \omega_x(h_n), \end{aligned}$$

while $|x(t)-\left(x(t_i)\frac{t_{i+1}-t}{\Delta t_i}+x(t_{i+1})\frac{t-t_i}{\Delta t_i}\right)|\le\omega_x(h_n)$, so that with (5.21),

$$\|\bar{Q}_n x - x\| \le (1+5q_n^2/4)\omega_x(h_n), \qquad Q.E.D.$$

Corollary 1. *If $\bar{Q}_n$ is restricted to $Y=C^{(2)}[0,1]$, normed by*

$$\|x\|_Y = |\, x(0)\,| + |\, x(1)\,| + \|x^{(2)}\|_\infty,$$

then $\bar{Q}_n \to I$, provided lim $h_n = 0$.

Proof. We have from (5.18) that $|\,(\bar{Q}_n x^{(2)}(t_i)\,| \le 9\|x^{(2)}\|_\infty$, $i=0,\dots,n$, since $x(t_{j-1},t_j,t_{j+1}) = x^{(2)}(\xi)/2$ for some $\xi\in(t_{j-1},t_{j+1})$, $j=1,\dots,n-1$. As $(\bar{Q}_n x)^{(2)}$ is a broken line with vertices at t_i, $i=0,\dots,n$, we get

$$\|(\bar{Q}_n x)^{(2)}\|_\infty = \max_i |\,(\bar{Q}_n x)^{(2)}(t_i)\,| \le 9\|x^{(2)}\|_\infty,$$

implying $\|\bar{Q}_n\|\le 9$. As lim $h_n=0$ implies that $\overline{\lim}\, X_n = Y$, the corollary follows from Lemma 1.2, Q.E.D.

Corollary 2. *If $\bar{Q}_n$ is restricted to $Y=C^{(1)}[0,1]$, normed by*

$$\|x\|_Y = |\, x(0)\,| + \|x^{(1)}\|_\infty,$$

then $\bar{Q}_n \to I$, provided lim $h_n=0$ *and there exists $c<1$ so that*

$$\max\{\Delta t_{0,n}/t_{2,n},\quad \Delta t_{n-1,n}/(1-t_{n-2,n})\}\le c,\quad n=3,4,\dots \tag{5.23}$$

Proof. Any $y\in X_n$ satisfies (cf., e.g., [3])

$$\begin{aligned}(\Delta t_i)y'_{i-1} + 2(\Delta t_{i-1}+\Delta t_i)y'_i + (\Delta t_{i-1})y'_{i+1} &=\\ = 3\Big((\Delta t_i)\,y\,(t_{i-1},t_i) + (\Delta t_{i-1})\,y\,(t_i,t_{i+1})\Big),&\quad i=1,.,n-1,\end{aligned} \tag{5.24}$$

– (we dropped again the reference to n in $t_{i,n}$ and set $y'_i = y^{(1)}(t_i)$) –, while the absence of a knot at t_1 and at t_{n-1} gives

$$(\Delta t_1)^2 y'_0 + \big((\Delta t_1)^2 - (\Delta t_0)^2\big)y'_1 - (\Delta t_0)^2 y'_2 = 2\Big((\Delta t_1)^2 y(t_0,t_1) - (\Delta t_0)^2 y(t_1,t_2)\Big), \tag{5.25}$$

with an analogous equation involving y'_{n-2}, y'_{n-1}, y'_n. Combine these with the first and last equation of (5.24) to get

$$\begin{aligned}(\Delta t_0+\Delta t_1)^2 y'_1 + (\Delta t_0)\,(\Delta t_0+\Delta t_1)y'_2 &=\\ = (\Delta t_1)^2 y(t_0,t_1) + (\Delta t_0)\,(2\Delta t_0 + 3\Delta t_1)\,y\,(t_1,t_2),&\end{aligned} \tag{5.26a}$$

$$\begin{aligned}(\Delta t_1)\,(\Delta t_0+\Delta t_1)y'_0 + (\Delta t_0+\Delta t_1)^2 y'_1 &=\\ = (\Delta t_1)\,(3\Delta t_1 + 2\Delta t_0)\,y\,(t_0,t_1) + (\Delta t_0)^2 y(t_1,t_2),&\end{aligned} \tag{5.26b}$$

and a pair of analogous equations involving y'_{n-2}, y'_{n-1}, y'_n. Adjoin (5.26a) and its counterpart to (5.24) to get, for any $y \in X_n$, the system

(5.27)

$$y'_1 + \frac{\Delta t_0}{\Delta t_0 + \Delta t_1} y'_2 = \\ = \Big((\Delta t_1)^2 y(t_0, t_1) + (\Delta t_0)\,(2\Delta t_0 + 3\Delta t_1)\, y\,(t_1, t_2)\Big)/(\Delta t_0 + \Delta t_1)^2,$$

$$\frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i} y'_{i-1} + 2y'_i + \frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i} y'_{i+1} = \\ = 3\Big((\Delta t_i)\, y\,(t_{i-1}, t_i) + (\Delta t_{i-1})\, y\,(t_i, t_{i+1})\Big)/(\Delta t_{i-1} + \Delta t_i), \quad i = 2, \ldots, n-2,$$

$$\frac{\Delta t_{n-1}}{\Delta t_{n-2} + \Delta t_{n-1}} y'_{n-2} + y'_{n-1} = \Big((\Delta t_{n-1})\,(2\Delta t_{n-1} + 3\Delta t_{n-2}) \cdot y(t_{n-2}, t_{n-1}) + \\ + (\Delta t_{n-2})^2 y(t_{n-1}, t_n)\Big)/(\Delta t_{n-2} + \Delta t_{n-1})^2,$$

whose matrix is diagonally dominant, giving the lower bound $(1-c)$ for the modulus of any of its eigenvalues. Much as in the proof of the theorem, this gives

$$|y'_i\,| \le 3\|x\|_Y/(1-c), \quad i = 1, \ldots, n-1,$$
$$|y'_i\,| \le 5\|x\|_Y/(1-c)^2, \quad i = 0, n,$$

whenever $y = \bar{Q}_n x$, $x \in Y$. As for $t \in [t_i, t_{i+1}]$, since

(5.28)

$$y^{(1)}\,(t) = x(t_i, t_{i+1})\, 6\,(t - t_i)\,(t - t_{i+1})/(\Delta t_i)^2 + y'_i(3t - 2t_i - t_{i+1})\,(t - t_{i+1})/(\Delta t_i)^2 + \\ + y'_{i+1}(3t - 2t_{i+1} - t_i)\,(t - t_i)/(\Delta t_i)^2,$$

$i = 0, \ldots, n-1$, it follows that

$$\|y^{(1)}\|_\infty \le \frac{3}{2}\|x^{(1)}\|_\infty + \max_i |\, y'_i\,|,$$

so that $\|\bar{Q}_n x\|_Y \le \frac{13}{2}\|x\|_Y/(1-c)^2$ for all $x \in Y$, showing the uniform boundedness of $\{\bar{Q}_n\}$. The corollary now follows from Lemma 1.2, Q.E.D.

Corollary 3. *If* $x \in \mathrm{Lip}^{(3)}[0,1]$, *then*

$$\|(\bar{Q}_n x - x)^{(j)}\|_\infty = O(h_n^{4-j}), \quad j = 0, 1, 2. \tag{5.29}$$

Proof. Under the assumption on x, there exists, by (5.14), $x_0 \in X_n$, such that

$$\|x^{(2)} - x_0^{(2)}\|_\infty = O(h_n^2).$$

Therefore $\|(\bar{Q}_n x)^{(2)} - x^{(2)}\|_\infty = O(h_n^2)$, by Corollary 1 and Lemma 1.1. But then, for $t \in [t_i, t_{i+1}]$,

$$(\bar{Q}_n x - x)\,(t) = \int_{t_i}^{t_{i+1}} G_i(t,s)\,(\bar{Q}_n x - x)^{(2)}\,(s)\,ds,$$

where

$$G_i(t,s) = -(\Delta t_i)^{-1} \begin{cases} (t-t_i)\,(t_{i+1}-s), & t \le s \\ \\ (s-t_i)\,(t_{i+1}-t), & t \ge s \end{cases},$$

from which one computes directly the estimates

$$\|\bar{Q}_n x - x\|_\infty \le \frac{1}{8} h_n^2 \|(\bar{Q}_n x - x)^{(2)}\|_\infty,$$

$$\|(\bar{Q}_n x - x)^{(1)}\|_\infty \le \frac{1}{32} h_n \|(\bar{Q}_n x - x)^{(2)}\|_\infty,$$

which concludes the proof, Q.E.D.

We note that the corollaries remain true if we set $\Delta t_{0,n} = \Delta t_{n-1,n} = 0$. This amounts to replacing the linear functionals $\delta_{t_{1,n}}$, $\delta_{t_{n-1,n}}$ in Λ_n by $\delta_0^{(1)}$, $\delta_1^{(1)}$. This is made precise in the following

Theorem 5.2. *For $n = 1, 2, \ldots$, let $X_n = S_3(0, t_{1,n}, \ldots, t_{n-1,n}, 1)$ with $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = 1$; set $h_n = \max\{\Delta t_{i,n} : i = 0, \ldots, n-1\}$ and let $\Lambda_n = \langle\{\delta_{t_{0,n}}, \ldots, \delta_{t_{n,n}}, \delta_0^{(1)}, \delta_1^{(1)}\}\rangle$, considered as a subspace of the topological dual of the Banach space $X = C^{(1)}[0,1]$ with norm*

$$\|x\|_X = | x(0) | + \|x^{(1)}\|_\infty, \quad \text{all } x \in X.$$

Then X_n and Λ_n define a projector $\bar{Q}'_n$ and $\|\bar{Q}'_n\| \le 13/2$, for all n. Hence $\bar{Q}'_n \to I$, provided lim $h_n = 0$. Also, for $x \in C^{(2)}[0,1]$, $\|(\bar{Q}'_n x)^{(2)}\|_\infty \le 9\|x^{(2)}\|_\infty$, and

$$(5.30) \qquad \|(\bar{Q}'_n x - x)^{(j)}\|_\infty = O(h_n^{4-j}), \quad \text{all } x \in \text{Lip}^{(3)}\,[0,1], \quad j = 0, 1, 2.$$

Proof. On X, $\bar{Q}'_n$ is the strong limit of the sequence $\{\bar{Q}_{n,i}\}$, where $\bar{Q}_{n,i}$ is the projector given by X_n and $\Lambda_{n,i} = \langle\{\delta_{t_{0,n}}, \ldots, \delta_{t_{n,n}}, i(\delta_{1/i} - \delta_0), i(\delta_1 - \delta_{1-1/i})\}\rangle$. For each $n \ge 1$, and for all i large enough, $\bar{Q}_{n,i}$ is of the type described in the preceding Theorem 5.1, hence the results in Corollaries 1, 2, 3 apply to $\bar{Q}_{n,i}$ for i large enough and therefore to their strong limit, $\bar{Q}'_n$. Note that the constant c in Corollary 2 can be taken arbitrarily small for i large enough so that we can set $c = 0$ for $\bar{Q}'_n$, Q.E.D.

Remark. Theorem 5.1 has been proved for the case of interpolation by periodic splines in [15]; the proof given here is an adaptation of the argument in [15]. The approximation properties of $\bar{Q}'_n$ of Theorem 5.2 have been studied in [2], where (5.30) has been obtained under the additional assumption that $x \in C^{(4)}[0,1]$ and that the sequence $\{q_n\}$ is bounded. Finally, it was asserted in [3] that the conclusions of Theorem 5.1 hold true for the sequence $\{\bar{Q}'_n\}$ of Theorem 5.2, an obvious mistake as the definition of $\bar{Q}'_n$ involves the linear functionals $\delta_0^{(1)}$, $\delta_1^{(1)}$, which fail to be bounded with respect to the norm $\|x\| = \|x\|_\infty$. Still, the argument in [3] shows that $\{\bar{Q}''\}$ is uniformly bounded on the Banach spase $C[0,1]$ (provided $\{q_n\}$ is bounded), where $\bar{Q}''$ is given by $\bar{X}_n = \{x \in X_n : x^{(1)}(0) = x^{(1)}(1) = 0\}$ and $\bar{\Lambda}_n = \langle\{\delta_{t_{i,n}}\}_0^n\rangle$. But the results of Corollaries 1, 2, 3, do not apply to $\bar{Q}''_n$ because of the arbitrary assignment of the boundary derivatives.

The use of Galerkin's method is, as will be seen in Section 8, connected with quite a different projector from $C[0,1]$ to a subspace of cubic splines, which we will discuss in the following

Theorem 5.3. *For $n = 3, 4, \ldots$, let $X_n = S_3\,(0, 1/n, \ldots, (n-1)/n, 1)$, i.e., we assume equidistant knots, and let $\bar X_n = \{x \in X_n : x(0) = x(1) = 0\}$. Define λ_i by*

$$\lambda_i x = \int_0^1 x(t)\, y_i(t)\, dt, \quad i = 0, \ldots, n,$$

where

$$y_i(t+ih) = y_0\,(t) = h^{-1} \cdot \begin{cases} (t+h)_+, & t < 0, \\ (h-t)_+, & t \ge 0, \end{cases} \tag{5.31}$$

and $h = h_n = 1/n$, and set $\Lambda_n = \langle \{\lambda_i\}_0^n \rangle$, considered as a subspace of the topological dual of the Banach space $\bar X = \{x \in C[0,1] : x(0) = x(1) = 0\}$. Then $\bar X_n$ and Λ_n define a projector $\bar Q_n$ on $\bar X$, and $\|\bar Q_n\| \le 22.5$, for all n. Hence $\bar Q_n \to I$.

Proof. Define a basis for X_n by

$$x_i(t) = (4h) g_4(t_{i-2}, \ldots, t_{i+2}; t), \quad i = -1, \ldots, n+1, \tag{5.32}$$

where $g_4(s;t) = (s-t)_+^3/6$ and $t_j = jh$, $j = -3, \ldots, n+3$ (cf. Lemma 5.2). So

$$x_i(t+ih) = x_0(t) = (6h^3)^{-1} \cdot \begin{cases} (t+2h)_+^3 \;\; - 4(t+h)_+^3, & t < 0 \\ (2h-t)_+^3 \;\; - 4(h-t)_+^3, & t \ge 0, \end{cases} \tag{5.33}$$

and, by Lemma 5.3 (or by direct computation), $\sum x_i(t) = \sum \mid x_i(t) \mid = 1$, for $t \in [0,1]$, therefore

$$\| \sum_{-1}^{n+1} a_i x_i(t) \| \le \max_i \mid a_i \mid \;. \tag{5.34}$$

From this, we get a basis for $\bar X_n$ by

$$\begin{aligned} &\bar x_0 = x_0 - 4x_{-1}, \quad \bar x_1 = x_1 - x_{-1}, \quad \bar x_{n-1} = x_{n-1} - x_{n+1}, \\ &\bar x_n = x_n - 4x_{n+1}, \quad \bar x_i = x_i, \quad i = 2, \ldots, n-2. \end{aligned} \tag{5.35}$$

Note that $\sum x_i(t) = \sum \mid x_i(t) \mid \le 1$, so that

$$\| \sum_0^n a_i \bar x_i \| \le \max_i |a_i| \,. \tag{5.34$'$}$$

It is possible to show directly that the matrix $A = \{\lambda_i \bar x_j\}_{i,j=1}^n$ is non-singular: From the condition that $(\bar Q_n x - x) \in \Lambda_n^-$, one finds for $\bar Q_n x = \sum a_i \bar x_i$,

$$\begin{array}{lr} 17a_0 + 18a_1 + a_2 & = 120h^{-1}\lambda_0 x, \\ 22a_0 + 65a_1 + 26a_2 + a_3 & = 120h^{-1}\lambda_1 x, \\ a_{i-2} + 26a_{i-1} + 66a_i + 26a_{i+1} + a_{i+2} & = 120h^{-1}\lambda_i x, \quad i = 2, \ldots, n-2, \\ a_{n-3} + 26a_{n-2} + 65a_{n-1} + 22a_n & = 120h^{-1}\lambda_{n-1} x, \\ a_{n-2} + 18a_{n-1} + 17a_n & = 120h^{-1}\lambda_n x. \end{array} \tag{5.36}$$

One computes that $\|\lambda_i\| = h$, $i = 1, \ldots, n-1$, and $\|\lambda_0\| = \|\lambda_n\| = h/2$. Let now i be such that

$$|a_i| = \max\{|a_j| : j = 0, \ldots, n\}\,.$$

Then, if $2 \le i \le n-2$, one gets from (5.36)

$$120\|x\| \ge |120h^{-1}\lambda_i x| \ge 66|a_i| - (1 + 26 + 26 + 1)\,|a_i| = 12|a_i|;$$

and if $i = 1$ or $i = n-1$, one gets

$$120\|x\| \ge |120h^{-1}\lambda_i x| \ge 65|a_i| - (22 + 26 + 1)|a_i| = 16|a_i|;$$

and if $i = 0$ or $i = n$, say, $i = 0$, then, subtracting (1/4) of the second equation of (5.36) from the first, on gets

$$11.5a_0 + 1.75a_1 - 5.5a_2 - 0.25a_3 = 120h^{-1}\lambda_0 x - 30h^{-1}\lambda_1 x,$$

and therefore

$$90\|x\| \ge |30h^{-1}\,(4\lambda_0 - \lambda_1)x| \ge 11.5|a_i| - (1.75 + 5.5 + .25)|a_i| = 4|a_i|.$$

From this, we conclude that $\bar{Q}_n$ is defined for all n and, with (5.34′), that $\|\bar{Q}_n\| \le 22.5$, Q.E.D.

Corollary. *With the notation of the theorem, let* $Y_n = \{x^{(2)} : x \in \bar{X}_n\}$*, and define the linear functionals* μ_i*,* $i = 0, \ldots, n$*, on the Banach space* $Y = C[0,1]$ *by*

$$\mu_i y = \int_0^1 y(t)\bar{x}_i(t)\,dt, \quad i = 0, \ldots, n, \tag{5.37}$$

where $\bar{x}_i$ *is given by (5.35) and (5.33). Then* Y_n *and* $\bar{\Lambda}_n = \langle\{\mu_i\}_0^n\rangle$ *define a projector* $\bar{P}_n$ *on* Y *for all* n*, and* $\{\bar{P}_n\}$ *is uniformly bounded, hence* $\bar{P}_n \to I$*.*

Proof. We note that $\{y_i\}_0^n$, given by (5.31), is a basis for Y_n, that $\|\sum a_i y_i\| = \max_i |a_i|$ and that (with (5.6)),

$$\|\mu_i\| = \int_0^1 \bar{x}_i\,(t)\,dt \le 4h/6, \quad i = 0, \ldots, n.$$

The matrix $\{\mu_i y_j\}_{i,j=0}^n$ is clearly the transpose of the matrix $A = \{\lambda_i \bar{x}_j\}_{i,j=0}^n$ of the theorem which was shown there to be non-singular, hence $\bar{P}_n$ is defined for all n. By an argument entirely analogous to the one given in the proof of the theorem, one also shows the existence of $c > 0$ such that $\|\bar{P}_n\| \le c$, all n, Q.E.D.

Remark. Note that $\bar{P}_n = M_1\bar{Q}_n M_1^{-1}$, where M_1 is the linear map from $X = \{x \in C^{(2)}\,[0,1] : x(0) = x(1) = 0\}$ to $Y = C[0,1]$,

$$M_1 x = x^{(2)}, \quad \text{all } x \in X.$$

If X is normed by

$$\|x\|_X = \|x^{(2)}\|_\infty, \quad \text{all } x \in X, \tag{5.38}$$

and Y is normed in the usual way, then M_1 is an isometry. The preceding corollary proves, therefore, that the sequence $\{\bar{Q}_n\}$ is uniformly bounded on X with respect to the norm (5.38).

CHAPTER 3: APPLICATION TO TWO POINT BOUNDARY VALUE PROBLEMS

6. Preliminary considerations

In the following sections, we apply the results of Sections 4 and 5 to the numerical solution of an ordinary linear second order differential equation with two point boundary conditions.

M denotes the differential operator given by

$$(6.1)\qquad (Mx)\,(t) = x^{(2)}\,(t) + p_1(t)\,x^{(1)}\,(t) + p_0\,(t)\,x\,(t), \quad t \in [0,1],$$

with p_0, $p_1 \in C[0,1]$; we assume the linear functionals δ_0, δ_1 to be linearly independent over the nullspace of M, or, what is the same, we assume that M maps the linear space $X = \{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$ in a 1-1 manner onto $Y = C[0,1]$, so that M^{-1} exists.

Denote by $G(s,t)$ the Green's function of the problem,

$$(6.2)\qquad x(t) = \int_0^1 G(s,t)\,(Mx)\,(s)\,ds, \quad \text{all } x \in X, \quad t \in [0,1].$$

In order to apply Theorem 4.1, we split M into M_1 and M_2,

$$(6.3)\qquad \begin{aligned} &M = M_1 + M_2, \\ &M_1 x = x^{(2)}, \quad \text{all } x \in X. \end{aligned}$$

Then

$$(6.4)\qquad x(t) \equiv \int_0^1 G_1\,(t,s)\,(M_1 x)\,(s)\,ds, \quad \text{all } x \in X,$$

where

$$(6.5)\qquad G_1\,(t,s) = \begin{cases} t(s-1), & t \le s, \\ s(t-1), & t \ge s. \end{cases}$$

To make application of Theorem 4.1 and its corollaries possible, we must norm X and Y in such a way that M and M_1 and their inverses are bounded. This can be done in a number of ways, as will be seen shortly. But there are other considerations. If X_n and Λ_n are given, then Q_n, if at all defined, will be bounded if and only if $M^a[\Lambda_n]$ consists of continuous linear functionals. For example, in the Collocation method, Λ_n is given by $\langle\{\delta_{t_i}\}_1^n\rangle$, where $0 \le t_1 < \cdots < t_n \le 1$, so that, with $M = M_1$, we have $M^a[\Lambda_n] = \langle\{\delta_{t_i}^{(2)}\}_1^n\rangle$; therefore Q_n fails to be bounded if the norm

$$6.6)\qquad \|x\|_X = \|x\|_\infty, \quad \text{all } x \in X,$$

or the norm

$$\|x\|_X = \|x^{(1)}\|_\infty, \quad \text{all } x \in X, \tag{6.7}$$

is chosen on X, but is bounded with respect to the norm

$$\|x\|_X = \|x^{(2)}\|_\infty, \quad \text{all } x \in X. \tag{6.8}$$

If, on the other hand, we choose $\Lambda_n = \langle\{\lambda_i\}_1^n\rangle$, with

$$\lambda_i y = (M_1^{-1})^a \delta_{t_i} y = \int_0^1 G_1(t_i, t)\, y\,(t)\, dt, \quad \text{all } y \in Y, \quad i = 1, \dots, n,$$

then Q_n, if at all defined, is bounded with respect to any of the norms (6.6) – (6.8).

In the next two sections, we will, for simplicity, consider only the three norms (6.6) – (6.8) for X and will norm Y always in such a way that M_1 becomes an isometry, i.e.

$$\|y\|_Y = \|M_1^{-1} y\|_X, \quad \text{all } y \in Y. \tag{6.9}$$

For these cases, we establish the applicability of Theorem 4.1 in the following two lemmata.

Lemma 6.1. *Let X be the Banach space $\{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$ with norm (6.8). Then M, given by (6.1), is a bounded linear map from X onto the Banach space $Y = C[0,1]$. Further, M^{-1} is bounded and $M_2 = M - M_1$ is totally bounded.*

Proof. By (6.4), we have

$$\|x^{(j)}\|_\infty \le c_j \|x\|_X, \quad \text{all } x \in X, \quad j = 0, 1, 2, \tag{6.10}$$

with

$$c_j = \| \int_0^1 \mid (\partial/\partial t)^{(j)} G_1\,(t, s) \mid ds \|_\infty, \quad j = 0, 1, \quad c_2 = 1.$$

Hence

$$\|Mx\|_Y \le \|M_1 x\|_\infty + \|M_2 x\|_\infty \le \Big(1 + c_1\|p_1\|_\infty + c_0\|p_0\|_\infty\Big)\|x\|_X, \quad \text{all } x \in X,$$

showing that M (and therefore M_2) is bounded. Therefore, as M^{-1} exists, and X and Y are Banach spaces, M^{-1} is bounded, (cf., e.g., [10, Thm.2,p.471] or else infer it from (6.2)).

Further,

$$\mid (M_2 x)\,(s) - (M_2 x)\,(t) \mid \le \sum_{j=0}^{1} \Big(|p_j(s) - p_j(t) \mid \mid x^{(j)}\,(s) \mid \; + |p_j(t) \mid \mid x^{(j)}\,(s) - x^{(j)}\,(t) \mid \Big),$$

so that, with $|x^{(j)}\,(s) - x^{(j)}\,(t) \mid \le \mid s - t| \; \|x^{(j+1)}\|_\infty$, and with (6.10),

$$\mid (M_2 x)\,(s) - (M_2 x)\,(t) \mid \le \omega_{M_2}(\mid s - t \mid)\|x\|_X, \tag{6.11}$$

where

$$\omega_{M_2}\,(h) \equiv \sum_{j=0}^{1} \Big(\omega_{p_j}\,(h)\, c_j + \|p_j\|_\infty \cdot h \cdot c_{j+1}\Big), \quad h \ge 0. \tag{6.12}$$

Therefore, M_2 maps the unit ball of X into a bounded set of uniformly equicontinuous functions in $C[0,1]$, so that, as such sets are totally bounded (they are in fact precompact by Arzela's Theorem), M_2 is shown to be totally bounded, Q.E.D.

With Theorem 4.1, we have the

Corollary. *If $\{\bar{P}_n\}$ is a sequence of projectors of finite rank on the Banach space $Y = C[0,1]$ such that $\bar{P}_n \longrightarrow I$, where $\bar{P}_n$ is given by Y_n and Λ_n, all n, then the projection scheme for M given by $\{M_1^{-1}[Y_n]\}$ and $\{\Lambda_n\}$ is boundedly convergent.*

Lemma 6.2. *For $j = 0, 1$, let X be the normed linear space $\{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$ with norm*

$$\|x\|_X = \|x^{(j)}\|_\infty, \quad \text{all } x \in X,$$

let Y be the normed linear space $C[0,1]$ with norm

$$\|y\|_Y = \|M_1^{-1}y\|_X, \quad \text{all } y \in Y,$$

and let M be the linear map from X to Y given by

$$(Mx)(t) = x^{(2)}(t) + (p_1(t)\,x(t))^{(1)} + p_0(t)\,x(t), \quad \text{all } x \in X, \tag{6.1'}$$

i.e., we add to the earlier assumptions on M that $p_1 \in C^{(1)}[0,1]$. Then M is bounded, M^{-1} is bounded and $M_2 = M - M_1$ is totally bounded.

Proof. The boundedness of M follows from the existence of a constant c so that

$$\|M_2x\|_Y \le c\|x\|_X, \quad \text{all } x \in X. \tag{6.13}$$

Case $j = 0$: We have, for $x \in X$ and $s \in [0,1]$,

$$\begin{aligned}(M_1^{-1}M_2x)(s) &= \int_0^1 G_1(s,t)\Big((p_1(t)\,x(t))^{(1)} + p_0(t)\,x(t)\Big)\,dt = \\ &= \int_0^1 \Big(-p_1(t)\,(\partial/\partial t)\,G_1(s,t) + p_0(t)\,G_1(s,t)\Big)\,x(t)\,dt,\end{aligned} \tag{6.14}$$

so that

$$\|M_2x\|_Y = \|M_1^{-1}M_2x\|_\infty \le c\|x\|_\infty = c\|x\|_X, \quad \text{all } x \in X,$$

with

$$c = \Big\| \int_0^1 \big| -p_1(t)\,(\partial/\partial t)\,G_1(s,t) + p_0(t)\,G_1(s,t) \big| \,dt \Big\|_\infty .$$

Case $j = 1$: We have, for $x \in X$ and $s \in [0,1]$,

$$\begin{aligned}(M_1^{-1}M_2x)^{(1)}(s) =& p_1(s)\,x(s) + \int_0^s p_0(t)\,x(t)\,dt - \\ &- \int_0^1 p_1(t)\,x(t)\,dt - \int_0^1\int_0^t p_0(r)\,x(r)\,dr\,dt,\end{aligned} \tag{6.15}$$

so that

$$\|M_2x\|_Y = \|(M_1^{-1}M_2x)^{(1)}\|_\infty \le c\|x\|_\infty \le c\|x^{(1)}\|_\infty = c\|x\|_X, \tag{6.16}$$

for all $x \in X$, with $c = 2(\|p_0\|_\infty + \|p_1\|_\infty)$, using the fact that $\|x\|_\infty \leq \|x^{(1)}\|_\infty$.

The boundedness of M^{-1} follows from the existence of a constant d so that

$$\|M_2 x\|_Y \leq d\|Mx\|_Y, \quad \text{all } x \in X; \tag{6.17}$$

for, with $M^{-1}y = x$, we have

$$M^{-1}y = M_1^{-1}M_1 x = M_1^{-1}(Mx - M_2 x),$$

so that then

$$\|M^{-1}y\|_X = \|M_1^{-1}(Mx - M_2x)\|_X = \|Mx - M_2x\|_Y \leq (1+d)\,\|Mx\|_Y = (1+d)\,\|y\|_Y$$

for all $y \in Y$.

Case $j = 0$: We have from (6.14) that for $x \in X$ and $s \in [0,1]$,

$$(M_1^{-1}M_2x)\,(s) = \int_0^1 R(s,t) \int_0^1 G(t,r)\,(Mx)\,(r)\,dr\,dt,$$

where $G(s,t)$ is given in (6.2) and

$$R(s,t) = -p_1(t)\,(\partial/\partial t)\,G_1\,(s,t) + p_0(t)\,G_1\,(s,t).$$

After the appropriate integration by parts, we get that

$$\|M_2x\|_Y = \|M_1^{-1}M_2x\|_\infty \leq d\|M\_1^{-1}Mx\|_\infty = d\|Mx\|_Y, \quad \text{all } x \in X,$$

with

$$d = \| \int_0^1 |R(s,r) + \int_0^1 R(s,t)\,\Big((\partial/\partial r)\,(p_1(r)\,G\,(t,r)) - p_0(r)\,G\,(t,r)\Big)\,dt|dr\|_\infty\,.$$

Case $j = 1$: We have from (6.16) that, for all $x \in X$,

$$\|M_2x\|_Y \leq 2(\|p_0\|_\infty + \|p_1\|_\infty)\,\|x\|_\infty.$$

As

$$x(s) = \int_0^1 G(s,t)\,(Mx)\,(t)\,dt = -\int_0^1 (\partial/\partial t)\,(G(s,t))\,(M_1^{-1}Mx)^{(1)}\,(t)\,dt,$$

(6.17) follows for this case with

$$d = 2(\|p_0\|_\infty + \|p_1\|_\infty)\,\| \int_0^1 |\,(\partial/\partial t)\,G\,(s,t)\,|\,dt\|_\infty\,.$$

It remains to show that M_2 is totally bounded.

Case $j = 0$: By (6.16), we have

$$\|(M_1^{-1}M_2x)^{(1)}\|_\infty \le c\|x\|_\infty, \quad \text{all } x \in X,$$

so that

$$\omega_y(h) \le ch, \quad \text{all } h \ge 0, \quad \text{all } y \in F = \{M_1^{-1}M_2x : x \in X, \quad \|x\|_\infty \le 1\}.$$

The set F is therefore totally bounded in X, and so, as M_1 is an isometry, M_2 is totally bounded.

Case $j = 1$: In this case, the total boundedness of M_2 follows from the uniform equicontinuity of the set

$$F = \{(M_1^{-1}M_2x)^{(1)} : x \in X, \quad \|x^{(1)}\|_\infty \le 1\}.$$

For $y(s) = (M_1^{-1}M_2x)^{(1)}(s) \in F$, we have $y^{(1)}(s) = (M_2x)(s)$, hence, using $\|x\|_\infty \le \|x^{(1)}\|_\infty$,

$$\|y^{(1)}\|_\infty \le \|M_2x\|_\infty \le e\|x^{(1)}\|_\infty,$$

with $e = \|p_1\|_\infty + \|p_1^{(1)}\|_\infty + \|p_0\|_\infty$, so that

$$\omega_y(h) \le eh, \quad \text{all } h \ge 0, \quad \text{all } y \in F,$$

Q.E.D.

Corollary. *Let $0 \le j \le 2$, and let $\{X_n\}$ be a sequence of finite dimensional subspaces of the normed linear space $X = \{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$ with norm*

$$\|x\|_X = \|x^{(j)}\|_\infty, \quad \text{all } x \in X;$$

let $\{\Lambda_n\}$ be a corresponding sequence of finite dimensional subspaces of the topological dual Y' of the normed linear space $Y = C[0,1]$ with norm

$$\|y\|_Y = \|M_1^{-1}y\|_X, \quad \text{all } y \in Y,$$

where $M_1x = x^{(2)}$, all $x \in X$. Let M be the map from X to Y given by (6.1′) with $p_0 \in C[0,1]$ and $p_1 \in C^{(1)}[0,1]$, and assume that M^{-1} exists. If $\{X_n\}$ and $\{\Lambda_n\}$ give a boundedly convergent projection scheme for M_1, then $\{X_n\}$ and $\{\Lambda_n\}$ give a boundedly convergent projection scheme for M and, in that case,

$$\|x^{*(i)} - x^{*(i)}\|_\infty \le c \cdot \min_{z \in X_n} \|x^{*(j)} - z^{(j)}\|_\infty, \quad i = 0, \ldots, j, \quad \text{some } c.$$

7. Collocation and a higher order method

As a first application of the results of the previous section we consider the Collocation method using cubic splines for X_n. Specifically, let $X_n = \{x \in S_3(t_{0,n}, \ldots, t_{n,n} : x(0) = x(1) = 0\}$ and use the points $\{t_{i,n}\}_0^n$ as collocation points, i.e., choose $\Lambda_n = \langle \{\delta_{t_{i,n}}\}_0^n \rangle$. Then $M_1[X_n]$ and Λ_n define a projector $\bar{P}_n$ for all $n \geq 1$ which has been discussed in Lemma 5.4; $\bar{P}_n$ is just interpolation by broken lines.

Theorem 7.1. *In the Banach space $X = \{x \in C^{(2)}[0,1] : x(0) = x(1) = 0\}$ with norm (6.8), let*

$$X_n = \{x(t) \in S_3(0, t_{1,n}, \ldots, t_{n-1,n}, 1) : x(0) = x(1) = 0\},$$

where $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = 1$, set $h_n = \max\{\Delta t_{i,n} : i = 0,.,n-1\}$; further let $\Lambda_n = \langle \{\delta_{t_{i,n}}\}_0^n \rangle$ considered as a subspace of the topological dual of the Banach space $Y = C[0,1]$, $n = 1, 2, \ldots$. Then the projection scheme for M (as defined by (6.1)) given by $\{X_n\}$ and $\{\Lambda_n\}$ is boundedly convergent, provided lim $h_n = 0$. Specifically, we have for all large enough n,

$$(7.1)\quad \|x^* - x_n^*\|_X \leq \|M^{-1}\| \left(\omega_{y^*}(h_n) + \omega_{M_2}(h_n) \, \|M_0^{-1}\| \, (1 - \omega_{M_2}(h_n) \, \|M_0^{-1}\|)^{-1} \|y^*\|_\infty \right),$$

where ω_{M_2} is given by (6.12) and $M_0 = M|X_n$, so that x_n^* and its first two derivatives converge uniformly to x^* and its first two derivatives. Further,

$$(7.2)\qquad \|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n), \quad j = 0,1,2, \quad \text{if } y^*, p_0, p_1 \in \text{Lip}[0,1],$$

and

$$\|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n^2), \quad j = 0,1,2, \quad \text{if } x^* \in \text{Lip}^{(3)}[0,1].$$

Proof. Except for the convergence estimates (7.1) – (7.3), this theorem follows at once from the Corollary to Lemma 6.1 and from Lemma 5.4, since

$$M_1[X_n] = S_1(t_{0,n}, \ldots, t_{n,n}) \,.$$

Denote the projector given by $M_1[X_n]$ and Λ_n by $\bar{P}_n$, $n \geq 1$. Then, by (6.11) and (5.12),

$$\|M_2 - P_n M_2\| = \sup\{\|M_2 x - \bar{P}_n M_2 x\|_Y : x \in X, \ \|x\|_X \leq 1\} \leq \omega_{M_2}(h_n) \,,$$
$$\|y^* - \bar{P}_n y^*\|_Y \leq \omega_{y^*}(h_n), \qquad \|\bar{P}_n y^*\|_Y \leq \|y^*\|_\infty \,,$$

so that (7.1) follows from (4.9) in Corollary 1 to Theorem 4.1. The estimate (7.2) in turn is a consequence of (7.1) with the observation that if $p_0, p_1 \in \text{Lip}[0,1]$ then $\omega_{M_2}(h) = O(h)$. Finally, if $x^* \in \text{Lip}^{(3)}[0,1]$, then $x^{*(2)} \in \text{Lip}^{(1)}[0,1]$, so that by (5.14) there is an $x_0 \in X_n$ so that

$$\|x^{*(2)} - x_0^{*(2)}\|_\infty = O(h_n^2) \,.$$

Since the projection scheme is boundedly convergent, we have that $\{Q_n\}$ is uniformly bounded (recall that $x_n^* = Q_n x^*$) so that Lemma 1.1 implies

$$\|x^* - x_n^*\|_X = O(h_n^2),$$

from which (7.3) follows, Q.E.D.

The estimate (7.3) is of the same order as those attained under similar assumptions on x^* for the error in the solution of standard finite difference approximations to M (cf., e.g., [17, Thm 6.2]).

It is possible to give examples showing that this order of convergence is, in general, best possible for the Collocation method. Take, e.g., $M = M_1$, $y^*(t) = t^2$, and use uniform knot spacing, $t_{i,n} = ih$, $i = 0, \ldots, n$, $h = 1/n$. Then with $e(t) = x^*(t) - x^*_n(t)$, we have

$$e^{(2)}(t) = t(h-t) + 2h \sum_{i=1}^{n-1} (t - ih)_+ \,,$$

therefore

$$e(t) = e^{(1)}(0)\, t + ht^3/6 - t^4/12 + (h/3) \sum_{i=1}^{n-1} (t - ih)^3_+ \,,$$

so that, with $e(1) = 0$, we get $e^{(1)}(0) = -h^2/12$. Hence, with $n = 2k$, we get

$$\|e\|_\infty \geq | e(1/2) | = h^2/48 \,.$$

On the other hand, we know from Section 5, e.g., from Corollary 3 to Theorem 5.1, that

$$\text{(7.4)} \qquad \min\{\|x^* - x\|_\infty : x \in X_n\} = O(h_n^4), \quad \text{all } x^* \in \mathrm{Lip}^{(3)}[0,1] \,,$$

so that the Collocation method does not provide us with the best possible order of convergence in this circumstance.

In view of Lemma 1.1, the fault must lie with the projectors Q_n: If X is provided with the norm

$$\text{(7.5)} \qquad \|x\|_X = \|x\|_\infty, \quad \text{all } x \in X \,,$$

then the sequence $\{Q_n\}$ must fail to be uniformly bounded. Actually, by the remarks in the beginning of Section 6, the Q_n are not even bounded in this norm. This does not contradict the fact proved in Theorem 7.1 that $Q_n \to I$ with respect to (7.5), since X with that norm is not a Banach space. Because of the connection between finite difference methods and collocation methods discussed in Section 3, it seems therefore rather unlikely that there exist finite difference schemes which, under the assumption that $x^* \in \mathrm{Lip}^{(3)}[0,1]$, converge faster than $O(h_n^2)$.

To put it positively, we have from Lemma 1.1 and from (7.4) that for $x^* \in \mathrm{Lip}^{(3)}[0,1]$, $\|x^* - x^*_n\|_\infty = O(h_n^4)$, provided there exists $c > 0$ such that for all $x \in X$ and all n,

$$\|x^* - x^*_n\|_\infty \leq c\|x\|_\infty \,.$$

This amounts to saying that the projection scheme given by $\{X_n\}$ and $\{\Lambda_n\}$ should be boundedly convergent if X is provided with the norm (7.5). According to the Corollary to Lemma 6.2 and Corollary 4 of Theorem 4.1, such a projection scheme can be constructed from a sequence $\{\bar{Q}_n\}$ of projectors on X which is uniformly bounded and converges strongly to the identity with respect to the norm (7.5). Such a sequence, with $\bar{Q}_n[X]$ a set of cubic splines, is provided by Theorem 5.1.

Theorem 7.2. *For $n \geq 3$, let*

$$X_n = \{x \in S_3(0, t_{2,n}, t_{3,n}, \ldots, t_{n-2,n}, 1) : x(0) = x(1) = 0\},$$

where $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = 1$; set $h_n = \max\{\Delta t_{i,n} : i = 0, \ldots, n-1\}$ and $q_n = \max\{\Delta t_{i,n}/\Delta t_{j,n} : i, j = 0, \ldots, n-1\}$, and let $\Lambda_n = \langle\{\lambda_{i,n}\}_1^{n-1}\rangle \subset Y^a$, where

$$\lambda_{i,n} y = \int_0^1 y(t) g_{i,n}(t)\, dt, \quad \text{all } y \in Y,$$

(7.6)

$$g_{i,n}(t) = \begin{cases} (t - t_{i-1,n})_+/\Delta t_{i-1,n}, & t \leq t_{i,n} \\ (t_{i+1,n} - t)_+/\Delta t_{i,n}, & t \geq t_{i,n}, \end{cases} \quad i = 1, \ldots, n-1.$$

Then the projection scheme for M given by $\{X_n\}$ and $\{\Lambda_n\}$ defines an approximant x_n^* for all sufficiently large n, provided $\lim h_n = 0$, and

$$\lim \|x^{*(j)} - x_n^{*(j)}\|_\infty = 0, \quad j = 0, 1, 2, \tag{7.7}$$

and if $x^* \in \mathrm{Lip}^{(1+i)}[0,1]$, $i = 1, 2$, then

$$\|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n^i), \quad j = 0, 1, 2. \tag{7.8}$$

Under the additional assumption that $p_1 \in C^{(1)}[0,1]$, we have

(i) if for some $c < 1$ and all n,

$$\max\{\Delta t_{0,1}/t_{2,n}, \quad \Delta t_{n-1,n}/(1 - t_{n-2,n})\} \leq c, \tag{5.23}$$

then

$$\|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n), \quad j = 0, 1, \tag{7.9}$$

and if also $x^* \in \mathrm{Lip}^{(1+i)}[0,1]$, $i = 1, 2$, then

$$\|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n^{1+i}), \quad j = 0, 1; \tag{7.10}$$

(ii) if the sequence $\{q_n\}$ is bounded, then

$$\|x^* - x_n^*\|_\infty = O(h_n^2), \tag{7.11}$$

and if also $x^* \in \mathrm{Lip}^{(1+i)}[0,1]$, then

$$\|x^* - x_n^*\|_\infty = O(h_n^{2+i}), \quad i = 1, 2. \tag{7.12}$$

Proof. Let $\Sigma_n = \langle\{\delta_{t_{i,n}}\}_1^{n-1}\rangle \subset X^a$; then, by Theorem 5.1, X_n and Σ_n define a projector $\bar{Q}_n$ on X. Observe that

$$(M_1)^a [\Lambda_n] = \Sigma_n :$$

Indeed, Λ_n is also spanned by the set $\{\mu_{i,n}\}_1^{n-1}$, where

$$\mu_{i,n}y = \int_0^1 G_1\left(t_{i,n}, t\right) y\left(t\right) dt, \quad i = 1, \ldots, n-1,$$

and $G_1(s,t)$ is the Green's function for the problem $M_1 x = y^*$, $x(0) = x(1) = 0$, given in (6.5). With this

$$\begin{aligned}(M_1^a \mu_{i,n})x = \mu_{i,n}(M_1 x) = \int_0^1 G_1(t_{i,n}, t)\,(M_1 x)\,(t)\,dt = \\ = x(t_{i,n}) = \delta_{t_{i,n}} x, \quad \text{all } x \in X, \quad i = 1, \ldots, n-1,\end{aligned}$$

or, $M_1^a\left[\Lambda_n\right] = \Sigma_n$.

Now assume lim $h_n = 0$; then, by Corollary 1 to Theorem 5.1, the sequence $\{\bar{Q}_n\}$ is uniformly bounded and converges strongly to the identity on X with respect to the norm

$$\|x\|_X = \|x^{(2)}\|_\infty, \quad \text{all } x \in X. \tag{7.13}$$

By Corollary 4 to Theorem 4.1 and Lemma 6.1, the projection scheme for M given by $\{X_n\}$ and $\{\Lambda_n\}$ is therefore boundedly convergent with respect to the norm (7.13) on X and the norm

$$\|y\|_Y = \|M_1^{-1} y\|_X = \|y\|_\infty, \quad \text{all } y \in Y,$$

on Y, so that (7.7) follows. Since, with this, the sequence $\{Q_n\}$ is uniformly bounded with respect to (7.13), Lemma 5.4 provides the estimate (7.8).

Further, by Corollary 2 to Theorem 5.1, the sequence $\{\bar{Q}_n\}$ is uniformly bounded and converges strongly to the identity on X with respect to the norm

$$\|x\|_X = \|x^{(1)}\|_\infty, \quad \text{all } x \in X, \tag{7.14}$$

provided that (5.23) holds for some $c < 1$ and all n. Hence, by Corollary 4 to Theorem 4.1 and Lemma 6.2, the projection scheme for M given by $\{X_n\}$ and $\{\Lambda_n\}$ is also boundedly convergent with respect to the norm (7.14) on X and the norm

$$\|y\|_Y = \|M_1^{-1} y\|_X, \quad \text{all } y \in Y,$$

on Y, so that $\{Q_n\}$ is uniformly bounded with respect to the norm (7.14), and (7.9) and (7.10) follow from Corollary 3 of Theorem 5.1 and its proof.

Finally, by Theorem 5.1, $\{\bar{Q}_n\}$ is uniformly bounded and converges strongly to the identity on X with respect to the norm

$$\|x\|_X = \|x\|_\infty, \quad \text{all } x \in X, \tag{7.15}$$

provided $\{q_n\}$ is bounded; this, by a repeat of the preceding argument, gives (7.11) and (7.12), Q.E.D.

Remark. The projector $\bar{P}_n = M_1 \bar{Q}_n M_1^{-1}$, given by $M_1[X_n]$ and Λ_n, is almost orthogonal projection onto $M_1[X_n]$. In fact, the projector $\bar{P}'_n$, given by $M_1[X_n]$ and $\Lambda'_n = \langle\{\lambda'_1, \lambda_2, \ldots, \lambda_{n-2}, \lambda'_{n-1}\}\rangle$, where

$$\lambda'_1 y = \int_0^1 y(t)\,(t_{2,n} - t)_+\,dt, \quad \lambda'_{n-1} y = \int_0^1 y(t)\,(t - t_{n-2,n})_+\,dt,$$

(we suppressed the second subscript in $\lambda_{i,n}$), is orthogonal projection onto $M_1[X_n]$. The projector $\bar{Q}'_n = M_1^{-1}\bar{P}'_n M_1$ is discussed in Theorem 5.2 and, much as in the proof of the preceding theorem, one shows that the sequences $\{X_n\}$ and $\{\Lambda'_n\}$ give a projection scheme for M which defines the approximant x^*_n for all sufficiently large n; also, for $j = 0, 1$,

$$\|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n)\,,$$
$$\|x^{*(j)} - x_n^{*(j)}\|_\infty = O(h_n^3)\,, \quad \text{if } x^* \in \mathrm{Lip}^{(3)}\,[0,1]\,,$$

using Theorem 5.2 and Lemma 6.2. But $\bar{Q}'_n$ is not bounded with respect to the norm $\|x\|_X = \|x\|_\infty$, so that (7.11) and (7.12) cannot be proved for this projection scheme.

8. Galerkin's Method

Galerkin's method consists, as remarked earlier, in choosing, given $X_n = \langle \{x_i\}_1^n \rangle$, the linear functionals

$$\lambda_i y = \int_0^1 y(t)\, x_i\,(t)\, dt, \quad i = 1, \ldots, n \tag{8.1}$$

as a basis for Λ_n.

In case X_n consists of cubic splines with $n-1$ equidistant knots, Theorem 5.3 provides the facts sufficient to derive for Galerkin's method the same order of convergence estimates that are proved in Theorem 7.2 for the projection scheme discussed there. In particular, it is possible to show that, in this case, Galerkin's method gives approximants x_n^* to the solution x^* of $Mx = y^*$ satisfying

$$\|x^* - x_n^*\|_\infty = O(n^{-4}), \quad \text{if } x^* \in \text{Lip}^{(3)}[0,1].$$

Theorem 8.1. *For $n = 1, 2, \ldots$, let*

$$X_n = \{x \in S_3(0, 1/n, \ldots, (n-1)/n, 1) : x(0) = x(1) = 0\}.$$

Then, for all large enough n, the Galerkin approximant x_n^* in X_n to the solution x^* of $Mx = y^*$ exists and

$$\lim \|x^{*(j)} - x_n^{*(j)}\|_\infty = 0, \quad j = 0, 1, 2. \tag{8.2}$$

If M satisfies the additional assumption that $p_1 \in C^{(1)}[0,1]$, then

$$\|x^* - x_n^*\|_\infty = O(n^{-2}), \tag{8.3}$$

while, if also $x^* \in \text{Lip}^{(3)}[0,1]$, then

$$\|x^* - x_n^*\|_\infty = O(n^{-4}). \tag{8.4}$$

Proof. By the Corollary to Theorem 5.3, $M_1[X_n]$ and Λ_n define a projector $\bar{P}_n$ on the Banach space $Y = C[0,1]$ and $\bar{P}_n \to I$. Galerkin's method is therefore boundedly convergent with respect to the norm $\|x\|_X = \|x^{(2)}\|_\infty$ on X and the norm $\|y\|_Y = \|y\|_\infty$ on Y. This implies (8.2). But, by Theorem 5.3, the sequence $\{\bar{Q}_n\}$, with $\bar{Q}_n = M_1^{-1}\bar{P}_n M_1$, all n, is uniformly bounded and converges strongly to the identity on X with respect to the norm $\|x\|_X = \|x\|_\infty$ on X, which, by an argument as given in the proof of Theorem 7.2, implies (8.3) and (8.4), Q.E.D.

9. Least-squares and the Golomb-Weinberger method

We retain the definitions (6.1), (6.3) of the maps M and M_1, but extend X and Y slightly: Let $Y = L_2[0,1]$ with the usual norm, set

$$X = \{x \in C^{(1)}[0,1] : x(0) = x(1) = 0, \quad x^{(1)}\text{abs.cont.}\,, \quad x^{(2)} \in L_2[0,1]\}.$$

Then M_1 maps X in a 1-1 manner onto Y, and, by the assumptions on M, so does M. Define an inner product on X by

$$(x,z)_X = (M_1 x, M_1 z)_Y = \int_0^1 x^{(2)}(t) z^{(2)}(t)\,dt, \quad \text{all } x, z \in X,$$

so that both X and Y are Hilbert spaces and M_1 is a unitary operator.

If $\{X_n\}$ is any sequence of finite dimensional subspaces of X, then the Least-squares method using this sequence is the projection scheme given by $\{X_n\}$ and $\{M[X_n]\}$, where $M[X_n]$ is considered here as a subspace of Y'. In other words, the corresponding sequence $\{P_n\}$ of projectors on Y consists of orthogonal projections. Hence, the Least-squares method is convergent (with respect to the norms on X and Y as chosen) if and only if $\overline{\lim}\ X_n = X$.

Instead of giving rather obvious theorems concerning the order of approximation achieved by the Least-squares method using cubic splines, I prefer to sketch a (facetious) example in order to demonstrate that the Least-squares method is maligned in [5, Kap. III, §6.4] because of insufficient evidence. It is intimated there that Galerkin's method (or Ritz' method) is to be preferred to the Least-squares method since the latter tends to give poorer approximations to x^* than the former.

So, let $G(s,t)$ be the Green's function for the problem at hand, so that

$$x(s) = \int_0^1 G(s,t)\,(Mx)(t)\,dt, \quad s \in [0,1], \quad \text{all } x \in X\,.$$

Pick a partition of $[0,1]$, $0 < t_1 < \cdots < t_n < 1$, set

$$x_i(s) \equiv \int_0^1 G(s,t)\,G(t_i,t)\,dt, \quad i = 1,\ldots,n,$$

and take $X_n = \langle \{x_i\}_1^n \rangle$. Then the Least-squares method using X_n provides an approximant x_n^* to x^* which agrees with x^* at the points t_i, $i = 1,\ldots,n$, — as good a result as one can hope to get by any numerical method.

To prove this assertion, let P_n be the orthogonal projection onto $M[X_n]$. Then, for $i = 1,\ldots,n$,

$$x^*(t_i) - x_n^*(t_i) = \int_0^1 G(t_i,t)\left((I-P_n)y^*\right)(t)\,dt = 0\,,$$

since $G(t_i,t) = (Mx_i)(t) \in M[X_n]$, and $(I-P_n)y^*$ is orthogonal to all elements of $M[X_n]$.

The Golomb-Weinberger method is dual to the Least-squares method in the sense that it is the Least-squares method applied to the dual problem (cf. Section 2). Whereas in the

Least-squares method one starts with a subspace X_n of X and chooses $\Lambda_n \subset Y'$ in such a way that P_n, given by $M[X_n]$ and Λ_n, is orthogonal projection, in the Golomb-Weinberger method one starts with a subspace $\Lambda_n \subset Y'$ and chooses $X_n \subset X$ in such a way that Q_n, given by X_n and $M^a[\Lambda_n]$, becomes orthogonal projection (cf.,[19, Example 6.1]). It is clear that, given a sequence $\{\Lambda_n\}$ of finite dimensional subspaces of Y', the Golomb-Weinberger approximant x_n^* is defined for all n, and that the Golomb-Weinberger method is convergent (with respect to the norms on X and Y as chosen) if and only if $= \overline{\lim}\ \Lambda_n = Y'$.

Although in general it may be difficult to determine X_n for given Λ_n, this is rather simple in the present case: Let $\Lambda_n = \langle\{\lambda_i\}_1^n\rangle$, and set

$$K(s,t) = \int_0^1 G_1(s,u)G_1(t,u)\,du, \quad s,t \in [0,1],$$

where $G_1(s,t)$ is given by (6.5). Then, $\{x_i\}_1^n$, given by

$$x_i(s) \equiv \lambda_{i(t)} M_{(t)} K(s,t), \quad i = 1, \ldots, n,$$

is a basis for X_n.

To prove this, it is sufficient to show that for $x \in X$, for all $z \in X_n$, $(x,z)_X = 0$, if and only if for all $\lambda \in \Lambda_n$, $(M^a\lambda)x = 0$. But this follows from the fact that x_i is the functional representer for $M^a\lambda_i$ on the Hilbert space X, i.e.,

$$(x, x_i)_X = (M^a\lambda_i)x, \quad i = 1, \ldots, n, \quad \text{all } x \in X,$$

a proof of which may be found in [4].

The Golomb-Weinberger method derives its appeal from the fact that the following pointwise bound

$$(9.1) \qquad |x^*(s) - x_n^*(s)\,| \le \sqrt{(\|x^*\|^2 - \|x_n^*\|^2)} \cdot \min\{\|g_s - z\| : z \in X_n\}$$

holds for $s \in [0,1]$, where $g_s(t) = K(s,t)$. Under the assumption that $\|x^*\|$ (or a bound for it) is known, the right side of (9.1) can be computed from the numbers $\lambda_i y^*$, $i = 1, \ldots, n$, and the bound is sharp with respect to the information $\lambda_i y^*$, $i = 1, \ldots, n$ and $\|x^*\|$.

10. Computational considerations

It is stated in Section 2 that, for the numerical determination of the approximant x_n^*, one picks **convenient** bases, $\{x_i\}_0^n$ for X_n and $\{\lambda_i\}_0^n$ for Λ_n, and determines the coefficients $a_0^*, \ldots, a_n^*$, of x_n^* with respect to the basis $\{x_i\}_0^n$ as the solution to the system of $n+1$ equations,

$$\sum_{j=0}^{n} \lambda_i(Mx_j)a_j^* = \lambda_i y^*, \quad i = 0, \ldots, n, \tag{10.1}$$

the so-called generalized Galerkin equations. In this section, we derive a basis for X_n for the case that X_n consists of cubic splines, as was assumed in the previous sections.

According to Lemma 5.2, the set $\{x_i\}_{-1}^{n+1}$, given by

$$\begin{aligned} &x_i(t) \equiv g_4(t_{i-2}, \ldots, t_{i+2}; t), \quad i = -1, \ldots, n+1, \\ &g_4(s;t) \equiv (s-t)_+^3/6, \end{aligned}$$

is a basis for $S_3(0, t_1, \ldots, t_{n-1}, 1)$, where we use the auxiliary points $t_i = i/n$, $i = -3, \ldots, 0, n \ldots, n+3$. Written out explicitly, we have, for $i = -1, \ldots, n+1$,

$$x_i(t) = \begin{cases} (t-t_{i-2})_+^3 / \prod\limits_{j=i-1}^{i+2}(t_j - t_{i-2}) + (t - t_{i-1})_+^3 / \prod\limits_{\substack{j=i-2\\ j\neq i-1}}^{i+2}(t_j - t_{i-1}), & t \le t_i, \\ \\ (t_{i+2} - t)_+^3 / \prod\limits_{j=i-2}^{i+1}(t_{i+2} - t_j) + (t_{i+1} - t)_+^3 / \prod\limits_{\substack{j=i-2\\ j\neq i+1}}^{i+2}(t_{i+1} - t_j), & t \ge t_i. \end{cases} \tag{10.2}$$

From this, we get a basis $\{\bar{x}_i\}_0^n$ for

$$X_n = \{x \in S_3(0, t_1, \ldots, t_{n-1}, 1) : x(0) = x(1) = 0\}$$

much as in the proof of Theorem 5.3. Set

$$\begin{aligned} &\bar{x}_0 = x_0 - x_0(0)x_{-1}/x_{-1}(0), \quad \bar{x}_1 = x_1 - x_1(0)x_{-1}/x_{-1}(0), \\ &\bar{x}_n = x_n - x_n(0)x_{n+1}/x_{n+1}(0), \\ &\bar{x}_{n-1} = x_{n-1} - x_{n-1}(0)x_{n+1}/x_{n+1}(0), \\ &\bar{x}_i = x_i, \quad i = 2, \ldots, n-2. \end{aligned} \tag{10.3}$$

In the case of equidistant knots, formulae (10.2) and (10.3) simplify considerably (cf. the proof of Theorem 5.3).

This basis for X_n is doubly convenient. For one, the evaluation of $x = \sum a_i \bar{x}_i$ at any point involves at most four of the a_i's, since, by the Corollary to Lemma 5.1 (or else by (10.2)), $x_i(t) = 0$, whenever $t \notin (t_{i-2}, t_{i+2})$. But, because of this, the matrix of (10.1) is a band matrix, whenever the linear functionals λ_i are "local" in the sense that the number $\lambda_i y$ depends only on the values of y in some "small" interval. On checks, e.g., that for the Collocation method in Section 7, the matrix of (10.1) is tri-diagonal, for the higher order method of Theorem 7.2, the matrix is five-diagonal, while for Galerkin's method and the Least-squares method the matrix is seven-diagonal. The work necessary to solve (10.1) is therefore, in these cases, of the order of n rather than n^2.

In this general setup, it is impossible to state whether or not direct inversion of (10.1) is numerically stable or whether an iterative method is to be preferred. I am certain that for any particular procedure for solving (10.1) there is some differential operator M for which this procedure gives poor results. In any case, I have yet not studied this question.

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