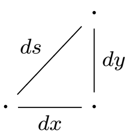
The Math Circle, Spring 2004

(Talks by Gordon Ritter)

What is Non-Euclidean Geometry?

Most geometries on the plane $\mathbb{R}^2$ are non-Euclidean. Let s denote arc length. Then Euclidean geometry arises from the formula for curve length

$$ds^2 = dx^2 + dy^2 \,,$$



which itself arises from applying the Pythagorean theorem to each segment in a piecewise-linear approximation to a curve. Given a definition of ds, one then measures length of a curve γ by integrating ds over the curve: $L = \int_\gamma ds$. However, one could perform the same integral with a different definition of ds. The most general degree two polynomial in dx and dy is

$$ds^2 = A(x,y)dx^2 + B(x,y)dxdy + C(x,y)dy^2 \tag{1}$$

Euclidean geometry arises from (1) through the very special choice $A = C = 1, B = 0$. For the Poincaré half-plane, B is again zero, and $A = C = y^{-2}$.

A definition of ds determines the length of any path γ by the formula $L = \int_\gamma ds$. Given points p, q, consider all paths originating from p and ending at q. It is of interest to know which path has the smallest value of L; such paths are called *geodesics*. For many choices of A, B, C, one may construct a plane geometry by designating the geodesics to be *lines*. By our definition of geodesic, this automatically satisfies the postulate that any two points must have a line containing them. Angles are defined as usual: when two differentiable curves intersect, measure the angle between their tangent lines at the point of intersection. A circle is the set of points equidistant from a given point.

Fundamental Theorem of Calculus

Theorem 1. *Let f be a bounded, continuous, real function on $[a,b]$. Let $\int_a^b f$ denote the (signed) area between f and the x-axis, over the indicated interval. Define $F(x) = \int_a^x f$, for $x \in [a,b]$. Then $F'(x) = f(x)$.*

Proof. Choose $h > 0$ small enough so that $x + h < b$, and note that

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f$$

Of course, $F'(x)$ is the limit as $h \to 0$ of this expression. Let $I = [x, x+h]$ and let $m = \min_I f, M = \max_I f$. Then

$$m = \frac{1}{h} mh = \frac{1}{h} \int_x^{x+h} m \leq \frac{1}{h} \int_x^{x+h} f \leq \frac{1}{h} \int_x^{x+h} M = \frac{1}{h} Mh = M$$

Therefore

$$m \leq \frac{F(x+h) - F(x)}{h} \leq M$$

Since f is continuous, it follows that as $h \to 0$, M and m approach each other and both approach the value $f(x)$. In other words,

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

□

The "integral evaluation shortcut" follows. Indeed, if G is any function such that $G' = f$, then let $H = F - G$. Clearly, $H' = 0$, so $H = c =$ constant. Then $F = G + c$ so $G(b) - G(a) = F(b) - F(a) = \int_a^b f$. Typically, we are given f, and asked to calculate $\int_a^b f$. If we can find G such that $G' = f$, then this calculation reduces to evaluating G at two points.

Hyperbolic Geodesics

Let $P = (x_1, y_1), Q = (x_2, y_2)$. Consider first the case $x_1 \neq x_2$, and construct the perpendicular bisector to $\overline{PQ}$, which we denote by $(c, 0)$. Use polar coordinates, $x = c + r\cos\theta, y = r\sin\theta$. In these coordinates, let the unknown geodesic be $r = f(\theta)$.

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$$

where a prime denotes derivative with respect to θ. For this geometry, $ds^2 = (dx^2 + dy^2)/y^2$, and

$$dx^2 + dy^2 = (r'^2 + r^2)d\theta^2$$

Let α and β denote the angles of P and Q with the positive x-axis, respectively. The hyperbolic length of the curve $r = f(\theta)$ from $\theta = \alpha$ to β is

$$L = \int ds = \int \frac{\sqrt{dx^2 + dy^2}}{y} = \int \frac{\sqrt{x'^2 + y'^2}}{y} d\theta = \int_\alpha^\beta \frac{\sqrt{r'^2 + r^2}}{r\sin\theta} d\theta$$

It will always hold that $\sqrt{r'^2 + r^2} \geq \sqrt{r^2} = r$. This means that among all possible curves $r = f(\theta)$ joining P to Q, the one which minimizes L is the one with $r' = 0$, so this inequality is saturated. This means that $r = f(\theta) = \text{constant}$, which is the polar equation of a circle centered at c. In this case, we can evaluate the integral explicitly, to find

$$L = \int_\alpha^\beta \csc\theta \, d\theta = \ln \left| \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} \right|$$

The case with $x_1 = x_2$ is handled similarly. Let the geodesic have equation $x = f(y)$, then

$$L = \int_{y_1}^{y_2} \frac{\sqrt{(f')^2 + 1}}{y} \, dy \geq \int_{y_1}^{y_2} \frac{dy}{y} = \ln \left| \frac{y_2}{y_1} \right|$$

So in both cases, the same calculation proves that the geodesics have the stated form, and gives the relevant distance formula.

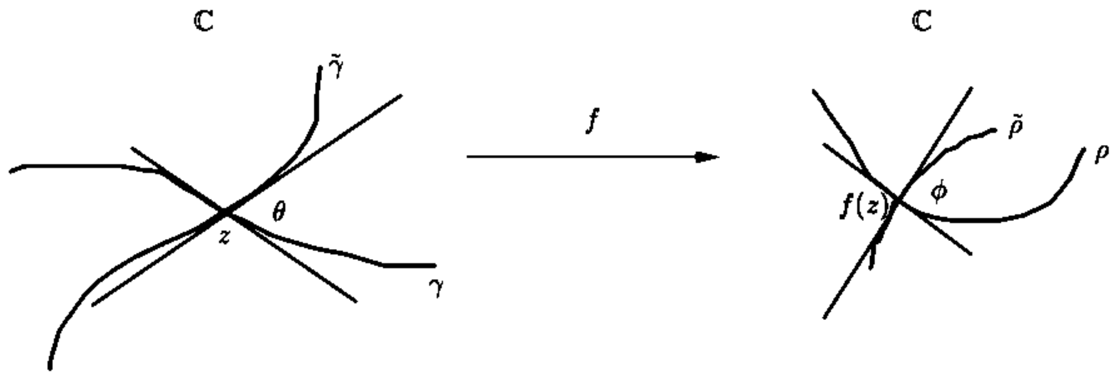
Conformal Mapping

Let $\gamma, \tilde{\gamma} : [a, b] \to \mathbb{C}$ be two smooth curves with $\gamma'(a) \neq 0 \neq \tilde{\gamma}'(a)$. Assume that $\gamma(a) = \tilde{\gamma}(a)$ (so there is an angle), and denote this point by z. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, differentiable[1] at z. Let $\rho(t) = f(\gamma(t))$ and $\tilde{\rho}(t) = f(\tilde{\gamma}(t))$. Then

$$\rho'(a) = \frac{d}{dt} f(\gamma(t))|_{t=a} = f'(z)\gamma'(a)$$

by the chain rule. Similarly, $\tilde{\rho}'(a) = f'(z)\tilde{\gamma}'(a)$. Let θ denote the angle between the curves $\gamma, \tilde{\gamma}$ at z, and let ϕ denote the angle between $\rho, \tilde{\rho}$ at $f(z)$. It follows that

$$\begin{aligned} \phi &= \arg\rho'(a) - \arg\tilde{\rho}'(a) = \arg f'(z) + \arg\gamma'(a) - (\arg f'(z) + \arg\tilde{\gamma}'(a)) \\ &= \arg\gamma'(a) - \arg\tilde{\gamma}'(a) = \theta \end{aligned}$$



[1]differentiable in the complex sense, meaning that $\lim_{w \to z} \dfrac{f(w) - f(z)}{w - z}$ exists

Mandelbrot and Julia Sets

Consider the sequence $z_{n+1} = z_n^2 + c$ where c is a complex number. Define the *Mandelbrot Set*

$$M = \{c : z_n \not\longrightarrow \infty \ \text{ with } \ z_0 = 0\}$$

Theorem 2. *If $|z_k| > 2$ for some k, then $\lim_{n\to\infty} z_n = \infty$ (hence c is outside M). The Mandelbrot set is contained in a disk of radius 2 centered at the origin.*

Proof. First note that if $z \in \mathbb{C}$ is such that

$$|z| > \max(2, |c|) \tag{2}$$

then

$$\begin{aligned} \frac{|f(z)|}{|z|} &= \frac{|z^2+c|}{|z|} \geq \frac{|z|^2-|c|}{|z|} = |z| - \frac{|c|}{|z|} \\ &> |z| - 1 \qquad\qquad\qquad (|z| > |c|) \\ &> 1 \qquad\qquad\qquad\qquad\; (|z| > 2) \end{aligned}$$

This implies that further iterations of $f(z)$ blow up. If $|c| \leq 2$ then z_k satisfies (2). On the other hand, if $|c| > 2$ then

$$|c^2 + c| \geq |c|^2 - |c| = |c|(|c| - 1) > |c| > 2$$

so z_2 satisfies (2). Thus if $|c| > 2$ the sequence always diverges, proving the second statement. □

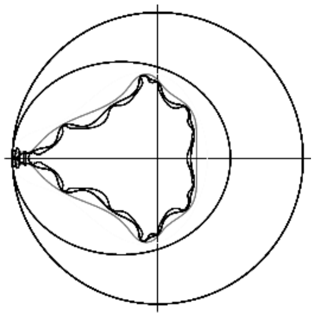
For points c outside M, define the *count* $n(c)$ to be the largest n such that $|z_n| < 2$. We can then study the sequence of nested regions with count $n = 1, 2, 3, \ldots$. How can we plot the boundaries of these regions? Boundary points saturate the inequality, i.e. $|z_n| = 2$. Now $z_1 = c, z_2 = c^2 + c, z_3 = c + (c + c^2)^2$, etc. So the first one is the set of points c such that $|c| = 2$, i.e. a circle. The second one is given by the equation

$$4 = (x^2 + y^2)[(x + 1)^2 + y^2]$$

Interestingly, this equation is a special case of a *Cassini oval*, that is the curve defined by the equation

$$[(x - a)^2 + y^2][(x - a)^2 + y^2] = b^4 \tag{3}$$

In words, (3) says that the product of the distances of (x, y) from two points at a distance $2a$ apart is b^2. For $a = b$, we get the classical lemniscate of Bernoulli and for this reason, the curves obtained by considering $n(c) = 2, 3$, etc. are called Mandelbrot set lemniscates.



Given a function $f : \mathbb{C} \to \mathbb{C}$, define $J(f)$, the Julia set of f, to be the set of all z_0 such that $z_n \not\longrightarrow \infty$, where the sequence is defined as before by $z_{n+1} = f(z_n)$. Define

$$A(\infty) = \{z : |f^{\circ n}(z)| \longrightarrow \infty\}$$

which is the set of unbounded orbits of f. (The notation $f^{\circ n}$ means $f \circ f \circ \ldots \circ f$, with a total of n compositions). A *critical point* of f is a point z_c which satisfies $f'(z_c) = 0$. Here is a fascinating result which we will not have time to prove.

Theorem 3. *$J(f)$ is connected $\iff$ no critical point of f lies in $A(\infty)$.*

Curvature

For a 1-dimensional curve in the Cartesian plane, the curvature is defined to be the inverse radius of the tangent (osculating) circle. The tangent circle can be defined by choosing three points on the curve, which uniquely define a circle, and looking at the limit as the spacing between the points approaches zero. While this has a nice geometrical interpretation, the formulas to calculate the relevant quantities in this picture are sufficiently complicated as to be not useful.

A much better way of calculating curvature of a plane curve is to re-interpret it as the rate of change of the tangent angle ϕ, as a function of arc length. Here, ϕ is the angle between a tangent to the curve and the horizontal, at each point. Defining s to be the arc length along the curve as measured from some reference point, the curvature is then

$$\kappa = \frac{d\phi}{ds}$$

This is a useful formula for computation. As an example, let's compute the curvature of a parabola of the form $y = Qx^2$. We will need to examine the relation between arc length s, and the horizontal coordinate x. We know that

$$s(x) = \int_0^x \sqrt{1 + (2Qt)^2}dt$$

The value of this integral is not important; we simply note that by the fundamental theorem of calculus,

$$\frac{ds}{dx} = \sqrt{1 + (2Qx)^2}$$

Also, $\tan\phi$ is the slope $2Qx$. Therefore

$$\kappa = \frac{d\phi}{ds} = (1 + (2Qx)^2)^{-1} 2Q \frac{d}{ds} x(s) = 2Q(1 + (2Qx)^2)^{-3/2}$$

Here, we have used the fact that $\frac{dx}{ds} = 1/\left(\frac{ds}{dx}\right)$. See the footnote for a proof. [2]

At the origin, the formula simplifies to

$$\kappa = 2Q$$

A second definition of curvature of a 2-surface embedded in 3-space is as follows. Choose coordinates so that the point p is at $(0, 0, 0)$. Locally in a neighborhood of p the surface is described by a function $z = f(x, y)$. Various planes perpendicular to the xy-plane and passing through p can now be constructed, and our surface intersects each such plane in a smooth curve. This gives an infinite family of curves formed by intersection. Each of these curves has some curvature κ at the point p. If we compute the minimum and maximum κ obtained by this process, then $\kappa_{min}\kappa_{max}$ will equal the curvature K we derived before. For a sphere, clearly $\kappa_{min} = \kappa_{max} = 1/\text{radius}$.

Aside from the plane $x = 0$, the others are described by equations of the form $y = mx$ (with z free to range through $\mathbb{R}$). Here, $\theta = \tan^{-1}(m)$ is the polar angle in the usual (r, θ) coordinates for the $z = 0$ plane. Let's call the plane with slope $m = \tan\theta$ the "θ-plane." If we introduce a Cartesian coordinate system in the θ-plane, the height is well described by z, but since the plane is rotated, the horizontal coordinate is r, not x. The equation of the curve in the θ-plane is therefore

$$z = f(r\cos\theta, r\sin\theta) = G_\theta(r)$$

where the last equality defines the function G_θ. As expected, for fixed θ, this gives z as a function of r. The curvature at p in the θ-plane is therefore the curvature of $G_\theta(r)$ at $r = 0$, which we will denote

$$\kappa(\theta) = \kappa[G_\theta]_{r=0}$$

This will, of course, depend on θ; we are interested in the minimum and the maximum.

[2] The latter identity can be proved as follows. A smooth function can be approximated locally near a point p by its tangent line at p. Therefore, $x(s) \approx ms + b$ for some constants m and b. Let us invert this function: $s \approx (x - b)/m$ which has slope $1/m$.

For a sphere, clearly $\kappa(\theta) = 1/R$. Let us compute this function explicitly for a nontrivial surface. An elliptic paraboloid has the defining property that the cross sections at fixed z are ellipses, while the cross sections at fixed x or y are parabolas. This fixes the parametric equations of the surface to be

$$x = a\sqrt{u}\cos\theta, \quad y = b\sqrt{u}\sin\theta, \quad z = u$$

for a, b constant. It's clear that

$$z = (\frac{x}{a})^2 + (\frac{y}{b})^2$$

which is an equivalent definition of the surface. Going to polar coordinates,

$$z = r^2[a^{-2}\cos^2\theta + b^{-2}\sin^2\theta]$$

Therefore, at fixed θ, $G_\theta(r)$ is a parabola Qr^2 whose coefficient depends on a, b, and θ.

$$G_\theta(r) = Qr^2, \qquad Q = (\frac{1}{a}\cos\theta)^2 + (\frac{1}{b}\sin\theta)^2$$

The curvature is that determined above for a parabola,

$$\kappa(\theta) = \kappa[G_\theta]_{r=0} = 2Q$$

To find the extreme values, note that Q is the distance from the origin of a point on an ellipse with minor and major axes given by the *inverses* a^{-1}, b^{-1}. We infer that

$$\kappa_{min}\kappa_{max} = \frac{1}{ab}$$

A Two-dimensional Black Hole

In 1915 Einstein published a set of equations which govern the action of strong gravitational fields. A solution to these equations is a metric for a four-dimensional spacetime. In this context, a *metric* is a formula for arc length on a curved space of the form (1), although (1) is written for two dimensions while any number of dimensions is possible. One year later in 1916, Schwarzschild found a metric which solves Einstein's equations and which represents a spherically symmetric gravitating body with infinite force at its center, and which shares many properties in common with phenomenological black holes, such as the existence of an event horizon, from whence no classical radiation can emerge.

Schwarzschild's solution is

$$ds^2 = -(1 - \frac{R_s}{r})dt^2 + \frac{dr^2}{(1 - R_s/r)} + r^2 d\Omega^2 \tag{4}$$

where $d\Omega^2$ is the standard round metric for a sphere,

$$d\Omega^2 = d\phi^2 + \sin^2\theta d\theta^2$$

Consider a simultaneity plane ($dt = 0$) through an equatorial plane ($d\theta = 0$). With these reductions, the Schwarzschild metric (4) takes the form

$$ds^2 = g_{rr}dr^2 + r^2 d\phi^2 \tag{5}$$

where $g_{rr} = (1 - R_s/r)^{-1}$. Now (5) describes some curved surface which we would like to visualize, by embedding it into 3-dimensional space. For this purpose, consider the usual cylindrical coordinates, in which the flat Euclidean metric is

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \tag{6}$$

A surface in these coordinates would be expressed by a function $z = z(r, \phi)$, and it is our goal to find this function explicitly for the surface (5). Suppose the surface is embedded in a way that is symmetric in the angular coordinate ϕ, which we expect is possible since the "space" part of the original metric (4) possesses spherical symmetry, and we only eliminated half of that symmetry by going to an equatorial plane. Then $z = z(r)$ and hence

$$dz^2 + dr^2 = (1 + (\frac{dz}{dr})^2)dr^2$$

Now, using (6) we have

$$ds^2 = (1 + (\frac{dz}{dr})^2)dr^2 + r^2 d\phi^2$$

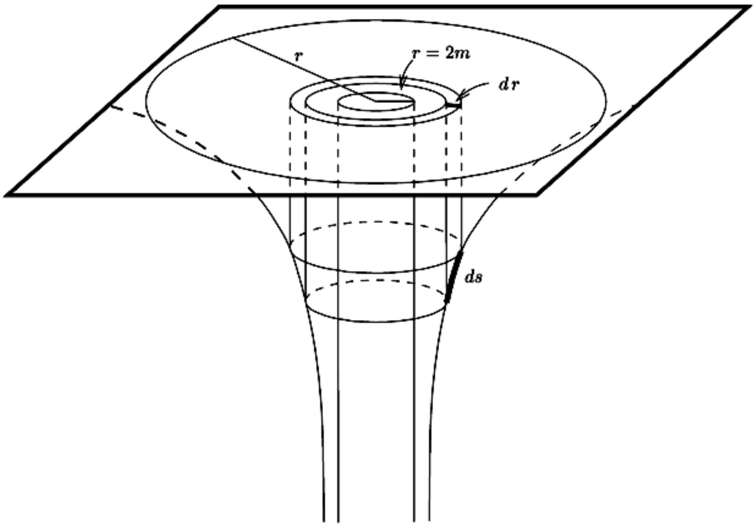
This implies that $g_{rr} = 1 + (\frac{dz}{dr})^2$ and (choosing the positive square root),

$$\frac{dz}{dr} = \sqrt{g_{rr} - 1} = \sqrt{\frac{R_s}{r - R_s}}$$

This implies that

$$z(r) = \int_{R_s}^{r} \sqrt{\frac{R_s}{r' - R_s}} dr' = \sqrt{R_s} \int_{R_s}^{r} \frac{dr'}{\sqrt{r' - R_s}} = \sqrt{4R_s(r - R_s)} \tag{7}$$

This solves the problem we set out to solve. From (7), we see that z^2 is a linear function of r, so $z(r, \phi)$ is just the surface of revolution of a half-parabola. The minimum value of r, which occurs at $z = 0$, is $r_{min} = R_s$. This is called the *Schwarzschild radius* and is the point at which the Schwarzschild solution breaks down (clearly, g_{rr} is singular at this value). Continuing the parabola to negative z in fact gives incorrect physics: you will not bounce back from the event horizon of a black hole! By a change to so-called Kruskal coordinates (or Eddington–Finkelstein coordinates), we could obtain a metric that is correct down to $r = 0$, in which we would see the true shape of the solution.



The paths traversed by light rays are geodesics of the relevant geometry, which is to say they are paths γ which minimize the length $L = \int_\gamma ds$. Previously we found the geodesics of a very simple toy model, the Poincaré half-plane. The same skills which were important there are useful in general: choosing a coordinate system adapted to the symmetries of the answer, and performing an estimate on the relevant integral.