Numerical solution for a kind of nonlinear telegraph equations using radial basis functions

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Outline

This report is devoted to the numerical computation of the nonlinear telegraph equations takes the following form:

$$u_{tt} + a_1 u_t = a_2 u_{xx} + f(u), \tag{1}$$

where a_1, a_2 are known constant coefficients, $f(u) = \alpha u^3 + \beta u^2 + \gamma u$, α β and γ are real constant. Eq.(1) referred to as second-order nonlinear telegraph equation.

Telegraph equation describes various phenomena in many applied fields, such as a planar random motion of a particle in fluid flow, transmission of electrical impulses in the axons of nerve and muscle cells, propagation of electromagnetic waves in superconducting media and propagation of pressure waves occurring in pulsatile blood flow in arteries[1, 2].

In the recent years much attention has been given in the literature to the development, analysis and implementation of stable numerical methods for the solution of second-order telegraph equations [3, 4]. The existence of time-bounded solution of nonlinear bounded perturbations of the telegraph equation with Neumann boundary conditions has recently been considered in [5]. The approach is based upon a Galerkin method combined with the use of some Lyapunov functionals. The exact solution of nonlinear telegraph equations are also given in many papers [6, 7].

Over the past several decades, many numerical methods have been developed to solve boundary-value problems involving ordinary and partial differential equations. Finite difference methods are known as the first techniques for solving partial differential equations [8, 9]. But the need to use large amount of CPU time in implicit finite difference schemes limit the applicability of these method. Furthermore, these methods provide the solution of the problem on mesh point only [10] and accuracy of these well known techniques is reduced in non-smooth and non-regular domains.

To avoid the mesh generation, in recent years meshless techniques have attracted attention of researchers. Some meshless schemes are the element free Galerkin method, the reproducing kernel particle, the point interpolation and etc. More descriptio see [11] and reference therein.

In last 25 years, the radical basis functions (RBFs) method is known as a powerful tool for scattered data interpolation problem. Because of the collection technique, this method does not need to evaluate any integral. The main advantage of numerical procedures which use RBFs over traditional techniques is meshless property of these methods. RBFs are used actively for solving partial differential equations. The examples see [12, 13].

In the last decade, the development of the RBFs as a truly meshless method for approximating the solutions of PDEs has drawn the attention of many researchers in science and engineering. Meshless method has became an important numerical computation method, and there are many academic monographs are published[14, 15].

We present a new numerical scheme to solve nonlinear telegraph problem using the meshfree method with Radial Basis Functions (RBFs). The results of numerical experiments are presented, and are compared with analytical solutions to confirm the good accuracy of the presented scheme.

The approximation of a distribution $u(\mathbf{x})$, using RBF, may be written as a linear combination of N radial functions, usually it takes the following form:

$$u(\mathbf{x}) \simeq \sum_{j=1}^{N} \lambda_j \varphi(\mathbf{x}, \mathbf{x}_j) + \psi(\mathbf{x}), \quad \text{for } \mathbf{x} \in {}^{\cdot} \subset \mathrm{R}^{\mathrm{d}}, \tag{2}$$

where N is the number of data points, $\mathbf{x} = (x_1, x_2, ..., x_d)$, d is the dimension of the problem, the λ's are coefficients to be determined and φ is the radial basis function. Eq. (2) can be written without the additional polynomial ψ. In that case, φ must be unconditional positive definite to guarantee the solvability of the resulting system (e.g. Gaussian or inverse multiquadrics). However, ψ is usually required when φ is conditionally positive definite, i.e, when φ has a polynomial growth towards infinity. We will use RBFs, which defined as:

$$\begin{aligned} \text{TPS} : \varphi(\mathbf{x}, \mathbf{x}_j) = \varphi(r_j) = r_j^{2m} \ln r_j, \quad m = 1, 2, 3 \cdots , \\ \text{IMQ} : \varphi(\mathbf{x}, \mathbf{x}_j) = \varphi(r_j) = \frac{1}{\sqrt{r_j^2 + c^2}}, \quad c > 0, \end{aligned} \tag{3}$$

where $r_j = \|\mathbf{x} - \mathbf{x}_j\|$ is the Euclidean norm. Since φ given by (3) is C^∞ continuous ,we can use it directly.

The IMQ radial basis function takes the form: $\frac{1}{\sqrt{r^2+c^2}}$, $c > 0$. The choice of parameter c is very important for the solution accuracy. Unsuitable parameter will produce the singular matrix. Moreover, how many nodes we choose can also affect the accuracy. The theory of radial basis function difference method can see[16, 17].

If $\mathcal{P}_q^d$ denotes the space of d-variate polynomial of order not exceeding than q, and letting the polynomials $P_1, P_2, ..., P_m$ be the basis of $\mathcal{P}_q^d$ in R^d, then the polynomial $\psi(\mathbf{x})$ in Eq. (2), is usually written in the following form:

$$\psi(\mathbf{x}) = \sum_{i=1}^{m} \zeta_i P_i(\mathbf{x}_j), \tag{4}$$

where $m = (q-1+d)!/(d!(q-1)!)$.

To get the coefficients $(\lambda_1, \lambda_2, ..., \lambda_N)$ and $(\zeta_1, \zeta_2, ..., \zeta_m)$, the collocation method is used. However, in addition to the N equations resulting from collocating (2) at the N points, an extra m equations are required. This is ensured by the m conditions for (2),

$$\sum_{j=1}^{N} \lambda_j P_i(\mathbf{x}_j) = 0, \quad i = 1, 2, ..., m. \tag{5}$$

In a similar representation as (2), for any linear partial differential operator $\mathcal{L}$, $\mathcal{L}u$ can be approximated by

$$\mathcal{L}u(\mathbf{x}) \simeq \sum_{j=1}^{N} \lambda_j \mathcal{L}\varphi(\mathbf{x}, \mathbf{x}_j) + \mathcal{L}\psi(\mathbf{x}). \tag{6}$$

We consider the following time-dependent problems:

$$u_{tt} + a_1 u_t = a_2 u_{xx} + f(u), \tag{7}$$

$$x \in \Omega \cup \partial\Omega = [a, b] \subset R, \;\; 0 < t \le T, \tag{8}$$

with initial condition

$$\begin{aligned} u(x, 0) &= g_1(x), \quad x \in \Omega, \\ u_t(x, 0) &= g_2(x), \quad x \in \Omega, \end{aligned} \tag{9}$$

and Dirichlet boundary condition

$$u(x, t) = h(x, t), \qquad x \in \partial\Omega, \; 0 < t \le T, \tag{10}$$

where a_1, a_2 are known constant coefficients, $f(u) = \alpha u^3 + \beta u^2 + \gamma u$ and α, β, γ are known constant coefficients, the function $u(x, t)$ is unknown.

First, let us discretize (7) according to the following θ-weighted scheme:

$$\frac{u(x,t+dt)-2u(x,t)+u(x,t-dt)}{(dt)^2}+a_1\frac{u(x,t+dt)-u(x,t-dt)}{2\cdot dt}$$
$$=a_2\theta\Delta u(x,t+dt)+a_2(1-\theta)\Delta u(x,t)+f(u(x,t+dt)), \tag{11}$$

where $0\le\theta\le 1$,and dt is the time step size and Δ is the Laplace operator, using the notation $u^n=u(x,t^n)$ where $t^n=t^{n-1}+dt$, we get

$$\begin{aligned}(1+a_1\frac{dt}{2})u^{n+1}-a_2(dt)^2\theta\Delta u^{n+1}&=2u^n+a_2(dt)^2(1-\theta)\Delta u^n\\&+(a_1\frac{dt}{2}-1)u^{n-1}+(dt)^2f^{n+1},\end{aligned} \tag{12}$$

where $f^{n+1}=f(u^{n+1})$.

Assuming that there are a total of $N-2$ interpolation points, $u^n(x)$ can be approximated by

$$u^n(x) = \sum_{j=1}^{N-2} \lambda_j^n \varphi(r_j) + \lambda_{N-1}^n x + \lambda_N^n, \tag{13}$$

To determine the coefficients $(\lambda_1, \lambda_2, \cdots, \lambda_{N-1}, \lambda_N)$, the collection method is used by applying (13) at every point x_i, $i = 1, 2, \cdots, N-2$. Thus we obtain

$$u^n(x_i) = \sum_{j=1}^{N-2} \lambda_j^n \varphi(r_{ij}) + \lambda_{N-1}^n x_i + \lambda_N^n, \tag{14}$$

where $r_{ij} = \sqrt{(x_i - x_j)^2}$. The additional conditions due to (5) can be written as

$$\sum_{j=1}^{N-2} \lambda_j^n = \sum_{j=1}^{N-2} \lambda_j^n x_j = 0. \tag{15}$$

Writing (14) together with (15) in a matrix form

$$[u]^n = \mathbf{A}[\lambda]^n$$

where $[u]^n = [u_1^n \ \ u_2^n \ \ \cdots \ \ u_{N-2}^n \ \ 0 \ \ 0]^T$, $[\lambda]^n = [\lambda_1^n \ \ \lambda_2^n \ \ \cdots \ \ \lambda_N^n]^T$ and $\mathbf{A} = [a_{ij}, 1 \leq i,j \leq N]$ is given as follows:

$$\mathbf{A} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1(N-2)} & x_1 & 1 & \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{(N-2)1} & \cdots & \varphi_{(N-2)(N-2)} & x_{N-2} & 1 & \\ x_1 & \cdots & x_{N-2} & 0 & 0 & \\ 1 & \cdots & 1 & 0 & 0 & \end{pmatrix}. \tag{16}$$

Assuming that there are $p < N-2$ internal points and $N-2-p$ boundary points, then the $N \times N$ matrix $\mathbf{A}$ can be split into:
$\mathbf{A} = \mathbf{A}_d + \mathbf{A}_b + \mathbf{A}_e$, where

$$\begin{aligned}
\mathbf{A}_d &= [a_{ij}\, for\ (1 \leq i \leq p, 1 \leq j \leq N) \text{ and } 0 \text{ elsewhere}] \\
\mathbf{A}_b &= [a_{ij}\, for\ (p+1 \leq i \leq N-2,\ 1 \leq j \leq N) \text{ and } 0 \text{ elsewhere}] \\
\mathbf{A}_e &= [a_{ij}\, for\ (N-1 \leq i \leq N,\ 1 \leq j \leq N) \text{ and } 0 \text{ elsewhere}]
\end{aligned} \tag{17}$$

Using the notation $\mathcal{L}\mathbf{A}$ to designate the matrix of the same dimension as $\mathbf{A}$ and containing the elements $\hat{a}_{ij}$ where $\hat{a}_{ij} = \mathcal{L}a_{ij}$, $1 \leq i, j \leq N$, then Eq. (12) together with the boundary conditions (10) can be written, in matrix form as

$$B[\lambda]^{n+1} = C[\lambda]^n + (a_1 \frac{dt}{2} - 1)u^{n-1} + (dt)^2 \cdot [f]^{n+1} + [H]^{n+1}, \tag{18}$$

where

$C = 2\mathbf{A}_d + a_2(1-\theta)(dt)^2\Delta\mathbf{A}_d$,
$B = (1 + a_1\frac{dt}{2})\mathbf{A}_d - a_2\theta(dt)^2\Delta\mathbf{A}_d + \mathbf{A}_b + \mathbf{A}_e$,
$[H]^{n+1} = [0 \ \cdots \ 0 \ h_{p+1}^{n+1} \ \cdots \ h_{N-2}^{n+1} \ 0 \ 0]^T$,
and $[f]^{n+1} = [f_1^{n+1} \ \cdots \ f_p^{n+1} \ 0 \ \cdots \ 0]^T$. Eq. (18) is obtained by combining (12), which applies to the domain points, while (10) applies to the boundary points.

At $n = 0$ the Eq. (18) has the following form:

$$B[\lambda]^1 = C[\lambda]^0 + (a_1\frac{dt}{2} - 1)u^{-1} + (dt)^2 \cdot [f]^1 + [H]^1. \tag{19}$$

To approximate u^{-1}, the second initial condition can be used. We discretise the second initial condition as

$$\frac{u^1(x) - u^{-1}(x)}{2 \cdot dt} = g_2(x), \qquad x \in \Omega. \tag{20}$$

Writing (19) together with (20) we have

$$(B+(1-a_1\frac{dt}{2})\mathbf{A}_d)[\lambda]^1 = C[\lambda]^0+(2-a_1dt)dt[G_2]+(dt)^2\cdot[f]^1+[H]^1, \tag{21}$$

where $[G_2] = [g_2^1,\cdots,g_2^p,0\cdots,0]^T$. Together with the initial condition (9) and (18), we can get all the λ, thus we can get the numerical solutions.

Example 1

In this example, $a_1 = 2$, $a_2 = 1$, $\alpha = 1$, $\beta = 0$ and $\gamma = -1$ Eq. (1) takes the following form:

$$u_{tt} + 2u_t = u_{xx} + u^3 - u, \quad 0 < x < 1, \ \ t > 0,$$

the analytical solution by [6] is

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{x}{8} + \frac{3t}{8} + 5),$$

We extract the boundary function and the initial conditions from the exact solution.

We use the radial basis functions (TPS, IMQ) for the discussed scheme. These results are obtained for $dx = 0.01$ and $dt = 0.001$. The L_∞, L_2 errors and Root-Mean-Square (RMS) of errors are obtained in Table 1 for $t = 0.5,\ 1,\ 1.5,\ 2,\ 3$. The results obtained in the table show the very good accuracy.

Table: Numerical errors using TPS ($m = 2$) and IMQ ($c = 0.01$) represented at different times, where $dx = 0.01, dt = 0.001, \theta = 1/2$.

t	L_∞ − error		L_2 − error		RMS	
	TPS	IMQ	TPS	IMQ	TPS	IMQ
0.5	1.014×10^{-5}	3.599×10^{-6}	7.434×10^{-6}	3.077×10^{-6}	7.325×10^{-6}	3.032×10^{-6}
1	1.668×10^{-5}	6.375×10^{-7}	1.209×10^{-5}	5.437×10^{-7}	1.192×10^{-5}	5.357×10^{-7}
1.5	1.087×10^{-5}	1.156×10^{-6}	7.769×10^{-6}	9.843×10^{-7}	7.655×10^{-6}	9.698×10^{-7}
2	3.633×10^{-6}	6.750×10^{-7}	2.606×10^{-6}	5.754×10^{-7}	2.568×10^{-6}	5.670×10^{-7}
3	2.159×10^{-6}	4.154×10^{-7}	1.580×10^{-6}	3.541×10^{-7}	1.557×10^{-6}	3.489×10^{-7}

We also give the analysis of the parameter c in IMQ for the results. In Table 2 The L_∞, L_2 errors and Root-Mean-Square (RMS) of errors with different c at time $t = 3$ are presented.

Table: Numerical errors using IMQ represented with different parameter c at time $t = 3$, where $dx = 0.01, dt = 0.001, \theta = 1/2$.

c	L_∞	L_2	RMS
0.01	4.154×10^{-7}	3.541×10^{-7}	3.489×10^{-7}
0.03	2.178×10^{-6}	1.591×10^{-6}	1.568×10^{-6}
0.05	2.163×10^{-6}	1.582×10^{-6}	1.559×10^{-6}
0.07	2.160×10^{-6}	1.581×10^{-6}	1.557×10^{-6}
0.09	2.159×10^{-6}	1.580×10^{-6}	1.557×10^{-6}
0.093	0.001	2.485×10^{-4}	2.449×10^{-4}
0.095	8.991	1.959	1.930

Example 2

In this example, we consider the nonlinear telegraph equation takes the following form:

$$u_{tt} + u_t = 2u_{xx} + u^3 - 2u, \quad 0 < x < 1, \;\; t > 0,$$

the analytical solution by [6] is

$$u(x,t) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \coth(x + \frac{3t}{2} + 5),$$

We extract the boundary function and the initial conditions from the exact solution.

We use the radial basis functions (TPS, IMQ) for the discussed scheme. These results are obtained for $dx = 0.01$ and $dt = 0.001$. The L_∞, L_2 errors and Root-Mean-Square (RMS) of errors are obtained in Table 3 for $t = 1,\ 2,\ 3,\ 4,\ 5$. The results obtained in the table show the very good accuracy.

Table: Numerical errors using TPS ($m = 2$) and IMQ ($c = 0.03$) represented at different times, where $dx = 0.01, dt = 0.001, \theta = 1/2$.

t	L_∞ − error		L_2 − error		RMS	
	TPS	IMQ	TPS	IMQ	TPS	IMQ
1	1.728×10^{-6}	2.199×10^{-6}	1.032×10^{-6}	1.340×10^{-6}	1.017×10^{-6}	1.320×10^{-6}
2	4.132×10^{-6}	2.302×10^{-6}	2.877×10^{-6}	1.610×10^{-6}	2.835×10^{-6}	1.587×10^{-6}
3	5.728×10^{-7}	2.399×10^{-6}	3.783×10^{-7}	1.662×10^{-6}	3.728×10^{-7}	1.637×10^{-6}
4	1.387×10^{-6}	1.133×10^{-6}	9.397×10^{-7}	8.134×10^{-7}	9.260×10^{-7}	8.014×10^{-7}
5	6.159×10^{-7}	1.563×10^{-7}	4.083×10^{-7}	9.434×10^{-8}	4.024×10^{-7}	9.296×10^{-8}

Similar to the previous example, we also give the analysis of the parameter c in IMQ for the results. In Table 4 The L_∞, L_2 errors and Root-Mean-Square (RMS) of errors with different c at time $t = 5$ are presented.

Table: Numerical errors using IMQ represented with different parameter c at time $t = 5$, where $dx = 0.01, dt = 0.001, \theta = 1/2$.

c	L_∞	L_2	RMS
0.01	5.848×10^{-7}	4.512×10^{-7}	4.446×10^{-7}
0.03	1.563×10^{-7}	9.134×10^{-8}	9.296×10^{-8}
0.05	1.968×10^{-7}	1.191×10^{-7}	1.173×10^{-7}
0.07	2.046×10^{-7}	1.242×10^{-7}	1.224×10^{-7}
0.09	2.061×10^{-7}	1.252×10^{-7}	1.234×10^{-7}
0.095	0.394	0.090	0.089

We proposed a numerical scheme to solve the nonlinear telegraph equation using the Kansa's method using Thin Plate Spline (TPS), Inverse Multiquadric (IMQ) radial basis function. The numerical results given in the previous section demonstrate the good accuracy of this scheme and also see that the choice of parameter c is very important for the solution accuracy. The method proposed can be extended to solves more important nonlinear partial differential equations.

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Thank You!