Abstract—This paper addresses the problems of discrete-time state and unknown input/fault estimation for continuous-time nonlinear systems with multiple unknown inputs. Taylor series expansion and a nonlinear transformation are used to convert the nonlinear continuous-time system into a discrete-time model. The conditions for the observability of unknown inputs w.r.t. outputs are discussed. The novelty lies in the formulation of multiple sliding-mode estimator for the states that are directly influenced by unknown inputs, which cannot be decoupled by nonlinear transformation. This framework allows for the estimation of unknown inputs from the multiple sliding modes. The existence of discrete-time sliding mode is guaranteed, and the relation between the boundary layer thickness and the sliding-mode gain design that will eliminate chattering and the boundedness conditions is obtained. The proposed technique can be applied for fault detection and isolation. Simulation results with application to three-phase motor are given to demonstrate the effectiveness of the proposed method.

Index Terms—Discrete-time systems, nonlinear systems, sliding-mode observers (SMOs), state estimation.

I. INTRODUCTION

Sliding mode control is a well established method for handling disturbances and modeling uncertainties through the concepts of sliding surface design and equivalent control [1], [2]. Based on the same concept, sliding-mode observers (SMOs) have been developed to robustly estimate the system states [2]–[11].

The Lyapunov-based approach of Walcott and Zak [3], [12] considered the problems of state observation in the presence of bounded uncertainties/unknown inputs based on a matching condition. The approach in [6] extended the design of [2] and designed the SMO such that the states affected by the unknown inputs are dealt with by the switching terms, and the method required the reduced-order system itself to be stable. The work in [9] extended the SMO design of [12] to Lipschitz nonlinear systems based on matching conditions [13], [2]. The equivalent control (average control [13])-based SMO was first proposed by Utkin [2], and its extensions to nonlinear systems were addressed in [7] and [14]. These methods chose the output estimation error and its higher order derivatives as the sliding surfaces. In practical applications, the higher order derivatives of estimation error have to be obtained through low-pass filtering [2], which, in general, introduces delay and may cause instability [15]. Moreover, there has been tremendous interest in the application of these techniques in real time for industrial applications in both continuous and discrete time [16]–[20].

However, discrete-time sliding-mode-based observer (DSMO) design has not received much attention, particularly for nonlinear systems due to the inherent problem associated with chattering. Discrete-time SMO design for linear systems was given in [21]. In [22], the concept of sliding lattice for discrete-time systems was introduced, and DSMO was designed for single-input–single-output (SISO) linear systems using the Lyapunov min–max method. Recently, DSMO for a class of SISO nonlinear uncertain systems was designed in [10] based on a new strategy for the existence of discrete-time sliding mode (DSM). The work in [10] also discussed the estimation of unknown input/disturbance from the sliding mode. In this paper, the strategy developed in [10] is employed for the existence of DSM.

In this paper, the observer design for a class of multiple-input–multiple-output nonlinear systems whose uncertainties can be modeled as multiple unknown inputs [7], [23] is examined. Through Taylor series expansion and the proposed new transformation, a discrete-time plant dynamic model is obtained. Under a structural assumption for the unknown inputs, the discretized observer with multiple sliding modes is combined, which are robust terms introduced to handle the unknown inputs. With the framework proposed in this paper, the estimation of unknown inputs is feasible from the multiple sliding modes to a certain level of accuracy. Compared with the existing methods in continuous-time domain [7], the design is performed under less conservative conditions. In this paper, it is shown that a proper design of the feedback estimation gain ensures the ultimate boundedness of the estimation error. The ultimate bound becomes smaller for a higher sampling rate. The estimation performance is significantly improved with the incorporation of sliding-mode terms. Through an inverse state transformation, the estimator for the original system is readily obtained.

II. PRELIMINARIES

Consider the class of continuous-time nonlinear systems that can be represented as

\[
\begin{align*}
\dot{x} &= f(x) + G(x, u) + \sum_{i=1}^{m} p_i(x) d_i(x, u, t) \\
y_j &= h_j(x), \quad \text{for } j = 1, \ldots, s
\end{align*}
\]  

(1)
where $x \in \mathcal{M}$ is a $C^\infty$ connected manifold of dimension $n$, $f(x)$ and $p_i(x)$, with $i = 1, \ldots, m$, are the smooth vector fields on $\mathcal{M}$, $h_j(x)$, with $j = 1, \ldots, s$, denotes the smooth functions from $\mathcal{M}$ to $\mathcal{R}$, and $G(x, u)$ is a vector field with smooth functions. $d_i(x, u, t)$ denotes the bounded uncertainties and/or disturbances in the system, i.e., $|d_i(x, u, t)| \leq \tilde{d}_i$ for some upper bound $\tilde{d}_i$. The unknown inputs can also be treated as faults in the input channel.

### A. Discrete-Time Plant Dynamics

By denoting the discrete-time index $(k)$ as the variable at time $t = kT_s$, where $T_s$ is the sampling period, the Taylor series expansion of $x$ about the variable “$k$” can be obtained as

$$x(k + 1) = x(k) + x(t)|_{t=kT_s} + O_1(T_s^2)$$

(2)

where $O_1(T_s^2) = (1/2!)x^{(2)}(kT_s)T_s^2 + \cdots + (1/!v!)x^{(v)}(kT_s)T_s^v + (1/((v + 1)!))x^{(v+1)}(\xi)T_s^{v+1}$, $\xi \in (kT_s, kT_s + T_s)$, and $v \geq 2$ are the higher order terms of the aforementioned expansion. A similar discretization for robust control of nonlinear plants is shown in [24] and [10]. This paper only considers the first order term and treats the higher order terms as disturbance.

### B. Nonlinear Transformation

In order to design the nonlinear unknown input observer using the traditional nonlinear transformation [25], [7] to decouple the unknown inputs, the system distribution vectors $p_1(x), \ldots, p_m(x)$ must satisfy the involutive property [7], [25]. The outputs should also have vector relative degree corresponding to $G(x, u)$ at each point $x_o \in \mathcal{M}$. These assumptions, in general, are conservative.

Instead of decoupling the unknown inputs, a new transformation is defined to deal with the disturbances directly.

**Assumption 1:** From the $s$ outputs, there are at least $q$ outputs with relative degrees $r_j = 1$ w.r.t. the unknown inputs $j = 1, \ldots, q$.

The proposed transformation is formed in two steps.

1) Choose $q$ outputs with relative degrees $r_j = 1$ to form the first part of the transformation. For simplicity of exposition, suppose that such outputs are given by the first $q$ outputs $y_1, \ldots, y_q$. For each of these outputs $y_j$, define $k_j$ and then form the following transformations:

$$\phi_j = [h_j(x) \ L_kh_j(x) \ \cdots \ L_k^{k_j-1}h_j(x)]^T$$

$$\phi_w = [\phi_1^T \ \phi_2^T \ \cdots \ \phi_q^T]^T$$

for $j = 1, \ldots, q$, and the Lie derivative is defined as $L_kh_j(x) = [(\partial h_j(x))/\partial x]f$.

2) The remaining outputs with relative degrees $r_j \geq 1$ are used to form the second part of the transformation via

$$\phi_0 = [h_{q+1}(x) \ \cdots \ h_s(x)]^T.$$

The selection of outputs $q$ and the values of $k_j$ in step 1) are to be done such that the desired transformation holds. That is, the outputs with relative degree one are divided between steps 1) and 2) so as to form $n$ functions to complete the transformation.

**Remark 1:** The first step of selecting $q$ outputs with relative degree one allows dealing with the disturbance inputs directly in the design of the observer. The proposed methodology exploits the structural properties when designing the multiple SMO to tackle the disturbance inputs. If the transformation can be formed with $q$ outputs, step 2) of the proposed transformation is not required. Step 2) provides the flexibility for the formation of $n$ functions to complete the transformation.

**Remark 2:** For outputs with relative degree $r_i > 2$ (if exists), the disturbance inputs are decoupled and are absent in the transformed domain. Thus, the observer can be designed easily, as the states are measurable directly as outputs. The complete transformation is given by

$$\dot{\tilde{x}} = \Phi(x) = [\phi_1^T \ \cdots \ \phi_q^T \ \phi_0^T]^T.$$  

(3)

Similar to (2), it can be evaluated from (1) that $\dot{\tilde{x}}(t)|_{t=kT_s} = [(\partial \Phi(x))/\partial x]|_{t=kT_s, x(t)|_{t=kT_s} = [(\partial \Phi(x))/\partial x]|_{t=kT_s, f(x(k)) + G(x(k), u(k)) + \sum_{i=1}^m p_i(x)d_i(x(k), u(k))}]$. With the aforementioned formulation and Taylor series expansion, the transformed system that can be divided into the two classes with the transformed coordinates is represented by

$$\dot{\tilde{x}}_j = \begin{bmatrix} x_1^j(k) & x_2^j(k) & \cdots & x_{k_j}^j(k) \end{bmatrix}^T = \phi_j$$

$$\tilde{x}_0 = \begin{bmatrix} x_1^0(k) & x_2^0(k) & \cdots & x_0^0(k) \end{bmatrix}^T = \phi_0$$

for $j = 1, \ldots, q$, and $e = s - q$.

### C. Transformed System

For the subsystems under the transformations $\phi_1, \ldots, \phi_q$, the following structure can be obtained:

$$\tilde{x}_j(k + 1) = \begin{bmatrix} x_1^j(k) \\ \vdots \\ x_{k_j-1}^j(k) \\ x_{k_j}^j(k) \end{bmatrix} + T_s \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \sum_{i=1}^m z_i^j(k_1) d_i(k)$$

$$+ T_s \begin{bmatrix} \mu_1^j(\tilde{x}(k), u(k)) \\ \vdots \\ \mu_{k_j-1}^j(\tilde{x}(k), u(k)) \\ \mu_k^j(\tilde{x}(k), u(k)) \end{bmatrix}$$

$$+ \begin{bmatrix} \alpha_1^j(T_s^2) \\ \alpha_2^j(T_s^2) \end{bmatrix}$$

$$y_j(k) = x_1^j(k)$$

(4)
where \( \mu_j^1(\tilde{x}, u) = (\partial L^T h_j)/\partial x) G(\Phi^{-1}(\tilde{x}), u) \), for \( j = 1, \ldots, q \) and \( v = 1, \ldots, k_j - 1 \), and \( \mu_j^2(\tilde{x}, u) = L^T h_j(\Phi^{-1}(\tilde{x}))) + ((\partial L^T h_j)/\partial x) G(\Phi^{-1}(\tilde{x}), u) \) for \( j = 1, \ldots, q \) and \( v = k_j \).

Moreover, \( z_j^T(\tilde{x}) = L_{p_j} L^T h_j(\Phi^{-1}(\tilde{x})) \) for \( j = 1, \ldots, q \), \( v = 1, \ldots, k_j \), and \( i = 1, \ldots, m \). \( \alpha^2_i(T^s) \), where \( i = 1, \ldots, k_j \), denotes the higher order terms from the Taylor series expansion. In the above content, the input argument of the time instant “(k)” has been neglected for simplicity. In the rest of this paper, the input argument will be time instant “(k)” if unspecified.

For the analysis of unknown inputs, the system can be rewritten in its more general form as

\[
\dot{x}_j(k + 1) = [I + A_j T_s] \bar{x}_j(k) + T_s \mu_j(\bar{x}(k), u(k))
\]

\[
+ T_s \sum_{i=1}^m \bar{Z}_i^j(\bar{x}(k)) d_i(k) + O_j(T^s)
\]

\[
y_j(k) = x_j^1(k) = C_j \bar{x}_j(k) \tag{5}
\]

where \( A_j = \begin{bmatrix} 0 & I_{(k_j-1)\times(k_j)} \\ 0_{1 \times (k_j)} \end{bmatrix} \) is an antishift constant matrix, and

\[
C_j = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}
\]

\[
\mu_j(x, u) = \begin{bmatrix} \mu_j^1 & \ldots & \mu_{j-1}^j & \mu_{k_j}^j \end{bmatrix}^T, \tag{6}
\]

\[
\bar{Z}_i^j(\bar{x}) = \begin{bmatrix} z_{1i}^j & \ldots & z_{(k_j-1)i}^j \end{bmatrix}^T, \tag{7}
\]

\[
O_j(T^s) = \begin{bmatrix} \alpha_{1i}(T^s) & \ldots & \alpha_{k_j-1}(T^s) & \alpha_{k_j}(T^s) \end{bmatrix}^T, \tag{8}
\]

for \( j = 1, \ldots, q \).

For ease of analysis, all the subsystems \( \bar{x}_1(k) \cdots \bar{x}_q(k) \) can be grouped together to form a single system as

\[
\bar{x}_w(k + 1) = A_w \bar{x}_w(k) + T_s \mu_w(\bar{x}(k), u(k))
\]

\[
+ T_s \sum_{i=1}^m \bar{Z}_i(\bar{x}(k)) d_i(k) + O_w(T^s) \tag{9}
\]

\[
y_w(k) = C_w \bar{x}_w(k) \tag{10}
\]

where \( \bar{x}_w(k) = [\bar{x}_1^T(k) \bar{x}_2^T(k) \cdots \bar{x}_q^T(k)]^T, A_w = \text{diag}[C_1, C_2, \ldots, C_q], A_w = \text{diag}[(I + A_1 T_s), (I + A_2 T_s), \ldots, (I + A_q T_s)], \mu_w(\bar{x}(k), u(k)) = [\mu_1^1 \cdots \mu_{k_j-1}^j \cdots \mu_{k_j}^j]^T, \) and \( \bar{Z}_i(\bar{x}(k)) = [\bar{Z}_i^1(\bar{x}(k)) \cdots \bar{Z}_i^{k_j-1}(\bar{x}(k)) \bar{Z}_i^k(\bar{x}(k))]^T. \)

Remark 3: For the subsystem under the transformation \( \phi_0 \), the transformed system has the structure (4) with \( k_j = 1 \).

To establish the result, the following additional assumptions are required.

Assumption 2: The mapping \( \Phi(x) \) is a diffeomorphism.

Assumption 3: The known functions \( f(x), G(x, u), \) and \( p_j(x) \) are bounded with respect to their arguments. The inputs of the nonlinear system (1) are bounded with some upper bounds. Furthermore, system (1) is assumed to be bounded-input–bounded-states stable.

**Assumption 4:** The nonlinear functions in the transferred system (5) have the following structure:

\[
\mu_j(\bar{x}(k), u(k)) = \begin{bmatrix} \mu_j^1(\bar{x}_1, \bar{x}_2, \bar{x}_0, u) \\ \vdots \\ \mu_j^q(\bar{x}_1, \bar{x}_2, \bar{x}_0, u) \\ \mu_j^{k_j+1}(\bar{x}_1, \bar{x}_2, \bar{x}_0, u) \end{bmatrix}
\]

\[
\bar{Z}_i^j(\bar{x}) = \begin{bmatrix} z_{11}^j(x_1, \bar{x}_2, \bar{x}_0) \\ \vdots \\ z_{k_j}^j(x_1, \bar{x}_2, \bar{x}_0) \end{bmatrix}
\]

for all \( j = 1, \ldots, q \), where \( \bar{x}_j(k) = (x_1(k), x_2(k), \ldots, x_{k_j-1}(k)) \), \( \bar{x}_j(k) = (\bar{x}_1(k), \bar{x}_2(k), \ldots, \bar{x}_{k_j-1}(k)) \), and \( \bar{x}_j(k) \) and \( \bar{x}_0(k) \) are regarded as inputs to the subsystem under consideration.

Then, the \( \bar{x}_1(k) \) subsystem is uniformly observable according to [26] for all inputs \( \bar{x}_1(k), \bar{x}_2(k), u(k), \) and \( d(k) \).

**Assumption 5:** The distribution vector \( \bar{Z}_i(\bar{x}) \) and the function \( \mu_w(\bar{x}, u) \) are Lipschitz functions of the argument \( \bar{x} \) for all \( i = 1, \ldots, m \).

**D. Observability of Unknown Inputs**

Essentially, the SMO is analyzed from the perspective of estimating the unknown inputs/disturbances and ensuring that accurate state estimation is still achievable in the presence of unknown inputs/disturbances. The analysis is based on the detectability (observability) of disturbances or unknown inputs from the output measurements.

Arising from the nonlinear transformation, the first state of every subsystem is measurable as output. All the first dynamics of subsystems are grouped together for ease of analysis in the following assumption.

**Assumption 6:** The number of disturbance inputs \( m \) in the transformed domain is less than or equal to the number of outputs \( q \), i.e., \( \nu(q, m) \). The dynamics of states that are measured as outputs of “m” subsystems have the following structure:

\[
\begin{bmatrix}
   y_1(k+1) \\
   y_2(k+1) \\
   \vdots \\
   y_m(k+1) \\
   \vdots \\
   y_{m+n}(k+1)
\end{bmatrix} = \begin{bmatrix}
   x_1^1(k+1) \\
   x_2^1(k+1) \\
   \vdots \\
   x_m^1(k+1) \\
   \vdots \\
   x_n^1(k+1)
\end{bmatrix} + \begin{bmatrix}
   1 & 0 & \ldots & 0 \\
   0 & 1 & \ldots & 0 \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & \ldots & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
   \mu_1(\bar{x}(k), u(k)) \\
   \vdots \\
   \mu_n(\bar{x}(k), u(k))
\end{bmatrix}
\]

\[
\begin{bmatrix}
   \mu_1(\bar{x}(k), u(k)) \\
   \vdots \\
   \mu_n(\bar{x}(k), u(k))
\end{bmatrix}
\]

\[
\begin{bmatrix}
   1 & 0 & \ldots & 0 \\
   0 & 1 & \ldots & 0 \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & \ldots & 1
\end{bmatrix}
\]

\[
+ [o_1^T(T^s) o_2^T(T^s) \cdots o_n^T(T^s)]^T \tag{11}
\]

---

1. The notion of observability of unknown inputs w.r.t. output measurements defined in this paper is different from the uniform observability [26] discussed earlier.
where \( z_i^T(\hat{x}) \), with \( i = 1, \ldots, m \), denotes the functions of \( \hat{Z}_i^T(k) \) corresponding to the dynamics of \( \dot{x}_i(k) \). Note that the diagonal entries \( z_i^T(\hat{x})(k) \) in the distribution matrix of unknown inputs in (11) can be made unity by normalizing the original elements in each column of the original disturbance distribution matrix.

III. MULTIPLE SMO DESIGN

For the subsystem (5) satisfying Assumptions 4–6, a discrete-time nonlinear state estimator with the robust terms can be designed as follows:

\[
\dot{\hat{x}}_j(k+1) = [I + A_j T_s] \dot{\hat{x}}_j(k) + T_s \mu_j \left( \dot{\hat{x}}(k), u(k) \right) + \sum_{i=1}^{m} \hat{Z}_i \left( \dot{\hat{x}}_i(k) \right) u_i^r(k) + L_j \left( y_j(k) - C_j \dot{\hat{x}}_j(k) \right) \tag{12}
\]

where

\[
L_j = \begin{bmatrix} l_1^T \, l_2^T \, \cdots \, l_{k-1}^T \, l_k^T \end{bmatrix}^T \tag{13}
\]

with \( l_1^T, l_2^T, \ldots, l_{k-1}^T, l_k^T > 0 \) being the properly chosen estimation gains such that the eigenvalues of \( I + A_j T_s - L_j C_j \) are within the unit circle. Here, \( u_i^r(k) \) is a scalar-valued robust term given by sliding-mode estimation

\[
u_i^r(k) = -\rho_i \text{sat}(e_i^r(k), \epsilon_i) \tag{14}
\]

for all \( i = 1, \ldots, m \), where \( \rho_i \) and \( \epsilon_i \) are the sliding-mode gain and the boundary layer, respectively, to be addressed in Section III-C1. A similar design for robust term was adopted in [10] and [11] for DSM observer design for SISO nonlinear systems. Here, the \( u_i^r(k) \) terms are introduced for added robustness of the system and to handle the uncertainties.

Remark 4: In the observer design, the disturbance inputs of the system are exactly replaced by the robust terms. The number of robust terms \( u_i^r(t) \) is equal to the number of unknown inputs \( \dot{d}_i(t) \), and they serve as “tracking elements” for the unknown inputs. Clearly, multiple robust terms (multiple sliding modes) are required to handle the multiple disturbances or unknown inputs present in the system.

A. Error Dynamics and Boundedness

The error dynamics for the combined subsystem will be of the form

\[
e_w(k+1) = \dot{x}_w(k+1) - \dot{x}_w(k+1) = [A_w - L_w C_w] e_w(k) + T_s \mu_w \left( \dot{\hat{x}}(k), u(k) \right) - T_s \mu_w \left( \dot{\hat{x}}(k), u(k) \right) - C_w T_s + \sum_{i=1}^{m} \hat{Z}_i \left( \dot{\hat{x}}_i(k) \right) u_i^r(k) + T_s \sum_{i=1}^{m} \hat{Z}_i \left( \dot{\hat{x}}_i(k) \right) \tag{15}
\]

where

\[
L_w = \text{diag}[L_1 \, L_2 \, \cdots \, L_q] \tag{16}
\]

denotes the properly chosen estimation gains such that the eigenvalues of \( M_w \geq [A_w - L_w C_w] \) are within the unit circle, and there exists a positive symmetric constant matrix \( Q_w = Q_w^T > 0 \) such that

\[
M_w^T Q_w M_w - Q_w = -I. \tag{17}
\]

The following Lemma 1 ensures the boundedness of the error dynamics by the Lyapunov function analysis.

Lemma 1: For the subsystem (5) satisfying Assumptions 4–6 and the estimator (12), \( \dot{x}(k) \) and \( e_j(k) \) are all bounded provided that

\[
\lambda_{\max}(Q_w) \leq \frac{1}{2} [2\|M_w\| + l_{mu} + (l_{mu})^2]^{-1} \tag{18}
\]

for some bounded function \( l_{mu} = T_s \mu + \sum_{i=1}^{m} l_i \rho_i \) with Lipschitz constants \( l_i \) and \( l_i^T \) and some upper bound \( \rho_i \).

Proof: See Appendix I.

Due to discretization and limited sampling period \( T_s \), the error will finally settle to a bound dictated by \( b_{0S}, T_s \), and disturbance bound \( d_t \). If the sampling period \( T_s \) is small, the error bound can be largely reduced.

B. Boundary Layer and Chattering-Free DSM [10]

For the convergent switching to occur and the DSM to exist, the following two conditions are to be satisfied [27]–[29]:

\[
[s(k+1) - s(k)] \text{sign}(s(k)) < 0 \tag{19}
\]

\[
[s(k+1) + s(k)] \text{sign}(s(k)) > 0. \tag{20}
\]

Here, \( s(k) \) is the sliding trajectory, and \( s(k) = 0 \) is the sliding manifold.

In this paper, for the existence of DSM and the design of boundary layer, the strategy developed in [10] is employed. The boundary layer forms an integral part of the observer design in the proposed design methodology. Furthermore, an analytical solution for the design of sliding-mode gain and the boundary layer thickness that completely eliminates the chattering and guarantees the existence of DSM can be designed. The DSM is achieved through the following saturation function:

\[
\text{sat}(\cdot, \epsilon) = \begin{cases} \cdot / \epsilon, & \text{if } |\cdot| \leq \epsilon \\ \text{sign}(\cdot), & \text{if } |\cdot| > \epsilon. \end{cases} \tag{21}
\]

For further details, please see [10]. In a standard sliding-mode phenomenon, a boundary layer is generally used to avoid excessive chattering over the sliding manifold in the continuous-time domain. The existing analysis of the DSM is performed outside the boundary layer, and no analytical methods are available for the design of boundary layer. The confinement of the trajectory to the boundary layer, after the trajectory enters the boundary layer, prevents chattering. This is crucial for the existence of DSM.
C. Existence of DSM

Although the estimation error is bounded, the main purpose of \(u_i^*(k)\) terms in (12) is to improve estimation accuracy. A two-step approach is adopted to improve accuracy by sliding-mode estimation.

1) Define the following sliding surfaces:
\[
e^1_s(k) = 0 \quad \forall i = 1, \ldots, m
\]  
(22)
for all the subsystems \(\hat{x}_i(k)\) \(\cdots \hat{x}_q(k)\).

2) Design the sliding-mode estimation as (14) to reach and maintain the sliding mode.

3) Ensure that the estimation error \(e_i(k)\) goes to zero in the sliding mode of \(e^1_s(k) = 0, e^1_s = 0, \ldots, e^1_m(k) = 0\).

The following Lemma 2 and Theorem 1 are devoted to these steps.

1) Sliding-Mode Gain Design: The sliding-mode gain is designed sequentially for all the subsystems. The dynamics of the measurement errors will be obtained from (11) according to (15) and Assumption 6. The error dynamics reach their corresponding sliding modes one by one sequentially. The reaching phase of the sliding modes is dependent on the structural dynamics of subsystems. As the states of \(\hat{x}_0(k)\) of subsystems are all measured as outputs, \(\hat{x}_i(k) \equiv \hat{x}_0(k)\).

The following lemma establishes the DSM and proves the convergence of \(e^1_s(k)\) to a bound inside the boundary layer \(\epsilon_j\).

Lemma 2: Consider the subsystem described by (5); the sliding-mode estimator of the form (12) ensures that the sliding surface \(e^1_s(k)\) satisfies (19) and (20) outside the boundary layer, i.e., \(|e^1_s(k)| > \epsilon_j\), and the trajectory \(e^1_s(k)\) is finally confined within the boundary layer once it enters the boundary layer, i.e., if \(|e^1_s(k) + i| < \epsilon_j\), \(\forall i \in N\), provided that the gain in (14) satisfies
\[
\rho_j^- < \rho_j < 2 \left(1 - \frac{1}{T^i_1}\right) \epsilon_j - \rho_j^-
\]  
(23)
and \(T_s\lim_{i \to 0} < T^i_1 < 1\), for Lipschitz constant \(\mu_{q_i}\) of \(\mu_j\), with \(\rho_j^- = \sup |\delta_j(k)|\); \(\delta_j(k) = T_s e^2_s(k) + \sum_{i=1}^{q} |z_{i}^{1}_1(\hat{x}(k))u^i_l - T_s z_{i}^{1}_1(\hat{x}(k))d_i(k)| - T_s d_j(k)) - \alpha^2_s(T_s)\) for all \(i = 1, \ldots, q\) subsystems.

Proof: Similar to the proof of Lemma 2 in [10] with error dynamics of \(e^1_s(k) + 1\) from (15).

Remark 5: Due to limitation in sampling rate \(T_s^{-1}\), the error is finally confined to a bound inside the boundary layer \(\epsilon_j\), which is a function of sampling period \(T_s\), disturbance \(d_j(k)\), and higher order dynamics \(O_j (T^2_s)\). Moreover, from (23), \(\rho_j^+\) is a function of \(T_s\); hence, \(\rho_j\) can be sufficiently small when sampling rate \(1/T_s\) is sufficiently high.

Considering the proposed DSM with quasi-sliding-mode band [8], the former is designed for linear systems, and the boundary layer thickness is dependent on the disturbance bounds. By incorporating the mean of the disturbance bound into the control law, the sliding surface is confined to the maximum spread of the disturbance. In our case, only the upper bound of the disturbances is known, and thus, the boundary layer thickness according to [8] can be evaluated as \(\epsilon_j > \sup |\delta_j(k)|\), where \(\delta_j(k)\) accounts for all the uncertainties and higher order terms. If the lower and upper bounds of disturbance are known, the boundary layer thickness can be reduced to half the spread of the disturbance. Moreover, the control law does not contain any forcing term. For the uncertainties that are subject to quick changes, the method proposed in [8] may not be accurate.

In our design, for the trajectory outside the boundary layer, “sign” term is still present to force the trajectory to reach the boundary layer. In order to design the sliding-mode gain, we require \(2(1 - l^i_1)\epsilon_j - \sup |\delta_j(k)| > 0\). Since \(\rho_j^- = \sup |\delta_j(k)|\), therefore \(\epsilon_j > (\sup |\delta_j(k)|)/(2(1 - l^i_1))\). Since \(l^i_1 < 1\), an appropriate selection of \(l^i_2\) and \(T_s\) ensures a small boundary layer.

2) Dynamics in the Sliding Mode: Since the conditions in Lemma 2 guarantee the sliding mode, the corresponding subsystems error dynamics \(e_i(k)\) also converge sequentially in the sliding modes of \(e^1_s = 0, e^1_s = 0, \ldots, e^1_m(k) = 0\). The equivalent control for the sliding dynamics is evaluated sequentially. In the following analysis, the subscript \(d\) is included for the variables in the sliding mode. Consider the error dynamics \(e^1_s(k)\) in the sliding mode. Once the trajectory reaches the sliding mode and stays in its vicinity, \(e^1_s(k + 1) = 0\) and \(e^1_s(k) = 0\), \(\dot{x}_1^1 = \dot{x}_1\), the equivalent control [2] of \(u_i^e(k)\) can be obtained as
\[
0 = T_s e^2_{1,d}(k) + u^e_{eq}(k) - T_s d_1(k) - \alpha^2_s(T^2_s) \\
+ \mu^1_1(\dot{x}_1^1(k), \dot{x}_0(k), u(k)) - \mu^1_1(x_1^1(k), \dot{x}(k), u(k))
\]
where \(u^e_{eq}(k)\) represents the equivalent control dynamics of \(u^e_{1}(k)\). According to structural Assumption 6, in the sliding mode of \(e^1_s(k) = 0\), \(\mu^1_1(x_1^1(k), \dot{x}_0(k), u(k)) \equiv \mu^1_1(x_1^1(k), \dot{x}_0(k), u(k))\). Therefore,
\[
u^e_{eq}(k) = T_s d_1(k) - T_s e^2_{1,d}(k) + \alpha^2_s(T^2_s).
\]
Similarly, for the trajectory \(e^1_s(k)\), in the sliding mode \(e^1_s(k) = 0\) and \(e^1_s(k) = 0\),
\[
\nu^e_{eq}(k) = T_s d_2(k) - T_s e^2_{2,d}(k) - T_s z^2_{11}(\dot{x}(k))e^1_{2,d}(k) \\
+ T_s z^2_{11}(\dot{x}(k))\alpha^2_s(T^2_s) + \alpha^2_s(T^2_s)
\]
Therefore, in the sliding modes of \(e^1_s(k) = 0, e^1_s(k) = 0, \ldots, e^1_m(k) = 0\), the equivalent control of \(u^e_{eq}(k)\) can be evaluated. In short, the equivalent control of \(e^1_s(k) = 0\) can be represented as
\[
u^e_{eq}(k) = T_s d_i(k) + \alpha^2_s(T^2_s) \\
- T_s \chi_i e^1_{2,d}(k), \ldots, e^1_{2,d}(k), \dot{x}w(k)
\]
(24)
for all \(i = 1, \ldots, m\), where \(\chi_i(\cdot, k)\) is a function of second-order dynamics of the subsystems and \(\alpha^2_s(T^2_s)\) accounts for the higher order terms that result from the successive evaluation.

3) Error Convergence in the Sliding Mode: Substituting the evaluated equivalent control elements (24), the error

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dynamics (15) in the multiple sliding mode of \( e_i^1(k) = 0 \), \( e_i^2(k) = 0, \ldots, e_i^m(k) = 0 \) will be of the form

\[
e_i^d(k+1) = M_w e_i^d(k) + T_s \mu_w (\hat{x}(k), u(k)) \]

\[- T_s \mu_w (\hat{x}(k), u(k)) \]

\[+ \sum_{i=1}^{m} \left( \tilde{Z}_i (\hat{x}(k)) - \hat{Z}_i (\tilde{x}(k)) \right) u_i^e(k) \]

\[+ \sum_{i=1}^{m} Z_i (\tilde{x}(k)) [\alpha_i^e (T_s^2) - \chi_i (\cdot, k)] - O_w (T_s^2) \]  

(25)

From (25), it can be clearly seen that the equivalent control completely cancels the unknown inputs in the error dynamics. Moreover, the equivalent controls evaluated here are only for the purpose of analysis. The following theorem further examines the condition for the asymptotical stability of the estimator error.

Theorem 1: For the subsystem (5) satisfying Assumptions 4–6, the estimator (12) with the sliding-mode gain (14) ensures that the estimation error is asymptotically stable in the sliding modes of \( e_i^1(k) = 0, \forall i = 1, \ldots, m \), provided that the gain \( L_w \) satisfies \( T_s \mu_1 < T_s^2 < 1 \), (17), (18), and

\[ \lambda_{\max} (Q_w) \leq (\Delta_1)^{-1} \]  

(26)

for a positive constant \( \Delta_1 \) [See (31) of Appendix II for details].

Proof: See Appendix II.

IV. UNKNOWN INPUT ESTIMATION FROM MULTIPLE SLIDING MODES

The equivalent control \( u_i^e(k) \) information can be used to reconstruct the unknown inputs to a certain level of accuracy. Once the trajectories attain the DSM and all the states converge to the true states, \( \tilde{x}_w^d(k) \rightarrow \tilde{x}_w^d(k) \). Therefore, \( e_i^1(d, k) \approx 0, e_i^2(d, k) \approx 0, \ldots, e_i^m(d, k) \approx 0 \) and \( \chi_i(e_i^1(d, k), \ldots, e_i^m(d, k), \tilde{x}_w(k)) \rightarrow 0 \) for all \( i = 1, \ldots, m \). Therefore, it can be approximated from (24) that

\[ u_i^e(k) \approx T_s d_m(k) + \alpha_i^e (T_s^2) \]  

(27)

Although ideal sliding mode cannot be attained in DSM, the unknown input can be approximated. By neglecting the higher order terms, it can be shown that

\[ \hat{d}_i(k) \approx \frac{u_i^e(k)}{T_s} \quad \forall i = 1, \ldots, m. \]  

(28)

Remark 6: The gain \( L_w \) should be designed such that it meets the two conditions (18) and (26) and the Lyapunov condition (17). Under similar conditions, the procedure for gain design using LMI approach was discussed in [10].

Remark 7: By means of inverse transformation, the observer in the original space can be obtained as

\[ \hat{x}(k+1) = \hat{x}(k) + T_s [f(\hat{x}(k)) + G(\hat{x}(k), u)] + \frac{\partial \Phi(x)}{\partial x}_{=\hat{x}(k)}^{-1} \]

\[ \times \left[ L_1 (y_1(k) - \hat{x}_1^2(k)) + \sum_{i=1}^{m} \hat{Z}_i \rho_i \text{sat}(y_i(k) - \hat{x}_1^2(k), \epsilon_i) \right] \]

\[ \times \left[ L_2 (y_2(k) - \hat{x}_2^2(k)) + \sum_{i=1}^{m} \hat{Z}_i \rho_i \text{sat}(y_i(k) - \hat{x}_2^2(k), \epsilon_i) \right] \]

V. ILLUSTRATIVE EXAMPLE

Consider the three-phase motor model [7, 30] described by the following nonlinear equations:

\[ \dot{x} = \begin{bmatrix} x_2 \\ \mu_1 x_1 - A_2 x_3 \sin x_1 - A_3 x_3 \sin(2x_1) \\ -D_1 x_3 + D_2 \cos x_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0.4 & 0.3 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \]  

(29)

where \( x = [x_1 \ x_2 \ x_3]^T \), with \( x_1, x_2, \) and \( x_3 \) representing the rotor angle, speed deviation, and field flux linkage, respectively, and \( u_1 \) and \( u_2 \) are known inputs and correspond to mechanical power input and field voltage, respectively. The unknown inputs \( d_i \) represent uncertainties or faults in the system.

The output functions are \( y_1 = h_1(x) = x_1 \) and \( y_2 = h_2(x) = x_3 \). Both the outputs have relative degree one. For the subsystem under \( h_1(x) \), select \( k_1 = 2 \), and therefore, \( k_2 = 1 \). The transformation for the change of coordinates is selected as

\[ x_1^1 = x_1 \quad x_2^1 = L_T h_1(x) = x_2 \quad x_3^1 = x_3. \]

The outputs in the transformed domain are \( y_1 = x_1^1 \) and \( y_2 = x_2^1 \). As the output equations are linear, the transformation is linear; the Jacobian and inverse Jacobian of transformations are identity matrices.

The discrete-time system in the observation space for system (29) under the new coordinates can be expressed as follows:

\[ \begin{bmatrix} x_1^1(k+1) \\ x_2^1(k+1) \end{bmatrix} = \begin{bmatrix} x_1^1(k) + T_s x_2^1(k) \\ x_3^1(k) \end{bmatrix} + T_s \mu_1^1 (\tilde{x}_1^1(k), u(k)) \\ + T_s \mu_2^1 (\tilde{x}_1^1(k), u(k)) \]

\[ + T_s \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} d_1(k) \\ d_2(k) \end{bmatrix} + O_1(T_s) \]

\[ x_2^1(k+1) = x_2^1(k) + T_s \mu_2^2 (\tilde{x}_1^2(k), u(k)) + 0.2T_s d_1(k) \]

\[ + T_s d_2(k) + C_2(T_s) \]

where \( \mu_1^1 (\tilde{x}_1^1(k), u(k)) = x_1^2(k); \mu_2^1 (\tilde{x}_1^1(k), u(k)) = -A_1 x_1^3(k) - A_2 x_1^3(k) \sin(x_1^1(k)) - A_3 \sin(2x_1^1(k)) + u_1(k); \) and \( \mu_2^2 (\tilde{x}_1^2(k), u(k)) = -D_1 x_2^1(k) + D_2 \cos(x_1^3(k)) + u_2(k). \)
From the above structure, it is clear that the system satisfies the observability conditions in Assumption 6. In the subsystems, $x_1^1$ is influenced by unknown input $d_1$ alone, whereas $x_1^2$ is affected by both the unknown inputs.

The observer is designed separately for the two subsystems. The SMO for the system can be designed according to (12) as

$$
\begin{align*}
\begin{bmatrix}
\dot{x}_1^1(k+1) \\
\dot{x}_2^1(k+1)
\end{bmatrix} &= 
\begin{bmatrix}
1 & T_s \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1^1(k) \\
x_2^1(k)
\end{bmatrix} + 
\begin{bmatrix}
T_s \mu_1 \left(\tilde{x}(k), u(k)\right) \\
T_s \mu_2 \left(\tilde{x}(k), u(k)\right)
\end{bmatrix} \\
&+ 
\begin{bmatrix}
T_1^1 \left(y_1(k) - \dot{x}_1^1(k)\right) \\
T_2^1 \left(y_1(k) - \dot{x}_1^1(k)\right)
\end{bmatrix} \\
&+ 
\begin{bmatrix}
1 \\
0.4
\end{bmatrix} u_1^1(k) + 
\begin{bmatrix}
0 \\
0.3
\end{bmatrix} u_2^2(k)
\end{align*}
$$

$$
\begin{align*}
\dot{x}_2^1(k+1) &= \dot{x}_1^2(k) + T_s \mu_2 \left(\tilde{x}(k), u(k)\right) \\
&+ T_2^1 \left(y_2(k) - \dot{x}_2^1(k)\right) + 0.2 \cdot u_1^1(k) + u_2^2(k).
\end{align*}
$$

Comparing the observer with the system, the unknown inputs are replaced by their corresponding robust terms and appended with a feedback term. The robust terms are designed according to (14)

$$
\begin{align*}
u_1^1(k) &= \rho_1 \text{sat} \left(y_1(k) - \dot{x}_1^1(k), \epsilon_1\right) \\
u_2^2(k) &= \rho_2 \text{sat} \left(y_2(k) - \dot{x}_2^1(k), \epsilon_2\right).
\end{align*}
$$

Since the inverse transformation is identity, the observer system can be written directly in the original system equations as shown at the bottom of the page.

The unknown inputs can be estimated from the sliding mode according to (28) as follows:

$$
\begin{align*}
\dot{d}_1(k) &= \frac{1}{T_s} \rho_1 \left(y_1(k) - \dot{x}_1^1(k)\right) \\
\dot{d}_2(k) &= \frac{1}{T_s} \rho_2 \left(y_2(k) - \dot{x}_2^1(k)\right).
\end{align*}
$$

The following values are chosen for the simulation: $A_1 = 0.2703$, $A_2 = 1.01$, $A_3 = -1.5$, $D_1 = 1.5$, $D_2 = 1.5$, $u_1 = 5$, $u_2 = 1$, $x(0) = [2 \ 0 \ 3]^T$, $y(0) = [0 \ 5 \ 0]^T$, $d_1(t) = 0.8 \sin(0.4 \pi t)$, and $d_2(t)$ is a square signal of amplitude 1.0 with a period of six. The sampling period $(T_s)$ is chosen to be 0.04. The estimation gains for subsystems are $l_1^1 = 0.4$, $l_2^1 = 0.3$, and $l_2^2 = 0.4$.

Based on the bounds of states and inputs, the Lipschitz constants and bounds are calculated. In the sliding mode of

$$
\begin{align*}
\dot{x}(k+1) &= \dot{x}(k) + T_s \left[
\begin{array}{c}
\dot{x}_2^2(k) \\
- A_1 \dot{x}_2^1(k) - A_2 \dot{x}_3^1(k) \sin \dot{x}_1^1(k) - A_3 \dot{x}_3^1(k) \sin \left(2 \dot{x}_1^1(k)\right) \\
- D_1 \dot{x}_3^1(k) + D_2 \cos \left(\dot{x}_1^1(k)\right)
\end{array}
\right] \\
&+ T_s \left[
\begin{bmatrix}
0 \\
u_1^1(k)
\end{bmatrix}
\right] + 
\begin{bmatrix}
l_1^1 \\
l_2^1 \\
l_2^2
\end{bmatrix} \left[
\begin{array}{c}
y_1(k) - \dot{x}_1^1(k) \\
y_1(k) - \dot{x}_1^1(k) \\
y_2(k) - \dot{x}_2^1(k)
\end{array}
\right] + 
\begin{bmatrix}
1 & 0 \\
0.4 & 0.3 \\
0.2 & 0.1
\end{bmatrix} \left[
\begin{array}{c}
u_1^1(k) \\
u_2^2(k)
\end{array}
\right]
\end{align*}
$$
converged in the sliding modes of $e_1 = 0$ and $e_2 = 0$. For the $x_2$ subsystem, in the multiple sliding modes, $x_2$ overcomes the uncertainties in the system and tracks the actual trajectory, as shown in Fig. 2. The unknown inputs are reconstructed from the multiple DSM and are shown in Figs. 4 and 5.

**VI. CONCLUSION**

A DSM observer with multiple sliding modes for a class of continuous-time nonlinear systems is developed. The Taylor’s series expansion together with a nonlinear transformation is used to discretize the system. With the proposed discrete-time multiple sliding-mode design, the uncertainties and disturbances can be tracked, and a good estimation of the states and unknown inputs can be obtained. Detailed analysis shows that the estimation converges under estimation feedback gain, sliding-mode gain, and the boundary layer condition.

**DSM with boundary layer:** With $\rho_1 = \rho_2 = 0.26$, $\epsilon_1 = \epsilon_2 = 0.3$, and $\rho_1' = 0.06$, $\rho_2' = 0.05$, the condition (23) is satisfied. Chattering is completely eliminated, and the trajectory is confined inside the boundary layer, as shown in Fig. 6. If condition (23) is not satisfied, e.g., by selecting $\rho_1' = \rho_2' = 0.45$, chattering occurs across the sliding manifold, as shown in Fig. 7. Hence, the condition (23) plays a crucial role in the design of DSM.
APPENDIX I

PROOF OF LEMMA 1

Based on Assumption 5, $\mu_w(\hat{x}(k), u(k))$ and $\hat{Z}_w(\hat{x}(k))$ are the Lipschitz functions such that $\|\mu_w(\hat{x}(k), u(k)) - \mu_w(\tilde{x}(k), u(k))\| \leq I_d(u_w(k))$ and $\|\hat{Z}_w(\hat{x}(k)) - \hat{Z}_w(\tilde{x}(k))\| \leq l_{\tilde{Z}}(u_w(k))$. Since disturbances and states are bounded, $|d_i(k)| \leq d_i$ and $|\hat{O}_w(T_s^d)| \leq b_2^d$ for the upper bounds $d_i$ and $b_2^d$, respectively. Therefore, with Assumption 3, it can be analyzed that $\|\hat{Z}_i(\hat{x}(k))u_i^w(k) - T_s\tilde{d}_i(k)\| \leq l_{\tilde{Z}}(\tilde{d}_i T_s + \rho_i)$ for some upper bound $b_{\tilde{Z}_i}$. Using the Lyapunov function $V_1^w = e_1^T(k)Q_w e_1(k)$, the difference of Lyapunov function can be computed as

$$V_1^w(k + 1) - V_1^w(k) = e_1^T(k + 1)Q_w e_1(k + 1) - e_1^T(k)Q_w e_1(k) - 2e_1^T(k)M_wQ_w e_1(k) - 2e_1^T(k)M_wQ_w e_1(k)$$

where $\Omega(k) = T_s\mu_w(\hat{x}(k), u(k)) - T_s\mu_w(\hat{x}(k), u(k)) = \sum_{i=1}^{q} Z_i(\hat{x}(k))u_i(k) - \sum_{i=1}^{q} \hat{Z}_i(\hat{x}(k))d_i(k)T_s$ according to (15). Using the aforementioned evaluated bounds

$$\Omega(k) \leq \left[ T_s\mu_w + \sum_{i=1}^{q} l_{\tilde{Z}}(k) \rho_i \right] \|e(k)\|$$

where $l_{\mu_w} = T_s\mu_w + \sum_{i=1}^{q} l_{\tilde{Z}}(k) \rho_i$ and $b_{\tilde{Z}_i} = \sum_{i=1}^{q} b_{\tilde{Z}_i}(\tilde{d}_i T_s + \rho_i)$. Based on the aforementioned evaluated bounds, together with (17), the difference of Lyapunov function can be further simplified to the form of $V_1(k + 1) - V_1(k) \leq -[e_1^T(k)]^2 + 2\lambda_{\max}(Q_w)\|M_w\|\|e_1(k)\| + [l_{\mu_w}(k)\|e_1(k)\|] + b_{\tilde{Z}_i}(k) + b_{\tilde{Z}_i}^2$ of $\|e_1(k)\| \leq [b_{\tilde{Z}_i}(k) + b_{\tilde{Z}_i}^2] + 2|l_{\mu_w}(k)\|\|e_1(k)\|\|e_1(k)\|$ of $\|e_1(k)\| \leq [b_{\tilde{Z}_i}(k) + b_{\tilde{Z}_i}^2]$. For simplicity, it can be written as

$$V_1(k + 1) - V_1(k) \leq -C_1\|e_1(k)\|^2 + C_2\|e_1(k)\|^2 + C_3$$

where $K_1 \triangleq 2\|M_w\|l_{\mu_w} + (l_{\mu_w})^2$, $C_1 \triangleq [1 - \lambda_{\max}(Q_w)K_1]$,

$$C_2 \triangleq 2\lambda_{\max}(Q_w)\|M_w\|l_{\mu_w} + (l_{\mu_w})^2,$$

and $C_3 \triangleq \lambda_{\max}(Q_w)(b_{\tilde{Z}_i}^2 + b_{\tilde{Z}_i}^2)$ are positive constants. The estimation gain $L_w$ should be designed such that condition (18) is satisfied, and therefore, $C_1 > 0$. Hence, the quadratic equation is asymptotically bounded. Hence, $e_1(k)$ is bounded. Since $\hat{x}_w(k)$ is bounded, $\tilde{x}_w(k)$ is also bounded.


APPENDIX II

PROOF OF THEOREM 1

For the error dynamics (25), the following denotations ease the analysis:

$$\Theta(k) = T_s\mu_w(\hat{x}(k), u(k)) - T_s\mu_w(\hat{x}(k), u(k)) + \sum_{i=1}^{m} \left( \hat{Z}_i(\hat{x}(k)) - \hat{Z}_i(\tilde{x}(k)) \right) u_{eq}(k) - O_j(T_s)$$

$$\Gamma(k) = \sum_{i=1}^{m} \hat{Z}_i(\hat{x}(k)) [\Theta_{eq}^T(T_s) - T_s\chi_i(\cdot, k)]$$

Since $e_{2,1}^i(\tilde{x}(k))$ and $d_i$ are bounded, it can be shown that $u_{eq}^i \leq b_{eq}$ for some upper bound $b_{eq}$. Moreover, $\chi_i(\cdot, k)$ corresponds to the second-error dynamics of all subsystems; therefore, from (24),

$$\left\| T_s\sum_{i=1}^{m} \hat{Z}_i(\hat{x}(k)) \chi_i(\cdot, k) \right\| \leq T_s \sum_{i=1}^{m} b_{\hat{Z}_i} b_{eq}, e_{eq}(k)$$

where $b_{\hat{Z}_i}$ is the bounded constant of the reduced-order $\hat{Z}_i(\hat{x}(k))$ in the multiple sliding mode and $b_{eq}$ is the equivalent bounded constant of $\chi_i(\cdot, k)$. Similar to the analysis in the Proof of Lemma 1, it can be obtained using the aforementioned evaluations that

$$\left\| \Theta(k) \right\| \leq \left( T_s\mu_w + \sum_{i=1}^{m} l_{\tilde{Z}_i} b_{\hat{Z}_i} \right) \|e(k)\| + b_{\tilde{Z}_i}^2$$

where $l_{\mu_w} = T_s\mu_w + \sum_{i=1}^{m} l_{\tilde{Z}_i} b_{\hat{Z}_i}$, $b_{\hat{Z}_i} = T_s\sum_{i=1}^{m} b_{\hat{Z}_i} b_{\hat{Z}_i}$, and $b_{\tilde{Z}_i} = \sum_{i=1}^{m} b_{\tilde{Z}_i} b_{\hat{Z}_i}$. By choosing a Lyapunov function as $V_2(k) = (e_2^T(k))Q_w e_2(k)$ and similar to the analysis in the proof of Lemma 1, it can be shown that $\|e(k)\| \geq \|e_2(k)\|$. In the multiple DSM of $\epsilon_1 = 0, \ldots, \epsilon_1^m = 0$, the reduced-order error dynamics, i.e., $\|e(k)\| \geq \|e_2(k)\|$, can be obtained. Using (25) and the aforementioned results, the difference of Lyapunov function can be evaluated similar to the analysis of Lemma 1 as

$$V_2(k + 1) - V_2(k) \leq -[e_2^T(k)]^2 + [e_2^T(k)]^2 + 2\lambda_{\max}(Q_w)(l_{\mu_w}^2 + 2\|M_w\|l_{\mu_w} + b_{\mu_w}) + 2\lambda_{\max}(Q_w)((b_{\tilde{Z}_i}^2 + b_{\tilde{Z}_i}^2) + 2|l_{\mu_w}(k)\|\|e_1(k)\|\|e_1(k)\|\|e_1(k)\|$ of $\|e_1(k)\| \leq [b_{\tilde{Z}_i}(k) + b_{\tilde{Z}_i}^2]$. It can be written in the form of

$$V_2(k + 1) - V_2(k) \leq -\left[ 1 - \lambda_{\max}(Q_w) \Delta_1 \right] \|e_2^T(k)\|^2 + \lambda_{\max}(Q_w)\|e_2^T(k)\|^2 + 2\|M_w\|l_{\mu_w} + b_{\mu_w}) + 2b_{\mu_w}^2$$

where $\Delta_1 \triangleq 2\lambda_{\max}(Q_w)\|M_w\|l_{\mu_w} + (l_{\mu_w})^2$, and $\lambda_{\max}(Q_w)\|e_2^T(k)\|^2 + 2\lambda_{\max}(Q_w)\|e_2^T(k)\|^2 + 2b_{\mu_w}^2$. It can be written in the form of

$$V_2(k + 1) - V_2(k) \leq -\left[ 1 - \lambda_{\max}(Q_w) \Delta_2 \right] \|e_2^T(k)\|^2 + \lambda_{\max}(Q_w)\|e_2^T(k)\|^2 + \Delta_3$$

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where \( \Delta_1 \triangleq \left[ (l_{\mu Z})^2 + 2l_{\mu Z} + 2l_{\beta Z}b_{\mu Z} \right] \), \( \Delta_2 \triangleq \left[ 2l_{\mu Z} + 2l_{\beta Z} + 2l_{\mu Z}b_{\beta Z} + 2l_{\beta Z}b_{\mu Z} \right] \), and \( \Delta_3 \triangleq \lambda_{\max}(Q_w) \left( b_{\beta Z}^2 + 2l_{\beta Z}b_{\mu Z} + 2l_{\mu Z}b_{\beta Z} \right) \) are positive constants that explicitly depend on the sampling period \( T_s \). Hence, the condition (26) guarantees the convergence. For further details please see [31].

Remark 8: In fact, the positive constants \( C_1 \) and \( C_2 \) in Lemma 1 and \( \Delta_1, \Delta_2, \) and \( \Delta_3 \) in Theorem 1 can be rewritten as \( C_1 = T_s \times f_{\Delta_1}(\cdot), C_2 = T_s \times f_{\Delta_2}(\cdot), \) and \( \Delta_1 = T_s \times f_{\Delta_1}(\cdot), \) for \( i = 1, 2, 3, \) where \( f_{\Delta_1}(\cdot), f_{\Delta_2}(\cdot), \) and \( f_{\Delta_1}(\cdot) \) are polynomial functions of \( T_s \). Hence, the selection of a lower sampling period \( T_s \) decreases the bounds in inequalities (18) and (26), as well as the ultimate bounds for error convergence.

REFERENCES

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