SINGULAR OBSERVER-BASED COMPENSATORS AND THE
DETERMINISTIC SEPARATION PRINCIPLE∗

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Abstract. The concept of observer-based compensators (OBCs) and the deterministic separation principle have long been cornerstones of modern control theory. In our paper, we extend these ideas via singular system theory to encompass a wider variety of rational compensators. We show that the separation principle generalizes in a natural way to OBCs of all orders, including the static case. We pay particular attention to the problems of properness and conjugate symmetry. Our intention is to provide an introductory account of this framework and to suggest possible avenues of further development.

Key words. observer-based compensators, separation principle, singular systems

AMS subject classifications. 93, 34, 15

DOI. 10.1137/100799356

1. Introduction. The deterministic “separation principle” has been a cornerstone of modern control theory for decades. (See, e.g., [2, section 7-5] and [9, sections 4.2 and 4.3].) Traditionally, this result has been applied to feedback compensators which are strictly proper or belong to a limited class of nonstrictly proper (“reduced-order”) systems. Further generalizations of the theory have been proposed, such as in [2] and most notably by Schumacher [1]. Although our results are reminiscent of those in [1], our approach is quite different. We will point out the similarity of our conclusions where appropriate.

We believe that the singular system methodology has the potential to offer new insight into the eigenvalue assignment problem with reduced compensator order. Thus our work is also distantly related to the classical “Q-parametrization” of stabilizing controllers as in [11, Chapter 5] as well as more recent work in the area, such as [12] and [13]. The obvious difference is that these papers are couched in the language of polynomial rings and rational functions, while we take a state-space approach. Hence our development may be more accessible to nonspecialists. Only the test of time will reveal which theory yields the more useful results. This paper is intended to serve as an introduction to our theory.

We begin with a brief summary of the main ideas surrounding observer-based control. Consider the controllable and observable (i.e., minimal) plant

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx,
\end{align*}
\]

where \(m, n, p \in \mathbb{N}, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \) and \(C \in \mathbb{R}^{p \times n}.\) For convenience, we also assume that

\[
\text{rank} B = m, \quad \text{rank} C = p.
\]

∗Received by the editors June 18, 2010; accepted for publication (in revised form) April 19, 2012; published electronically July 24, 2012.
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There is no loss of generality in assuming (2), since the input and output vectors \( u \) and \( y \) can always be redefined accordingly. Associated with the plant is the family of full-order observers

\[
\dot{z} = (A - LC) z + Bu + Ly
\]

and state-estimate feedback laws

\[
u = -Kz.
\]

For each \( K \) and \( L \), (3) and (4) may be combined to yield the observer-based compensator (OBC)

\[
\dot{z} = (A - BK - LC) z + Ly,
\]

\[
u = -Kz,
\]

with transfer function

\[
G_c(s) = -K (sI - A + BK + LC)^{-1} L.
\]

The corresponding closed-loop system is

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
A & -BK \\
LC & A - BK - LC
\end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.
\]

Rewriting (7) in terms of \( x \) and the estimation error \( e = x - z \), we obtain

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} = \begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.
\]

Hence, the problem of assigning closed-loop eigenvalues splits into a pair of state-feedback problems, where \( K \) and \( L \) are chosen to yield desired eigenvalues in \( A - BK \) and \( A - LC \).

It should be noted that the assignment of closed-loop eigenvalues may not be arbitrary, depending on the problem constraints. For example, restricting \( K \) and \( L \) to be real forces the spectra of \( A - BK, A - LC \), and, hence, (7) to be conjugate-symmetric. Thus, if \( n \) is odd, the closed-loop system must have at least two real eigenvalues. More generally, a typical real-world design would allow complex \( K \) and \( L \), as long as \( G_c \) has real coefficients. This is illustrated by the following example.

**Example 1.** Let \( (A, B, C) = (0, 1, 1) \). If \( K \) and \( L \) are real, then the two closed-loop eigenvalues must be real. However, if we merely require the coefficients of

\[
G_c(s) = \frac{-KL}{s + K + L}
\]

to be real, then we may set \( L = K^* \), yielding closed-loop eigenvalues \( A - BK = -K \) and \( A - LC = -K^* \). Hence, any conjugate-symmetric closed-loop spectrum is achievable. We also note that setting \( K = 0 \) yields \( G_c(s) = 0 \), so any spectrum of the form \( \{0, -L\} \) is also achievable, where \( L \) is arbitrary complex. The latter observation is of trivial utility, since both eigenvalues are hidden modes of the compensator.
A useful generalization of the structure (5)–(6) is obtained by including all similar systems in the analysis. The class of all such systems can be characterized in several ways. First, we consider general \( n \)th order compensators

\[
\begin{align*}
\dot{z} &= Fz + Gy, \\
u &= Hz,
\end{align*}
\]

where \( F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times p}, \) and \( H \in \mathbb{R}^{m \times n}, \) and the corresponding closed-loop systems

\[
\begin{bmatrix}
\dot{x} \\
z
\end{bmatrix} = F_{cl} \begin{bmatrix}
x \\
z
\end{bmatrix},
\]

where

\[
F_{cl} = \begin{bmatrix}
A & BH \\
GC & F
\end{bmatrix}.
\]

Denoting the image space of a matrix by "Im," we say a subspace \( S \subset \mathbb{C}^{2n} \) is nonaxial if

\[
S \cap \text{Im} \begin{bmatrix}
I_n \\
0
\end{bmatrix} = S \cap \text{Im} \begin{bmatrix}
0 \\
I_n
\end{bmatrix} = 0.
\]

Otherwise, \( S \) is axial. Note that \( \text{dim } S > n \) implies that \( S \) is axial. It is elementary to show that \( S \) is nonaxial iff it can be expressed as

\[
S = \text{Im} \begin{bmatrix}
U \\
V
\end{bmatrix},
\]

where \( U \in \mathbb{C}^{n \times r} \) and \( V \in \mathbb{C}^{q \times r} \) have independent columns for some \( r = \text{dim } S. \) Thus, if \( S \) is \( n \)-dimensional, nonaxiality is equivalent to nonsingularity of \( U \) and \( V. \)

**Theorem 2.** Let \((F, G, H) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{m \times n}. \) The following are equivalent:

1. There exist \( K \in \mathbb{C}^{m \times n}, L \in \mathbb{C}^{n \times p}, \) and nonsingular \( T \in \mathbb{C}^{n \times n} \) such that

\[
(F, G, H) = (T^{-1} (A - BK - LC) T, T^{-1} L, -KT).
\]

2. There exists nonsingular \( T \) such that

\[
TGCT - AT + TF - BH = 0.
\]

3. \( F_{cl} \) has an \( n \)-dimensional, nonaxial, invariant subspace \( S. \) In this case, we may take

\[
S = \text{Im} \begin{bmatrix}
T \\
I
\end{bmatrix}.
\]

The equivalence of Theorem 2, parts (1) and (2) follows by direct calculation. The equivalence of (1) and (3) is more difficult and was proven in [1, Propositions 4.2 and 4.3].

It is important to note that, even allowing for similarity and complex \( T, \) not every \((F, G, H) \) is observer-based.
**Example 3.** Let \((A,B,C) = (0,1,1)\) and \((F,G,H) = (0,0,1)\). By Theorem 2, part (3), \((F,G,H)\) can be expressed in the form (12) iff

\[
F_{cl} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

has a nonaxial eigenvector. But \(F_{cl}\) has only one independent eigenvector, which is axial.

One might argue that Example 3 is not entirely convincing, since the compensator transfer function \(G_c = 0\) is also realized by \((F,G,H) = (1,0,1)\). For this triple, \(F_{cl}\) does have a nonaxial eigenvector

\[
S = \text{Im} \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

leading to \(T = 1\), \(K = -HT^{-1} = -1\), and \(L = TG = 0\).

A more difficult question is whether a rational transfer function \(G_c\) (with McMillan degree at most \(n\)) exists which is not realizable by any observer-based system (12). Although requiring considerable effort, the next example answers this question in the affirmative.

**Example 4.** Let \(n = 2\), \((A,B,C) = (0,I,I)\), and

\[
G_c(s) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

For any realization \((F,G,H)\) of \(G_c\),

\[
F_{cl} = \begin{bmatrix} 0 & H \\ G & F \end{bmatrix}.
\]

One minimal realization of \(G_c\) is

\[
F_{22} = 0, \quad G_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

so, by the Kalman decomposition, every realization is similar to one of the forms

\[
F = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\]

or

\[
F = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & c \\ 0 & d \end{bmatrix}.
\]

Using (13),

\[
F_{cl} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ c & d & a & b \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

which has characteristic polynomial

\[
\Delta(s) = s^3(s - a).
\]
The following analysis shows that no such $F_{cl}$ has a 2-dimensional nonaxial invariant subspace. Thus, by Theorem 2, no observer-based realization of $G_c$ exists.

If $a \neq 0$, then $F_{cl}$ has two independent eigenvectors

$$v_0 = \begin{bmatrix} -a \\ 0 \\ c \\ 0 \end{bmatrix}, \quad v_a = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. $$

Although

$$S = \text{Im} \left[ \begin{bmatrix} v_0 & v_a \end{bmatrix} \right]$$

is $F_{cl}$-invariant, it is axial. The only other choice for $S$ is

$$S = \text{Ker} F_{cl}^2 = \text{Ker} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ac & ad + b & a^2 & c + ab \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Im} \begin{bmatrix} -a & 1 \\ 0 & 0 \\ c & b \\ 0 & -a \end{bmatrix},$$

which is also axial. If $a = 0$ and $c \neq 0$, then $F_{cl}$ has only one independent eigenvector

$$v_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

so there is a single 2-dimensional invariant subspace $S$. But $S$ contains $v_0$, so it is axial. Finally, if $a = c = 0$, then $F_{cl}$ has two 2-dimensional invariant subspaces. One is

$$S = \text{Ker} F_{cl} = \text{Im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which is axial. The other is obtained by selecting $w_0 \in \text{Ker} F_{cl}^2 - \text{Ker} F_{cl}$, generating the eigenvector $v_0 = F_{cl} w_0$, and setting $S = \text{Im} \left[ \begin{bmatrix} v_0 & w_0 \end{bmatrix} \right]$. This yields the form

$$w_0 = \begin{bmatrix} x \\ 0 \\ y \\ 1 \end{bmatrix}, \quad v_0 = \begin{bmatrix} 1 \\ 0 \\ b \end{bmatrix},$$

so $S$ is axial.

Now consider the form (14). Here we obtain

$$F_{cl} = \begin{bmatrix} 0 & 0 & 1 & c \\ 0 & 0 & 0 & d \\ 0 & 1 & 0 & a \\ 0 & 0 & 0 & b \end{bmatrix}$$
and \( \Delta(s) = s^3(s-b) \). If \( b \neq 0 \), then \( F_{cl} \) has two independent eigenvectors

\[
v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_b = \begin{bmatrix} ab + b^2c + d \\ b^2d \\ b(ab + d) \end{bmatrix}.
\]

Then

\[
S = \text{Im} \left[ \begin{array}{c} v_0 \\ v_b \end{array} \right]
\]

is axial. The only other choice for \( S \) is

\[
S = \text{Ker} F_{cl}^2 = \text{Ker} \begin{bmatrix} 0 & 1 & 0 & a + bc \\ 0 & 0 & 0 & bd \\ 0 & 0 & 0 & ab + d \\ 0 & 0 & 0 & b^2 \end{bmatrix} = \text{Im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

which is also axial. If \( b = 0 \) and \( d \neq 0 \), then \( F_{cl} \) has only one independent eigenvector

\[
v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

so there is a single 2-dimensional invariant subspace \( S \), which is axial. Finally, if \( b = d = 0 \), \( F_{cl} \) has two 2-dimensional invariant subspaces. One is

\[
S = \text{Ker} F_{cl} = \text{Im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -a \\ -c \\ 1 \end{bmatrix},
\]

which is axial. The other is obtained by selecting \( w_0 \in \text{Ker} F_{cl}^2 - \text{Ker} F_{cl} \), generating the eigenvector \( v_0 = F_{cl}w_0 \), and setting \( S = \text{Im} \left[ \begin{array}{c} v_0 \\ w_0 \end{array} \right] \). This yields the form

\[
w_0 = \begin{bmatrix} x \\ -ay \\ 1 - cy \\ y \end{bmatrix}, \quad v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

so \( S \) is axial.

2. Singular system fundamentals. In order to generalize the ideas in section 1, we need certain basic facts from singular system theory. (See [3, Chapter XII] for detailed information on pencils and their associated differential equations.) Let \( q \in \mathbb{N} \), and consider a matrix pencil \((E, F) \in \mathbb{C}^{q \times q} \times \mathbb{C}^{q \times q}\). Define the characteristic polynomial of \((E, F)\) to be

\[
\Delta(s) = \det(sE - F).
\]

If \( \Delta \neq 0 \), we say \((E, F)\) is regular. In this case, \( \deg \Delta \leq \text{rank} E \). Now consider the group action

\[
\text{GL}^2(q, \mathbb{C}) \times (\mathbb{C}^{q \times q} \times \mathbb{C}^{q \times q}) \to \mathbb{C}^{q \times q} \times \mathbb{C}^{q \times q}
\]
defined by
\[(M, N) \cdot (E, F) = (MEN, MFN).\]

We say \((E_1, F_1)\) and \((E_2, F_2)\) are equivalent if they lie in the same orbit under "\(\cdot\)". It is easy to see that regularity is closed under equivalence.

Matrix pencils lead naturally to singular systems
\[(15) \quad E \dot{z} = Fz + Gy,\]
\[u = Hz.\]

These generalize (9) according to the imbedding
\[(16) \quad (F, G, H) \mapsto (I, F, G, H).\]

System equivalence can be extended to (15) by defining
\[\Omega_q = C^{q \times q} \times C^{q \times q} \times C^{q \times p} \times C^{m \times q}\]
and the group action
\[GL^2(q, C) \times \Omega_q \rightarrow \Omega_q\]
according to
\[(17) \quad (M, N) \cdot (E, F, G, H) = (MEN, MFN, MG, HN).\]

We say a system \((E, F, G, H)\) is regular if \((E, F)\) is regular. Two systems \((E_1, F_1, G_1, H_1)\) and \((E_2, F_2, G_2, H_2)\) are equivalent if they lie in the same orbit. Note that regularity of systems is closed under (17) and that regularity is required in order to write the transfer function
\[(18) \quad \mathcal{G}_c(s) = H (sE - F)^{-1} G.\]

Also,
\[HN (sMEN - MFN)^{-1} MG = H (sE - F)^{-1} G,\]
so \(\mathcal{G}_c\) is invariant under equivalence. As in (9)–(10), a compensator (15) may be combined with the plant (1) to yield the closed-loop pencil
\[(19) \quad (E_{cl}, F_{cl}) = \left( \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \right)\]
with characteristic polynomial
\[\Delta_{cl}(s) = \det (sE_{cl} - F_{cl}).\]

We may also consider regularity of \((E_{cl}, F_{cl})\), depending on whether \(\Delta_{cl} \neq 0\).

Since (15) involves a differential equation, one inevitably confronts the question of admissibility of initial conditions and inputs. These issues have been treated extensively in references such as [3, Chapter XII, section 7], and [10, Chapter 22]. In particular, it is shown that, as long as the system pencil is regular, its input-output map is well defined over a large class of signals. In our paper, we are really only
interested in the algebra surrounding singular systems, building on the results of [6] and [7].

According to the Weierstrass decomposition, every regular orbit contains pencils of the form

\[(MEN, MFN) = \left( \begin{bmatrix} I_r & 0 \\ 0 & E_f \end{bmatrix}, \begin{bmatrix} F_s & 0 \\ 0 & I_{q-r} \end{bmatrix} \right)\]

for some \( r \in [0, q] \). If \( r < q \), then \( E_f \) is nilpotent. The characteristic polynomial may thus be written

\[
\Delta(s) = \frac{\det(sI - F_s) \det(sE_f - I)}{\det(MN)} = (-1)^{q-r} \det(sI - F_s). \tag{21}
\]

One can show that, if \((E, F)\) is real, we may constrain \( M \) and \( N \) in (20) to be real. For systems \((E, F, G, H)\), we may set

\[
\begin{bmatrix} \overline{G_s} \\ \overline{G_f} \end{bmatrix} = MG, \quad \begin{bmatrix} H_s & H_f \end{bmatrix} = HN,
\]

yielding the subsystems \((I, F_s, G_s, H_s)\) and \((E_f, I, G_f, H_f)\). Similarly, the transfer function may be uniquely decomposed via polynomial division to obtain

\[
\mathcal{G}_c = \mathcal{G}_{cs} + \mathcal{G}_{cf} \tag{23}
\]

with \( \mathcal{G}_{cs} \) strictly proper and \( \mathcal{G}_{cf} \) polynomial. If \( \mathcal{G}_c \) has real coefficients, so do \( \mathcal{G}_{cs} \) and \( \mathcal{G}_{cf} \). This decomposition is consistent with (20), since we may write

\[
\mathcal{G}_{cs}(s) = H_s (sI - F_s)^{-1} G_s, \tag{24}
\]

As with any polynomial,

\[
\frac{1}{s} \mathcal{G}_{cf} \left( \frac{1}{s} \right) = \frac{1}{s} H_f \left( \frac{1}{s} E_f - I \right)^{-1} G_f = -H_f(sI - E_f)^{-1} G_f \tag{25}
\]

is strictly proper with all poles equal to 0.

If \((E, F)\) is regular and \( \text{rank} E = \deg \Delta \), we say \((E, F)\) has unit index and write \( \text{ind}(E, F) = 1 \). In this case, we also write \( \text{ind}(E, F, G, H) = 1 \). For the case \( F = I \), it is easy to show that \( \text{ind}(E, I) = 1 \) iff \( E \) has unit index in the classical sense, i.e., \( \text{rank} E = \text{rank} E^2 \).

The next result will be useful in addressing the design of proper compensators in sections 3, 5, and 6.

**Proposition 5.** The following are equivalent:

(1) \( \text{ind}(E, F) = 1 \).

(2) \((sE - F)^{-1}\) is proper.

(3) \( \text{ind}(E_{cl}, F_{cl}) = 1 \).
Proof. (1) \iff (2): From (20) and (21), \( \text{rank} E = r + \text{rank} E_f \) and \( \text{deg} \Delta = r \), so \( \text{ind} (E, F) = 1 \) iff \( E_f = 0 \). Also from (20),

\[
(sE - F)^{-1} = N \begin{bmatrix}
(sI - F_s)^{-1} & 0 \\
0 & -\sum_{i=0}^{q-r-1} s^i E_f'
\end{bmatrix} M
\]

is proper iff \( E_f = 0 \).

(1) \iff (3): As an alternative to the Weierstrass decomposition (20), we may choose nonsingular \( M \) and \( N \) such that

\[
\begin{bmatrix}
F_{11} & F_{22} \\
F_{21} & F_{22}
\end{bmatrix} = M F N,
\begin{bmatrix}
G_1 \\
G_2
\end{bmatrix} = M G,
\begin{bmatrix}
H_1 & H_2
\end{bmatrix} = H N.
\]

Then \( \text{rank} E = \rho \) and

\[
\text{deg} \Delta = \text{deg} \det \begin{bmatrix}
sI - F_{11} & -F_{12} \\
-F_{21} & -F_{22}
\end{bmatrix} \leq \rho
\]

with equality iff \( F_{22} \) is nonsingular. Hence, nonsingularity of \( F_{22} \) is equivalent to (1).

Applying the same argument to the closed-loop pencil, the transformation

\[
\begin{bmatrix}
I & 0 \\
0 & M
\end{bmatrix} \begin{bmatrix}
I & 0 \\
0 & E
\end{bmatrix} \begin{bmatrix}
I & 0 \\
0 & N
\end{bmatrix} = \begin{bmatrix}
I_n & 0 & 0 \\
0 & I_\rho & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
I & 0 \\
0 & M
\end{bmatrix} \begin{bmatrix}
A & BH & 0 \\
G C & F
\end{bmatrix} \begin{bmatrix}
I & 0 \\
0 & N
\end{bmatrix} = \begin{bmatrix}
A & BH_1 & BH_2 \\
G_1 C & F_{11} & F_{22} \\
G_2 C & F_{21} & F_{22}
\end{bmatrix}
\]

shows that nonsingularity of \( F_{22} \) is also equivalent to (3).

In view of Proposition 5 and (18), unit index compensators always have proper transfer functions \( G_c \) and lead to unit index closed-loop systems. This fact is fundamental to the design of a well-behaved control system.

State-space realization theory for strictly proper transfer functions is well established. (See, e.g., [2, Chapter 6].) A natural extension of the theory for arbitrary rational functions can be constructed based on the singular system concept. (See [4] for a summary.) Recall that the McMillan degree \( \mu \) of a strictly proper rational matrix \( G_c(s) \) is the least common denominator of all its minors. The degree \( \mu \) is equal to the smallest dimension over all state-space realizations of \( G_c(s) \). A realization of \( G_c(s) \) is controllable and observable iff it has dimension \( \mu (G_c(s)) \) (justifying the term “minimal”). McMillan degree can be extended to arbitrary rational matrices according to

\[
\mu (G_c(s)) = \mu (G_c(s)) + \mu \left( \frac{1}{s} G_c(s) \right).
\]

In view of (25), the two terms on the right determine (nonunique) minimal realizations \( (F_s, G_s, H_s) \) and \( (E_f, G_f, H_f) \) with \( E_f \) nilpotent. By (23)–(25), the form (20) is a
realization of $G_c$. It can be shown that (1) the system (20), (22) is a minimal realization of $G_c$, (2) a realization of $G_c$ is controllable and observable (in the sense of [5]) iff it is minimal, and (3) the minimal realizations of $G_c$ constitute exactly one orbit in $\Omega_{\mu}(G_c)$.

We will also make use of the Kalman decomposition for state-space systems. (See, e.g., [2, section 5-5].) One form of this result states that, for any $(F,G,H)$, there exists a nonsingular $T$ such that

$$ T^{-1}FT = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix}, \quad T^{-1}G = \begin{bmatrix} G_1 \\ G_2 \\ 0 \end{bmatrix}, \quad HT = \begin{bmatrix} 0 & H_2 & H_3 \end{bmatrix}, $$

where $(F_{22}, G_2, H_2)$ is minimal. In the singular system context, such a transformation may be applied to $(F_s, G_s, H_s)$ and $(E_f, G_f, H_f)$ separately and combined via (20) to yield

$$ (26) \quad \begin{bmatrix} T_s^{-1} & 0 \\ 0 & T_f^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & E_f \end{bmatrix}\begin{bmatrix} T_s & 0 \\ 0 & T_f \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & E_{f11} & E_{f12} & E_{f13} \\ 0 & 0 & 0 & E_{f22} & E_{f23} & 0 \\ 0 & 0 & 0 & 0 & E_{f33} \end{bmatrix}, $$

$$ \begin{bmatrix} T_s^{-1} & 0 \\ 0 & T_f^{-1} \end{bmatrix}\begin{bmatrix} F_s & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} T_s & 0 \\ 0 & T_f \end{bmatrix} = \begin{bmatrix} F_{s11} & F_{s12} & F_{s13} & 0 & 0 & 0 \\ 0 & F_{s22} & F_{s23} & 0 & 0 & 0 \\ 0 & 0 & F_{s33} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, $$

$$ \begin{bmatrix} T_s^{-1} & 0 \\ 0 & T_f^{-1} \end{bmatrix}\begin{bmatrix} G_s \\ G_f \end{bmatrix} = \begin{bmatrix} G_{s1} \\ G_{s2} \\ 0 \\ G_{f1} \\ G_{f2} \\ 0 \end{bmatrix}, $$

$$ (27) \quad \begin{bmatrix} H_s & H_f \\ 0 & T_f \end{bmatrix} = \begin{bmatrix} 0 & H_{s2} & H_{s3} & 0 & H_{f2} & H_{f3} \end{bmatrix}. $$

Then $(F_{s22}, G_{s2}, H_{s2})$ and $(E_{f22}, G_{f2}, H_{f2})$ are both minimal. Writing

$$ E_{ii} = \begin{bmatrix} I & 0 \\ 0 & E_{fii} \end{bmatrix}, \quad F_{ii} = \begin{bmatrix} F_{sii} & 0 \\ 0 & I \end{bmatrix}, $$

$$ E_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & E_{fij} \end{bmatrix}, \quad F_{ij} = \begin{bmatrix} F_{sij} & 0 \\ 0 & 0 \end{bmatrix}, \quad i > j, $$

$$ G_i = \begin{bmatrix} G_{si} \\ G_{fi} \end{bmatrix}; \quad i = 1, 2; \quad H_j = \begin{bmatrix} H_{sj} & H_{fj} \end{bmatrix}; \quad j = 2, 3,
and permuting (26)–(27) (an equivalence transformation) leads to
\[
M \mathcal{E} N = \begin{bmatrix}
E_{11} & E_{12} & E_{13} \\
0 & E_{22} & E_{23} \\
0 & 0 & E_{33}
\end{bmatrix}, \quad M \mathcal{F} N = \begin{bmatrix}
F_{11} & F_{12} & F_{13} \\
0 & F_{22} & F_{23} \\
0 & 0 & F_{33}
\end{bmatrix},
\]

\[
M G = \begin{bmatrix}
G_1 \\
G_2 \\
0
\end{bmatrix}, \quad H N = \begin{bmatrix}
0 & H_2 & H_3
\end{bmatrix}
\]

for some nonsingular \( M \) and \( N \). The subsystem \((E_{22}, F_{22}, G_2, H_2)\) is a minimal realization of \( \mathcal{G}_c \). It is crucial to note that in both the Weierstrass form and the Kalman decomposition, if \((E, F, G, H)\) is real, then the transformations \( M, N, T_s, \) and \( T_f \) may also be taken to be real. Hence, real \((E, F, G, H)\) leads to a real decomposition (28)–(29). This fact will be required in section 4.

We conclude with a further examination of the real systems in \( \Omega_q \).

**Proposition 6.** Consider any regular orbit \( W \subset \Omega_q \) and the corresponding transfer function \( \mathcal{G}_c \).

1. \( W \) contains a nonreal system.
2. If \( W \) contains a real system, then \( \mathcal{G}_c \) has real coefficients.
3. If \( \mathcal{G}_c \) has real coefficients and \( W \) is minimal, then \( W \) contains a real system.

**Proof.** (1) If \((E, F, G, H) \in W\) is real, we may choose real \((M, N)\) to put \((E, F)\) into Weierstrass form (20)–(22). If \( r \neq 0 \), choose any nonsingular, nonreal \( M_s \in \mathbb{C}^{r \times r} \) and set
\[
M_1 = \begin{bmatrix}
M_s & 0 \\
0 & I
\end{bmatrix} M.
\]

Then
\[
M_1 E N = \begin{bmatrix}
M_s & 0 \\
0 & E_f
\end{bmatrix},
\]

which is nonreal. If \( r = 0 \), then \( q - r > 0 \), so we may choose a nonsingular, nonreal \( M_f \in \mathbb{C}^{q-r \times q-r} \) and set
\[
M_1 = \begin{bmatrix}
I & 0 \\
0 & M_f
\end{bmatrix} M.
\]

Then
\[
M_1 F N = \begin{bmatrix}
F_s & 0 \\
0 & M_f
\end{bmatrix},
\]

which is nonreal. In either case, \((M_1, N) \cdot (E, F, G, H)\) is nonreal.

(2) This is obvious from (18).

(3) From minimality, \( \mathcal{G}_c \) corresponds to a single orbit, namely \( W \). Write \( \mathcal{G}_c = \mathcal{G}_{cs} + \mathcal{G}_{cf} \) as in (23), and recall that \( \mathcal{G}_{cs} \) and \( \mathcal{G}_{cf} \) have real coefficients. Letting \( q = \mu(\mathcal{G}_c) \), we may choose real minimal realizations \((I, F_s, G_s, H_s)\) and \((E_f, I, G_f, H_f)\) of these transfer functions, having dimensions \( r \) and \( q - r \), respectively. Then the real Weierstrass form (20)–(22) is a minimal realization of \( \mathcal{G}_c \) and, therefore, lies in \( W \).  \( \square \)
3. Singular OBCs. In this section, we extend the classical ideas summarized in section 1. In particular, we show that the separation of eigenvalues exhibited by (8) can be generalized to a larger class of compensators. Our approach is motivated by results we obtained in [6] and [7] pertaining to high-gain feedback: Consider compensators (5)–(6), where $K$ and $L$ are replaced by sequences $K_k$ and $L_k$. Premultiplying (5) by nonsingular matrices $M_k$ and applying coordinate changes 

$$z = N_k w$$

yields

$$M_k N_k w = M_k (A - BK_k - L_k C) N_k w + M_k L_k y,$$

$$u = -K_k N_k w.$$ 

According to [6, Theorem 3.1] and a dual argument, for any 4-tuple $(X_c, Y_c, X_o, Y_o)$ with appropriate dimensions, the sequences $M_k, L_k, N_k,$ and $K_k$ can be chosen to achieve

$$(N_k, K_k N_k, M_k, M_k L_k) \rightarrow (X_c, Y_c, X_o, Y_o).$$

Thus we arrive at the structure of the OBC

$$(32) \quad X_oX_c w = (X_oAX_c - X_oBY_c - Y_oCX_c) w + Y_o y,$$

$$u = -Y_c w.$$ 

It is important to note that the high-gain concepts which lead naturally to the study of (32) do not constitute a rigorous justification, since the coordinate change (30) may degenerate to a singular matrix. Nevertheless, ample proof of the utility of (32) will be supplied below, even when $X_c$ is singular.

Henceforth, we will express an OBC as a 4-tuple

$$(33) \quad (E, F, G, H) = (X_oX_c, X_oAX_c - X_oBY_c - Y_oCX_c, Y_c, -Y_c),$$

where

$$(X_c, Y_c, X_o, Y_o) \in \mathbb{C}^{n \times q} \times \mathbb{C}^{m \times q} \times \mathbb{C}^{q \times n} \times \mathbb{C}^{q \times p}.$$ 

Although we are mainly interested in the case $q = n$, many of our results apply equally well to $q \neq n$. If $q \neq n$ or if either $X_c$ or $X_o$ is singular, we say that (33) is a singular OBC. Denote by $OBC_q$ all OBCs in $\Omega_q$. For any $(M, N) \in GL^2(q, \mathbb{C})$, the mapping of parameters

$$(X_c, Y_c, X_o, Y_o) \mapsto (X_c N, Y_c N, MX_o, MY_o)$$

is equivalent to applying the group action (17) to (33). Thus $OBC_q$ is closed under system equivalence.

For $q = n$, it is easy to relate the form (33) to the classical observer-based structure (12). Here we must invoke the identification (16). Let $OBC_n \subset OBC_\infty$ be the points with nonsingular $X_c$ and $X_o$. (This notation will be justified in section 5.)

**Proposition 7.** (1) For any fixed $(K, L) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p}$, the classical OBCs (12) all lie in the same orbit in $OBC_n^\infty$ under the group action (17).

(2) Every orbit in $OBC_n^\infty$ contains a system of the form (12).
Proof. (1) For each $T$, set $M = T^{-1}$ and $N = T$. Then


(2) Let

$$(X_o X_c, X_o AX_c - X_o BY_c - Y_o CX_c, Y_o, -Y_c) \in OBC_n^\infty.$$ Then $X_c$ and $X_o$ are nonsingular, so we may set

$$M = X_o^{-1}, \quad N = X_c^{-1}, \quad K = Y_c X_c^{-1}, \quad L = X_o^{-1} Y_o,$$ yielding

$$(M, N) \cdot (X_o X_c, X_o AX_c - X_o BY_c - Y_o CX_c, Y_o, -Y_c) = (I, A - BK - LC, L, -K).$$

In particular, Proposition 7 tells us that the classical OBC orbits are in 1-1 correspondence with the orbits in $OBC_n^\infty$.

For any OBC (33), we obtain the closed-loop pencil

$$(E_{cl}, F_{cl}) = \left( \begin{bmatrix} I_n & 0 \\ 0 & X_o X_c \end{bmatrix}, \begin{bmatrix} A & -BY_c \\ Y_o C & X_o AX_c - X_o BY_c - Y_o CX_c \end{bmatrix} \right).$$

Setting

$$M_{cl} = \begin{bmatrix} I_n & 0 \\ X_o & -I_q \end{bmatrix}, \quad N_{cl} = \begin{bmatrix} X_c & I_n \\ I_q & 0 \end{bmatrix}$$

yields

$$(M_{cl} E_{cl} N_{cl}, M_{cl} F_{cl} N_{cl}) = \left( \begin{bmatrix} X_c & I_n \\ 0 & X_o \end{bmatrix}, \begin{bmatrix} AX_c - BY_c & A \\ 0 & X_o A - Y_o C \end{bmatrix} \right).$$

In an algebraic sense, (35) may be interpreted as a generalized “separation principle,” although the placement of closed-loop eigenvalues may be problematic when the OBC is singular.

From the structure (35), the case $q > n$ leads to

$$\text{rank} (sE_{cl} - F_{cl}) = \text{rank} \begin{bmatrix} sX_c - (AX_c - BY_c) & sI - A \\ 0 & sX_o - (X_o A - Y_o C) \end{bmatrix} \leq \text{rank} (sX_c - (AX_c - BY_c)) + \text{rank} \begin{bmatrix} sI - A \\ sX_o - (X_o A - Y_o C) \end{bmatrix} \leq 2n < n + q,$$

so $\Delta_{cl} \equiv 0$. Hence, to obtain meaningful results, we henceforth restrict ourselves to $q \leq n$. The case $q = n$ is ultimately the most important. In this instance, we may define

$$\Delta_c (s) = \det (sX_c - (AX_c - BY_c)),$$

$$\Delta_o (s) = \det (sX_o - (X_o A - Y_o C)).$$
From (35), the closed-loop characteristic polynomial is $\Delta_{cl} = \Delta_c \Delta_o$. In particular, regularity of (34) is equivalent to regularity of $(X_c, AX_c - BY_c)$ and $(X_o, X_o A - Y_o C)$. Under regularity, calculation of the closed-loop eigenvalues amounts to finding the roots of $\Delta_c$ and $\Delta_o$. Let $ROBC_n \subset OBC_n$ be the set of points such that $(E_{cl}, F_{cl})$ is regular. A simple calculation shows that $ROBC_n$ is closed under system equivalence. It is a useful fact that closed-loop regularity places restrictions on the ranks of $X_c$ and $X_o$.

**Proposition 8.** If an OBC (33) belongs to $ROBC_n$, then

$$\text{rank } X_c \geq n - m$$

and

$$\text{rank } X_o \geq n - p.$$

**Proof.** Suppose $\Delta_c(\lambda) \neq 0$ and let

$$Z_c = (A - \lambda I) X_c - BY_c.$$

Then

$$\det Z_c = (-1)^n \det (\lambda X_c - (AX_c - BY_c)) \neq 0,$$

so

$$n = \text{rank } ((A - \lambda I) X_c - BY_c) \leq \text{rank } (A - \lambda I) X_c + \text{rank } BY_c \leq \text{rank } X_c + m.$$

The inequality $n \leq \text{rank } X_o + p$ is proven similarly.

Another useful result relates the ranks of $X_c, X_o,$ and $E_{cl}$.

**Lemma 9.** $\text{rank } E_{cl} \geq \text{rank } X_c + \text{rank } X_o$ with equality iff $\text{Ker } X_o \subset \text{Im } X_c$.

**Proof.** The inequality follows from (35). Let $M_c, N_c, M_o,$ and $N_o$ be nonsingular and such that

$$M_c X_c N_c = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad M_o X_o N_o = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

and set

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = M_c N_o.$$

Then

$$\begin{bmatrix} M_c & 0 \\ 0 & M_o \end{bmatrix} \begin{bmatrix} X_c & I \\ 0 & X_o \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & N_o \end{bmatrix} = \begin{bmatrix} I & 0 & P_{11} & P_{12} \\ 0 & 0 & P_{21} & P_{22} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so $\text{rank } E_{cl} = \text{rank } X_c + \text{rank } X_o$ iff $P_{22} = 0$. But

$$\text{Im } X_c = M_c^{-1} \text{Im } \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \text{Ker } X_o = N_o \begin{bmatrix} 0 \\ I \end{bmatrix},$$

so $\text{Ker } X_o \subset \text{Im } X_c$ iff

$$\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} = M_c N_o \begin{bmatrix} 0 \\ I \end{bmatrix} \subset \text{Im } \begin{bmatrix} I \\ 0 \end{bmatrix},$$

proving the result.

The case of OBCs where $\text{ind } (E, F) = 1$ is especially important, since such compensators have proper transfer functions $G_c$ and lead to unit index closed-loop systems.
(by Proposition 5). In particular, every unit index member of \( OBC_n \) also belongs to \( ROBC_n \). We can characterize these systems explicitly in terms of \( (X_c, Y_c, X_o, Y_o) \).

**Theorem 10.** A member of \( OBC_n \) has unit index iff

\[
\text{ind}(X_c, AX_c - BY_c) = 1,
\]

\[
\text{ind}(X_o, X_o A - Y_o C) = 1,
\]

\[
\text{Ker} X_o \subset \text{Im} X_c.
\]

**Proof.** From Proposition 5, it suffices to prove that (39)–(41) are equivalent to

\[
\text{ind}(E_{cl}, F_{cl}) = 1.
\]

We may assume \((E_{cl}, F_{cl})\) is in the form (35). Since (35) is block-triangular,

\[
\text{rank} E_{cl} \geq \text{rank} X_c + \text{rank} X_o \geq \deg \Delta_c + \deg \Delta_o = \deg \Delta_{cl}.
\]

The second inequality becomes equality iff (39) and (40) hold. From Lemma 9, the first becomes equality iff (41) holds.

For \( q = n \), the problem of choosing \((X_c, Y_c, X_o, Y_o)\) to achieve eigenvalue specifications may still appear daunting. We will show in Section 5 that this problem can be reduced to a more manageable form.

Although we have allowed \((X_c, Y_c, X_o, Y_o)\) to be arbitrary complex in the analysis above, we will also impose conjugate symmetry and investigate its consequences in the remaining sections. It is important to note that the development of Sections 1–3 applies equally well to discrete-time problems by substituting the left-shift of state variables for their derivatives throughout. Indeed, differential or difference equations are invoked merely as motivation, with the algebra being identical in both cases. We also observe that the algebra leading to the separation principle is very general, consisting essentially of the calculation (34)–(35). These operations can be carried out for matrices over any ring. The pursuit of such a theory would be in the spirit of “algebraic system theory.” (See, e.g., [8].)

**4. Invariant subspaces.** We are now in a position to generalize Theorem 2 to singular OBCs. Our basic approach is to investigate the invariant subspaces of the closed-loop pencil (cf. [1, Propositions 4.2 and 4.3]). In particular, this line of reasoning will lead to a discrete-time problem by substituting the left-shift of state variables for their derivatives throughout. Indeed, differential or difference equations are invoked merely as motivation, with the algebra being identical in both cases. We also observe that the algebra leading to the separation principle is very general, consisting essentially of the calculation (34)–(35). These operations can be carried out for matrices over any ring. The pursuit of such a theory would be in the spirit of “algebraic system theory.” (See, e.g., [8].)

For any \((P, Q) \in \mathbb{C}^{n+q \times q} \times \mathbb{C}^{n+q \times n+q}\), a subspace \( S \subset \mathbb{C}^{n+q} \) is \((P, Q)\)-invariant if

\[
\dim (PS + QS) \leq \dim S.
\]

In the context of pencils, nonaxiality is defined only for invariant subspaces. We say that a \((P, Q)\)-invariant subspace \( S \) is **nonaxial** if

\[
S \cap \text{Im} \begin{bmatrix} I_n \\ 0 \end{bmatrix} = (PS + QS) \cap \text{Im} \begin{bmatrix} 0 \\ I_q \end{bmatrix} = 0.
\]

Otherwise, \( S \) is **axial**. Note that nonaxiality implies \( \dim S \leq q \). Motivated by (36), we restrict the analysis to the case \( q \leq n \).

The following lemma shows that subspace invariance and nonaxiality for pencils generalize the corresponding definitions for matrices.

Lemma 11. Suppose \( P = I \).

1. \( S \) is \((I, Q)\)-invariant iff \( S \) is \( Q \)-invariant.

2. If \( S \) is \( Q \)-invariant and \( q = n \), then \( S \) is nonaxial iff (11) holds.

Proof. For \( P = I \), (42) becomes

\[
\dim (S + QS) \leq \dim S,
\]

which holds iff

\[
S + QS = S.
\]

This is equivalent to \( Q \)-invariance of \( S \). For \( q = n \), substituting (44) into (43) yields (11).

The next result provides a direct characterization of nonaxiality when \( \dim S = q \).

Lemma 12. If \( \dim S = q \), then \( S \) is nonaxial iff there exist \( U \in \mathbb{C}^{n \times q} \) and \( V \in \mathbb{C}^{q \times n} \) such that

\[
S = \text{Im} \left[ \begin{array}{c} U \\ I_q \end{array} \right], \quad PS + QS \subset \text{Im} \left[ \begin{array}{c} I_n \\ V \end{array} \right].
\]

Proof. (Sufficient) The result follows immediately from (43).

(Necessary) Write

\[
S = \text{Im} \left[ \begin{array}{c} U_1 \\ V_1 \end{array} \right],
\]

where \( U_1 \in \mathbb{C}^{n \times q} \), \( V_1 \in \mathbb{C}^{q \times q} \), and the matrix has full rank. From (43), \( V_1 \) is nonsingular, so we may set \( U = U_1 V_1^{-1} \). From (42),

\[
PS + QS = \text{Im} \left[ \begin{array}{c} U_2 \\ V_2 \end{array} \right],
\]

where \( U_2 \in \mathbb{C}^{n \times r} \) and \( V_2 \in \mathbb{C}^{q \times r} \) for some \( r \leq q \), and the matrix has full rank. From (43), \( U_2 \) has full rank, so there exists \( \bar{U}_2 \in \mathbb{C}^{n \times n-r} \) such that

\[
U_3 = \left[ \begin{array}{cc} U_2 & \bar{U}_2 \end{array} \right]
\]

is nonsingular. Let

\[
V_3 = \left[ \begin{array}{cc} V_2 & 0 \end{array} \right]
\]

and \( V = V_3 U_3^{-1} \). Then

\[
PS + QS \subset \text{Im} \left[ \begin{array}{c} U_3 \\ V_3 \end{array} \right] = \text{Im} \left[ \begin{array}{c} I_n \\ V \end{array} \right].
\]

Now we turn to the question of determining when a given system \((E, F, G, H) \in \Omega_q \) is an OBC. The next result generalizes Theorem 2.

Theorem 13. (1) If \((E_{cl}, F_{cl}) \) as in (19) has a \( q \)-dimensional, nonaxial, invariant subspace \( S \), then \((E, F, G, H) \in OBC_q \) as in (33) with

\[
S = \text{Im} \left[ \begin{array}{c} X_c \\ I_q \end{array} \right],
\]

and

\[
(S, (I, Q)) -\text{invariant iff } S \text{ is } Q \text{-invariant.}
\]

\[
PS + QS \subset \text{Im} \left[ \begin{array}{c} U_3 \\ V_3 \end{array} \right] = \text{Im} \left[ \begin{array}{c} I_n \\ V \end{array} \right].
\]

Proof. For \( P = I \), (42) becomes
(46) \[ E_{cl}S + F_{cl}S \subset \text{Im} \begin{bmatrix} I_n \\ X_o \end{bmatrix}. \]

(2) If \((E, F, G, H) \in OBC_q\) as in (33) and \(S\) is given by (45), then (46) holds, so \(S\) is \(q\)-dimensional, nonaxial, and \((E_{cl}, F_{cl})\)-invariant.

**Proof.** (1) From Lemma 12, there exist \(U\) and \(V\) such that
\[
\begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} = \begin{bmatrix} I \\ V \end{bmatrix} U,
\]
\[
\begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} = \begin{bmatrix} I \\ V \end{bmatrix} (AU + BH).
\]
Hence,
\[ E = VU, \quad F = VAU + VBH - GCU. \]

Setting \((X_c, Y_c, X_o, Y_o) = (U, -H, V, G)\) yields (33), (45), and (46).

(2) In view of (34), the result follows from the calculation
\[
E_{cl} \begin{bmatrix} X_c \\ I \end{bmatrix} = \begin{bmatrix} I \\ X_o \end{bmatrix} X_c,
\]
\[
F_{cl} \begin{bmatrix} X_c \\ I \end{bmatrix} = \begin{bmatrix} I \\ X_o \end{bmatrix} (AX_c - BY_c). \]

In practice, control engineers deal with real plants and need to design real compensators. In our context, we address two competing definitions of a “real OBC.” The first is that the compensator transfer function \(G_c\) has real coefficients and any realization in \(OBC_n\). The second is that \(G_c\) has a real realization in \(OBC_n\). Note that \((E, F, G, H)\) being real implies that \(G_c\) has real coefficients, so the second definition implies the first. For engineering problems, the weaker notion is adequate, since we need only a real \(G_c\) in order to physically implement the compensator. We will prove the surprising fact that *these two definitions are equivalent.* First, we need a lemma.

**Lemma 14.** If \(0 \leq q < n\) and \(V \in \mathbb{C}^{n \times q}\) has full rank, then there exist \(W \in \mathbb{R}^{n \times n-q}\) and nonsingular \(\Lambda \in \mathbb{R}^{n-q \times n-q}\) such that
\[
\det \begin{bmatrix} V & W \end{bmatrix} \neq 0,
\]
\[
\text{Im}(W\Lambda - AW) \subset \text{Im}B.
\]

**Proof.** We first prove that there exist \(\lambda_1 \in \mathbb{R}\) and \(w_1 \in \mathbb{R}^n\) such that
\[
\text{rank} \begin{bmatrix} V & w_1 \end{bmatrix} = q + 1,
\]
(47) \[
(\lambda_1 I - A) w_1 \in \text{Im}B.
\]
Suppose not. Then, for every \(\lambda_1 \in \mathbb{R}\), (47) has no solution \(w_1 \notin \text{Im}V\). If \(\lambda_1\) lies outside the spectrum \(\sigma(A)\) of \(A\), then
\[
(\lambda_1 I - A)^{-1} \text{Im}B \subset \text{Im}V.
\]
Let \( c \in \mathbb{C}^n - \{0\} \) be orthogonal to \( \text{Im} V \subset \mathbb{C}^n \). Denoting conjugate transpose by “\(^*\)”,

\[
c^* (\lambda_1 I - A)^{-1} B = 0.
\]

But \( \mathbb{R} - \sigma(A) \) is infinite, so

\[
\mathcal{G}(s) = c^* (sI - A)^{-1} B
\]

has infinitely many zeros, which implies \( \mathcal{G} \equiv 0 \). Since \((A, B)\) is controllable, any state vector \( x \in \mathbb{C}^n \) is reachable (using a complex input), so any output vector \( y \in \text{Im} c^* \) is also achievable. But \( \mathcal{G} \equiv 0 \) implies that only \( y = 0 \) is achievable, so \( \text{Im} c^* = 0 \), contradicting \( c \neq 0 \).

Proceeding inductively, we may construct \( \lambda_1, \ldots, \lambda_{n-q} \in \mathbb{R} - \{0\} \), \( w_1, \ldots, w_{n-q} \in \mathbb{R}^n \) such that

\[
\det \begin{bmatrix} V & w_1 & \cdots & w_{n-q} \end{bmatrix} \neq 0
\]

and \( (\lambda_j I - A) w_j \in \text{Im} B \) for \( j = 1, \ldots, n-q \). Setting

\[
\Lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-q} \end{bmatrix}, \quad W = \begin{bmatrix} w_1 & \cdots & w_{n-q} \end{bmatrix},
\]

we obtain

\[
\text{Im}(WA - AW) = \text{Im} \begin{bmatrix} (\lambda_1 I - A) w_1 & \cdots & (\lambda_{n-q} I - A) w_{n-q} \end{bmatrix} \subset \text{Im} B.
\]

**Theorem 15.** Let \( q = \mu(\mathcal{G}_c) \).

(1) \( \mathcal{G}_c \) has a realization in \( \text{OBC}_n \) iff it has a realization in \( \text{OBC}_q \).

(2) If \( \mathcal{G}_c \) has real coefficients and a realization in \( \text{OBC}_q \), then it has real realizations in \( \text{OBC}_q \) and \( \text{OBC}_n \).

**Proof.** (1) (Necessary) Let \( (E, F, G, H) \in \text{OBC}_n \) be a realization of \( \mathcal{G}_c \). Then there exist nonsingular \( M \) and \( N \) such that \((MEN, MFN, MG, HN) \in \text{OBC}_n \) has the form (29) with \((E_{22}, F_{22}, G_2, H_2)\) a minimal realization of \( \mathcal{G}_c \). From Theorem 13, part (2), there exist matrices \( U_i \) and \( V_i \) for \( i = 1, 2, 3 \) such that

\[
\begin{bmatrix}
I & 0 & 0 & 0 \\
0 & E_{11} & E_{12} & E_{13} \\
0 & 0 & E_{22} & E_{23} \\
0 & 0 & 0 & E_{33}
\end{bmatrix}
\begin{bmatrix}
U_1 & U_2 & U_3 \\
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
= \begin{bmatrix}
I \\
V_1 \\
V_2 \\
V_3
\end{bmatrix}
\begin{bmatrix}
U_1 & U_2 & U_3 \\
A & 0 & BH_2 \\
G_{1}C & F_{11} & F_{12} \\
G_{2}C & 0 & F_{23} \\
0 & 0 & 0 & F_{33}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I \\
V_1 \\
V_2 \\
V_3
\end{bmatrix}
\begin{bmatrix}
AU_1 & AU_2 + BH_2 & AU_3 + BH_3
\end{bmatrix}.
\]
Isolating rows 1 and 3 and column 2, we obtain

\[
\begin{bmatrix}
I & 0 \\
0 & E_{22}
\end{bmatrix}
\begin{bmatrix}
U_2 \\
I
\end{bmatrix}
= 
\begin{bmatrix}
I \\
V_2
\end{bmatrix}
U_2,
\]

\[
\begin{bmatrix}
A & BH_2 \\
GC_2 & F_{22}
\end{bmatrix}
\begin{bmatrix}
U_2 \\
I
\end{bmatrix}
= 
\begin{bmatrix}
I \\
V_2
\end{bmatrix}
(AU_2 + BH_2).
\]

From Theorem 13, part (1), \((E_{22}, F_{22}, G_2, H_2) \in OBC_q\).

(Sufficient) Let \((E_{22}, F_{22}, G_2, H_2) \in OBC_q\) and define

\[
\tilde{E}_{cl} = \begin{bmatrix}
I_n & 0 \\
0 & E_{22}
\end{bmatrix},
\tilde{F}_{cl} = \begin{bmatrix}
A & BH_2 \\
GC_2 & F_{22}
\end{bmatrix}.
\]

From Theorem 13, part (2), there exist \(U\) and \(V\) such that

\[
(48) \quad \tilde{E}_{cl} \tilde{S} + \tilde{F}_{cl} \tilde{S} \subset \text{Im} \begin{bmatrix}
I_n \\
V
\end{bmatrix},
\]

where

\[
(49) \quad \tilde{S} = \text{Im} \begin{bmatrix}
U \\
I_q
\end{bmatrix}
\]

is \((\tilde{E}_{cl}, \tilde{F}_{cl})\)-invariant. Hence,

\[
\tilde{E}_{cl} \tilde{S} + \tilde{F}_{cl} \tilde{S} \subset \text{Im} \begin{bmatrix}
P \\
Q
\end{bmatrix}
\]

for some \(P \in \mathbb{C}^{n \times q}\) with full rank and \(Q \in \mathbb{C}^{q \times q}\). Expressions (48) and (49) are equivalent to the matrix equations

\[
\begin{bmatrix}
I_n & 0 \\
0 & E_{22}
\end{bmatrix}
\begin{bmatrix}
U \\
I_q
\end{bmatrix}
= 
\begin{bmatrix}
P \\
Q
\end{bmatrix}\Phi,
\]

\[
\begin{bmatrix}
A & BH_2 \\
GC_2 & F_{22}
\end{bmatrix}
\begin{bmatrix}
U \\
I_q
\end{bmatrix}
= 
\begin{bmatrix}
P \\
Q
\end{bmatrix}\Psi
\]

for some \(\Phi\) and \(\Psi\). From Lemma 14, there exist \(W, \Lambda,\) and \(Z\) such that

\[
(50) \quad \det \begin{bmatrix}
P & W
\end{bmatrix} \neq 0,
\]

\[
WA - AW = BZ.
\]

Let

\[
(51) \quad E = \begin{bmatrix}
E_{22} & 0 \\
0 & I_{n-q}
\end{bmatrix}, \quad F = \begin{bmatrix}
F_{22} & -G_2CW \\
0 & \Lambda
\end{bmatrix},
\]

\[
(52) \quad G = \begin{bmatrix}
G_2 \\
0
\end{bmatrix}, \quad H = \begin{bmatrix}
H_2 & Z
\end{bmatrix}.
\]
The calculation
\[
\begin{bmatrix}
I & 0 & 0 \\
0 & E_{22} & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
U & W \\
I & 0 \\
0 & I
\end{bmatrix}
= \begin{bmatrix}
U & W \\
E_{22} & 0 \\
0 & I
\end{bmatrix}
= \begin{bmatrix}
P & W \\
Q & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Phi & 0 \\
0 & I
\end{bmatrix},
\]
shows that \(S\) is \((E_{cl}, F_{cl})\)-invariant. From (50) and Lemma 12, \(S\) is nonaxial. From Theorem 13, part (1), \((E, F, G, H) \in OBC_n\).

(2) In addition to the given realization in \(OBC_q\), \(G_c\) also has a real realization \((E_{22}, F_{22}, G_2, H_2) \in \Omega_q\). But \(G_c\) corresponds to only one orbit in \(\Omega_q\), so the two realizations are equivalent. Hence, \((E_{22}, F_{22}, G_2, H_2) \in OBC_q\). According to Lemma 14, \(W, \Lambda, \) and \(Z\) in (51)–(52) may be chosen real, making \((E, F, G, H) \in OBC_n\) real.

The most useful consequence of Theorem 15 is obtained by combining parts (1) and (2).

**Corollary 16.** If \(G_c\) has real coefficients and a realization in \(OBC_n\), then it has a real realization in \(OBC_n\).

In practice, since \(G_c\) invariably has real coefficients, Corollary 16 tells us that we need only look for an appropriate compensator from the real points in \(OBC_n\). This is a weaker condition than restricting \((X_c, Y_c, X_o, Y_o)\) to be real. Indeed, the next example demonstrates that some \(G_c\) with real coefficients can only be generated by nonreal \((X_c, Y_c, X_o, Y_o)\).

**Example 17.** Let \((A, B, C) = (0, 1, 1)\) and
\[
G_c(s) = -\frac{1}{s}.
\]
Setting
\[
(X_c, Y_c, X_o, Y_o) = (i, 1, -i, 1),
\]
we see that \(G_c\) is realized by the real \(OBC\)
\[
(E, F, G, H) = (1, 0, 1, -1).
\]
Since \(\mu(G_c) = 1\), all realizations of \(G_c\) lie in the orbit of \((E, F, G, H)\). Thus every realization has the form
\[
(\tilde{X}_c, \tilde{Y}_c, \tilde{X}_o, \tilde{Y}_o) = (MX_c, MY_c, X_oN, Y_oN),
\]
where $M, N \neq 0$. If $(\tilde{X}_c, \tilde{Y}_c, \tilde{X}_o, \tilde{Y}_o)$ is real, then $MX_c \in \mathbb{R}$. This implies $M = M_1i$ for some $M_1 \in \mathbb{R} - \{0\}$, yielding the contradiction

$$\tilde{Y}_c = MY_c = M_1i \notin \mathbb{R}.$$

The case of “static” compensators $G_c \in \mathbb{C}^{n \times n}$ is of special importance in control theory. Applying Theorem 15 to such a $G_c$ leads to the interesting result that every static compensator can be realized by an OBC (cf. [1, section V-D]).

**Corollary 18.** Every constant $G_c$ has a realization in $OBC_n$. If $G_c$ is real, then it has a real realization in $OBC_n$.

**Proof.** Let $q = \mu (G_c)$ and $\rho = \text{rank} G_c$, and factor $-G_c = H_2 G_2$, where $G_2 \in \mathbb{C}^{\rho \times p}$ and $H_2 \in \mathbb{C}^{m \times \rho}$. Then $G_2$ has independent columns and $H_2$ has independent rows. Setting $E_{22} = 0$ and $F_{22} = I$ yields

$$H_2 (sE_{22} - F_{22})^{-1} G_2 = -H_2 G_2 = G_c,$$

so $(0, I, G_2, H_2)$ is a realization of $G_c$. From (23), $G_c = G_{cf}$, so all minimal realizations of $G_c$ have the form $(E_f, I, G_f, H_f)$ with $E_f$ nilpotent. Since $G_c$ is constant, (24) indicates $G_c = -H_f G_f$. But the inner dimension of this factorization must be at least $\rho$, so $q = \rho$ and $(0, I_q, G_2, H_2) \in \Omega_q$. Since $BH_2$ has independent columns, there exists $W \in \mathbb{C}^{n \times n-q}$ such that

$$\det \begin{bmatrix} BH_2 & W \end{bmatrix} \neq 0.$$

Setting

$$S = \text{Im} \begin{bmatrix} 0 \\ I_q \end{bmatrix},$$

the calculation

$$\begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} 0 \\ I_q \end{bmatrix} = \begin{bmatrix} BH_2 & W \\ I_q & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} A \\ G_2 C \\ I_q \end{bmatrix} \begin{bmatrix} 0 \\ I_q \end{bmatrix} = \begin{bmatrix} BH_2 & W \\ I_q & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

shows that $S$ is a $q$-dimensional invariant subspace of the closed-loop pencil. We may set

$$U = 0, \quad V = \begin{bmatrix} I_q & 0 \end{bmatrix} \begin{bmatrix} BH_2 \\ W \end{bmatrix}^{-1}$$

in Lemma 12 to establish that $S$ is nonaxial. From Theorem 13, part (2),

$$(0, I, G_2, H_2) \in OBC_q.$$
5. $\lambda$-sections. Since $OBC_n$ and $ROBC_n$ are invariant under the group action (17), the definition of a singular OBC (33) contains considerable redundancy. In this section, we propose a simple way to remove this redundancy, bringing our development into better alignment with classical state-space theory.

Suppose we are willing to exclude from consideration all OBCs that generate a particular closed-loop eigenvalue $\lambda \in \mathbb{C}$ in (34). In control problems, this can typically be justified by the requirement of closed-loop stability. For example, in continuous-time problems involving asymptotic stability, $\lambda$ can be set to any value with $\text{Re}\lambda \geq 0$. For discrete-time, we may choose $|\lambda| \geq 1$. Other stability regions can be handled similarly.

Define the $\lambda$-domain

$$OBC^\lambda_n = \left\{ (X_oX_c, X_oAX_c - X_oBY_c - Y_oCX_c, Y_o, -Y_c) \in \Omega_n \mid \Delta_c(\lambda), \Delta_o(\lambda) \neq 0 \right\}.$$ 

From (35), the set $OBC^\lambda_n \subset OBC_n$ consists of all OBCs except those that generate $\lambda$ as a closed-loop eigenvalue. We may extend this idea in a natural way to include $\lambda = \infty$ by setting

$$OBC^\infty_n = \left\{ (X_oX_c, X_oAX_c - X_oBY_c - Y_oCX_c, Y_o, -Y_c) \in \Omega_n \mid \det X_c, \det X_o \neq 0 \right\}.$$ 

In view of Proposition 7, $OBC^\infty_n$ may be identified with the nonsingular OBCs of section 1.

Recall the definition of $ROBC_n$ from section 3. Obviously, $OBC^\lambda_n \subset ROBC_n$ for every $\lambda$. The next result shows that finitely many $OBC^\lambda_n$ cover $ROBC_n$.

**Proposition 19.** Let $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{C} \cup \{ \infty \}$ be distinct. Then

$$\bigcup_{i=1}^{n+1} OBC^\lambda_i = ROBC_n.$$ 

**Proof.** Suppose $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{C}$, and consider any point that lies in $ROBC_n - OBC^\lambda_i$ for every $i$. Then $\Delta_c(\lambda_i) = 0$ for every $i$, so $\deg \Delta_c > n$, contradicting (37). Now suppose $\lambda_j = \infty$. Then either $X_c$ or $X_o$ is singular. In the first case, $\Delta_c(\lambda_i) = 0$ for $i \neq j$, so $\deg \Delta_c \geq n$, which also contradicts (37). Singular $X_o$ is handled similarly by examining $\Delta_o$. $\Box$

For each $\lambda \neq \infty$, define the $\lambda$-section to be the set $\Sigma^\lambda_n \subset OBC_n$ of all members satisfying

$$X_o (A - \lambda I) - Y_o C = I.$$ 

Combining (37)–(38) with (53)–(54), we obtain

$$\Delta_c(\lambda) = \Delta_o(\lambda) = \det (-I) = (-1)^n,$$

so $\Sigma^\lambda_n \subset OBC^\lambda_n$. The solutions of (53) and (54) with nonsingular $X_c$ and $X_o$ all have the form

$$(X_c, Y_c) = \left( (A - \lambda I - BK)^{-1}, K (A - \lambda I - BK)^{-1} \right),$$ 

$$(X_o, Y_o) = \left( (A - \lambda I - LC)^{-1}, (A - \lambda I - LC)^{-1} L \right),$$
where $K$ and $L$ range over the matrices that make $A - \lambda I - BK$ and $A - \lambda I - LC$ nonsingular. The remaining solutions are obtained by taking $K$ and $L$ to infinity in all ways that yield convergence of $X_c$ and $X_o$. This amounts to a special case of the degenerating system transformation (31) with

$$M_k = (A - \lambda I - L_k C)^{-1}, \quad N_k = (A - \lambda I - BK_k)^{-1}.$$  

We may also define the $\infty$-section $\Sigma_n^\infty \subset \text{OBC}_n$ by replacing (53)–(54) with the conditions

$$\text{in (37) and (38)}, \quad F = X_c + X_o - X_o (A - 2\lambda I) X_c$$

in (33), and

$$M_{cl} E_{cl} N_{cl} = \begin{bmatrix} X_c & I \\ 0 & X_o \end{bmatrix},$$

$$M_{cl} F_{cl} N_{cl} = \begin{bmatrix} I + \lambda X_c & A \\ 0 & I + \lambda X_o \end{bmatrix}$$

in (35).

(2) Every compensator in $\text{OBC}_n^\infty$ is equivalent to a unique $(E, F, G, H) \in \Sigma_n^\infty$. In particular,

$$\Delta_c (s) = \det ((s - \lambda) X_c - I), \quad \Delta_o (s) = \det ((s - \lambda) X_o - I)$$

in (37) and (38),

$$F = A - BY_c - Y_o C$$

in (33), and

$$M_{cl} E_{cl} N_{cl} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix},$$

$$M_{cl} F_{cl} N_{cl} = \begin{bmatrix} A - BY_c & A \\ 0 & A - Y_o C \end{bmatrix}$$

in (35).

Proof. (1) Consider

$$\left( \bar{X}_o \bar{X}_c, \bar{X}_o A \bar{X}_c - \bar{X}_o B \bar{Y}_c - \bar{Y}_o C \bar{X}_c, \bar{Y}_o, -\bar{Y}_c \right) \in \text{OBC}_n^\lambda.$$
Then
\[
det \left( \lambda \tilde{X}_c - \left( A \tilde{X}_c - B \tilde{Y}_c \right) \right) \neq 0
\]
or equivalently,
\[
det \left( (A - \lambda I) \tilde{X}_c - B \tilde{Y}_c \right) \neq 0.
\]
Similarly,
\[
det \left( \tilde{X}_o (A - \lambda I) - \tilde{Y}_o C \right) \neq 0.
\]
Applying
\[
M = \left( \tilde{X}_o (A - \lambda I) - \tilde{Y}_o C \right)^{-1}, \quad N = \left( (A - \lambda I) \tilde{X}_c - B \tilde{Y}_c \right)^{-1}
\]
to (56) and setting
\[
(X_c, Y_c) = \left( \tilde{X}_c N, \tilde{Y}_c N \right), \quad (X_o, Y_o) = \left( M \tilde{X}_o, M \tilde{Y}_o \right),
\]
we obtain (33) with
\[
(A - \lambda I) X_c - BY_c = \left( (A - \lambda I) \tilde{X}_c - B \tilde{Y}_c \right) N = I,
\]
\[
X_o (A - \lambda I) - Y_o C = M \left( \tilde{X}_o (A - \lambda I) - \tilde{Y}_o C \right) = I.
\]
Hence, \((E, F, G, H) \in \Sigma^\lambda_n\).

To prove uniqueness, let \((E_1, F_1, G_1, H_1) \in \Sigma^\lambda_n\) be equivalent to \((E, F, G, H)\).

Then there exist nonsingular \(M\) and \(N\) such that
\[
(E_1, F_1, G_1, H_1) = (MX_oX_cN, M (X_oAX_c - X_oBY_c - Y_oCX_c) N, MY_o, -Y_cN).
\]

But this is the OBC generated by \((X_cN, Y_cN, MX_o, MY_o)\), so
\[
M = M (X_o (A - \lambda I) - Y_o C) = (MX_o) (A - \lambda I) - (MY_o) C = I, \tag{57}
\]
\[
N = ((A - \lambda I) X_c - BY_c) N = (A - \lambda I) (X_c N) - B (Y_c N) = I, \tag{58}
\]
which implies \((E_1, F_1, G_1, H_1) = (E, F, G, H)\).

For \((E, F, G, H) \in \Sigma^\lambda_n\), (53) and (54) imply
\[
AX_c - BY_c = I + \lambda X_c, \quad X_oA - Y_o C = I + \lambda X_o. \tag{59}
\]

Hence,
\[
sX_c - (AX_c - BY_c) = (s - \lambda) X_c - I, \quad sX_o - (X_oA - Y_o C) = (s - \lambda) X_o - I,
\]
from which we obtain the form of \( \Delta_c \) and \( \Delta_o \). From (59),

\[-Y_o C = I + X_o (\lambda - A) , \]

\[F = X_o AX_c - X_o BY_c - Y_o CX_c \]

\[= X_o ((A - \lambda I) X_c - BY_c) + \lambda X_o X_c + (I - X_o (A - \lambda I)) X_c \]

\[= X_o + X_c - X_o (A - 2\lambda I) X_c , \]

\[M_c F_{cd} N_{cl} = \begin{bmatrix} AX_c - BY_c & \lambda A \\ 0 & X_o A - Y_o C \end{bmatrix} = \begin{bmatrix} I + \lambda X_c & \lambda A \\ 0 & I + \lambda X_o \end{bmatrix}. \]

(2) Consider

\[(60) \quad (\tilde{X}_o \tilde{X}_c, \tilde{X}_o AX_c - \tilde{X}_o BY_c - \tilde{Y}_o C \tilde{X}_c, \tilde{Y}_o, -\tilde{Y}_c) \in OBC_n^\infty. \]

Then \( \det \tilde{X}_c, \det \tilde{X}_o \neq 0 \). Applying \( M = \tilde{X}_o^{-1} \) and \( N = \tilde{X}_c^{-1} \) to (60) and setting

\[(X_c, Y_c) = \left( \tilde{X}_c N, \tilde{Y}_c N \right) = \left( I, \tilde{Y}_c \tilde{X}_c^{-1} \right), \quad (X_o, Y_o) = \left( \tilde{M} \tilde{X}_o, \tilde{M} \tilde{Y}_o \right) = \left( I, \tilde{X}_o^{-1} \tilde{Y}_o \right), \]

we obtain (33) with \( X_c = X_o = I \), so \((E, F, G, H) \in \Sigma_n^\infty. \) Uniqueness is proven as in (1), but replacing (57)–(58) by

\[M = MX_o = I, \quad N = X_c N = I. \]

For \((E, F, G, H) \in \Sigma_n^\infty, \) the forms of \( \Delta_c, \Delta_o, F, M_c F_{cd} N_{cl}, \) and \( M_c F_{cd} N_{cl} \) follow immediately by substituting (55) into (37), (38), (33), and (35).

The form of \( \Delta_c \) and \( \Delta_o \) in Theorem 20, part (1) gives us the following result.

**Corollary 21.** For any compensator \((E, F, G, H) \in \Sigma_n^\lambda, \) the eigenvalues of \((E_{cd}, F_{cd}) \) all have the form \( \frac{1}{\eta} + \lambda, \) where \( \eta \) ranges over the nonzero eigenvalues of \( X_c \) and \( X_o. \)

Theorem 20 serves to simplify the design of the pencils \((X_c, AX_c - BY_c) \) and \((X_o, X_o A - Y_o C) \). For a given notion of stability, we first exclude an appropriate \( \lambda \) and restrict our attention to \( \Sigma_n^\lambda. \) If \( \lambda \neq \infty, \) we look for solutions of (53) and (54) such that the eigenvalues \( \eta \) of \( X_c \) and \( X_o \) place \( \frac{1}{\eta} + \lambda \) in desired locations. We have already explored the special case \( \lambda = 0 \) in some detail in [6] and [7]. For \( \lambda = \infty, \) the problem reduces to the classical state-feedback framework of section 1. The next result specializes Theorem 10 to \( \Sigma_n^\lambda. \)

**Theorem 22.** A member of \( \Sigma_n^\lambda \) has unit index iff \( \text{ind} X_c = \text{ind} X_o = 1, \) and \( \text{Ker} X_o \subset \text{Im} X_c. \)

**Proof.** If \( \lambda = \infty, \) the result follows trivially from (55). Let \( \lambda \neq \infty. \) From Theorem 20,

\[\text{deg} \Delta_c (s) = \text{deg} \Delta_c (s + \lambda) = \text{deg} \ det (sX_c - I) , \]

so (39) holds iff \( \text{ind} (X_c, I) = 1. \) Similarly, (40) is equivalent to \( \text{ind} (X_o, I) = 1. \) The result follows by Theorem 10.

We end this section with some basic information about \( \Sigma_n^\lambda \) for \( \lambda \neq 0. \)

**Proposition 23.** Let \( X_c \in \mathbb{C}^{m \times n}. \) The following are equivalent:

1. There exists \( Y_c \in \mathbb{C}^{m \times n} \) such that \((X_c, Y_c) \) satisfies (53).
(2) Im \((A - \lambda I) X_c - I\) \(\subset\) ImB.
(3) Im \(\begin{bmatrix} X_c \\ B \end{bmatrix}\) \(\subset\) Im \(\begin{bmatrix} A - \lambda I \\ I \end{bmatrix}\).
(4) rank \(\begin{bmatrix} X_c (A - \lambda I) - I & X_cB \end{bmatrix}\) = \(m\).

In this case, \(Y_c\) in (1) is unique, Ker\(X_c\) \(\subset\) ImB, and \((A - \lambda I)\)Im\(X_c\) + ImB = \(\mathbb{C}^n\).

Proof. (1)\(\iff\) (2): Condition (2) says that there exists \(Y_c\) such that \((A - \lambda I) X_c - I = BY_c\), which is the same as (1).

(2)\(\iff\) (3): Condition (3) is equivalent to saying that, for each \(x\), there exist \(y\) and \(z\) such that

\[(A - \lambda I) \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} x \\ X_c x \end{bmatrix} \quad (61)\]

Writing out the equations, we see that (61) is the same as

\[(A - \lambda I) X_c - I x = -Bz,\]

which is a restatement of (2).

(3)\(\iff\) (4): From matrix theory,

\[\text{rank} \begin{bmatrix} X_c (A - \lambda I) - I & X_cB \end{bmatrix} = \text{rank} \begin{bmatrix} X_c - I & A - \lambda I \end{bmatrix}\]

\[\geq \text{rank} \begin{bmatrix} A - \lambda I & B \\ I & 0 \end{bmatrix} = (2n - \text{rank} \begin{bmatrix} X_c & -I \end{bmatrix}) = (n + m) - (2n - n) = m\]

with equality iff

\[\text{Im} \begin{bmatrix} 1 \\ X_c \end{bmatrix} = \text{Ker} \begin{bmatrix} X_c & -I \end{bmatrix} \subset \text{Im} \begin{bmatrix} A - \lambda I & B \\ I & 0 \end{bmatrix}\].

Now assume (1) holds. To prove uniqueness of \(Y_c\), rewrite (53) as

\[BY_c = (A - \lambda I) X_c - I.\]

Since \(B\) has full rank, we may apply any left inverse \(B^\dagger\) of \(B\) to obtain

\[Y_c = B^\dagger ((A - \lambda I) X_c - I).\]

If \(x \in \text{Ker} X_c\), then

\[x = ((A - \lambda I) X_c - BY_c) x = -BY_c x \in \text{Im} B.\]

Finally,

\[(A - \lambda I) \text{Im} X_c + \text{Im} B \supset \text{Im} ((A - \lambda I) X_c) + \text{Im} (BY_c) \supset \text{Im} ((A - \lambda I) X_c - BY_c) = \mathbb{C}^n.\]

The dual result is proven similarly.

**Proposition 24.** Let \(X_o \in \mathbb{C}^{n \times n}\). The following are equivalent:

(1) There exists \(Y_o \in \mathbb{C}^{n \times p}\) such that \((X_o, Y_o)\) satisfies (54).
(2) Ker \((X_o (A - \lambda I) - I) \supset \text{Ker} C\).
(3) Ker \(\begin{bmatrix} 1 \\ X_o \end{bmatrix} \supset \text{Ker} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\).
(4) rank \(\begin{bmatrix} (A - \lambda I) X_o - I \end{bmatrix}_{C X_o} = p\).

In this case, \(Y_o\) in (1) is unique, \(\text{Im} X_o \supset \text{Ker} C\), and \((A - \lambda I)^{-1} (\text{Ker} X_o) \cap \text{Ker} C = 0\).
6. Design issues and conclusions. We are now ready to apply the principles established in sections 3–5 to a numerical design problem.

**Example 25.** Let

\[(A, B, C) = \left( \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).\]

We are interested in continuous-time stability, so we may set \(\lambda = 0\). The structure of points in \(\Sigma_0^3\) can be obtained as outlined in Propositions 23 and 24, yielding

\[(62) \quad X_c = \begin{bmatrix} \frac{1}{2} (y_{c1} + 4) & \frac{1}{2} (y_{c2} - 3) & \frac{1}{2} (y_{c3} + 1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y_c = \begin{bmatrix} y_{c1} & y_{c2} & y_{c3} \end{bmatrix}, \]

\[(63) \quad X_o = \begin{bmatrix} \frac{1}{2} (y_{o11} + 1) & y_{o12} - \frac{3}{2} (y_{o11} + 1) & \frac{1}{2} (y_{o11} + 1) \\ 2y_{o21} + 1 & y_{o22} - \frac{3}{2} y_{o21} & \frac{1}{2} y_{o21} \\ 2y_{o31} & y_{o32} - \frac{3}{2} y_{o31} + 1 & \frac{1}{2} y_{o31} \end{bmatrix}, \quad Y_o = \begin{bmatrix} y_{o11} & y_{o12} \\ y_{o21} & y_{o22} \\ y_{o31} & y_{o32} \end{bmatrix}.\]

If we wish to have \((E, F, G, H)\) real, we must restrict all parameters to be real, since \(Y_o = G\) and \(Y_c = -H\). For simplicity, we consider only compensators where \(\text{rank}X_c = 2\) and \(\text{rank}X_o = 1\), as in Proposition 8. This requires \(y_{c3} = y_{o11} = y_{o32} = -1\) and \(y_{o12} = y_{o31} = 0\). Applying the conditions of Theorem 22 yields the further constraints

\[(64) \quad y_{c2} \neq 3, \quad 3y_{o21} \neq 2y_{o22},\]

\[(65) \quad \begin{bmatrix} \frac{1}{2} y_{c1} + \frac{5}{2} y_{o21} + y_{o21} y_{c1} + y_{o22} + 2 \frac{1}{2} y_{c2} + y_{o21} y_{c2} - \frac{5}{2} y_{o21} - \frac{3}{2} \end{bmatrix} = X_o X_c = 0.\]

Equation (65) is solvable only when \(y_{c2} \neq \frac{5}{2}\), in which case

\[(66) \quad y_{o21} = \frac{y_{c2} - 3}{2y_{c2} - 5}, \quad y_{o22} = -\frac{1}{2} \frac{y_{c1} + 3y_{c2} - 5}{2y_{c2} - 5}.\]

Combining (64) and (66) restricts \(y_{c1} \neq -4\). Writing \(\alpha = y_{c1}\) and \(\beta = y_{c2}\) yields the 2-parameter family of 4-tuples

\[X_c = \begin{bmatrix} \frac{1}{2} (\alpha + 4) & \frac{1}{2} (\beta - 3) & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Y_c = \begin{bmatrix} \alpha & \beta & -1 \end{bmatrix}, \]

\[X_o = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} \frac{\beta - 3}{2y_{c2} - 5} & \frac{1}{2} \frac{\alpha + 3 \beta - 5}{2y_{c2} - 5} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Y_o = \begin{bmatrix} -\frac{1}{2} \frac{\beta - 3}{2y_{c2} - 5} & -\frac{1}{2} \frac{\alpha + 3 \beta - 5}{2y_{c2} - 5} & 0 \end{bmatrix}.\]
By Theorem 20, part (1) the resulting OBCs are given by

\[ E = 0, \quad F = X_c + X_o - X_o AX_c = \begin{bmatrix}
\frac{1}{2} (\alpha + 4) & \frac{1}{2} (\beta - 3) & 0 \\
\frac{3\alpha - \alpha\beta - 4\beta + 10}{2\beta - 5} & \frac{1}{2} (\beta - 3) & 0 \\
\frac{3\alpha - \alpha\beta - 4\beta + 10}{2\beta - 5} & 1 & - \frac{1}{2} \beta - 3
\end{bmatrix}, \]

\[ G = Y_o, \quad H = -Y_c. \]

The compensator \((E, F, G, H)\) is guaranteed to have unit index and thus a proper (in fact, static) transfer function

\[ G_c(s) = \frac{1}{(\alpha + 4)(\beta - 3)} \begin{bmatrix}
6\alpha - 2\alpha\beta + 4 & -\alpha^2 + \alpha - 3\alpha\beta + 10\beta - 4\beta^2 - 10
\end{bmatrix}. \]

The closed-loop eigenvalues are the roots of

\[ \Delta_c(s) = \frac{1}{2} (\beta - 3) s^2 + \frac{1}{2} (\alpha + 4) s - 1, \]

\[ \Delta_o(s) = -\frac{1}{2} \frac{\alpha + 4}{2\beta - 5} s - 1. \]

Hence, the closed-loop system is continuous-time (asymptotically) stable iff \( \alpha < -4 \) and \( \beta < \frac{5}{2} \).

Example 25 illustrates several issues that are typically confronted in the design process: (1) By Corollary 16, we may confine ourselves to solutions that produce real \((E, F, G, H)\). On the other hand, as demonstrated in Example 17, restricting \(X_c\) and \(X_o\) to be real limits the OBCs that can be generated. In the example, this problem did not arise, since \(X_c\) and \(X_o\) in (62)–(63) are polynomials in \(Y_c\) and \(Y_o\) with real coefficients. (2) Properness of \(G_c\) is achieved by applying the conditions of Theorem 22. (3) We had to resort to ad hoc methods to ensure closed-loop stability. Clearly, an extended theory is called for which would provide a comprehensive and systematic method to guarantee stability. Ideally, this would take the form of a parametrization of all real, proper points in \(\Sigma^o_n\) with desirable closed-loop eigenvalues. Developing such a theory is a topic for further research.

Another important issue confronting our work is the degree to which OBCs cover the space of all compensators. Besides the fact that we are working within the linear, time-invariant, state-space framework, Example 4 demonstrates that not every compensator is realizable by an OBC. This fact raises the concern that substantially better closed-loop performance might be achievable through a non-OBC. In an upcoming paper, we will lay such fears to rest by proving that \(OBC_n\) actually contains an open and dense subset of \(\Omega_n\). Hence, any optimal compensator can be approximated with OBCs, resulting in at least \(\varepsilon\)-optimal performance.

REFERENCES