The Adaptive Notch Filter with Controlled Null Width

Yung-Yi Wang
Department of Computer and Communications Engineering
St. John’s University
Taipei, Taiwan
Email: yywang@mail.sju.edu.tw

Abstract—The design of the finite impulse response (FIR) adaptive notch filter with controlled null width is expressed as a derivatively constrained quadratic optimization problem. The problem is then transformed into an unconstrained one by choosing a null matrix orthogonal to the derivative constraint matrix. In this paper, wavelet filters are employed to construct the null matrix. Taking advantage of the vanishing moment property of the wavelet filters, the proposed method can govern the null width of the notch filter for eliminating the intractable interference of the wavelet filters. In this paper, wavelet filters are employed to construct the null matrix. Taking advantage of the vanishing moment property of the wavelet filters, the proposed method can govern the null width of the notch filter for eliminating the intractable interference of the wavelet filters.

Keywords: notch filters, wavelet filters, reduced rank approximation

I. INTRODUCTION

The notch filters, which null out the unwanted interference of prescribed frequencies while retaining the integrity of the signal of interest, have found various applications in digital signal processing, such as digital image processing, parameter estimation, and etc. However, for the cases of the frequency components of the interference varying slightly with time as well as grouping as clusters at a center frequency, a notch filter with broad null width is required to carry out the interference rejection. Among the previous works in broadening the null width of the notch filter [1]-[2], the frequency domain method utilizing the minimum-mean-square-error (MMSE) [2] provides a simple yet effective approach on the design of the notch filter with controlled null width. The MMSE-based notch filter exploits high order derivative constraints with respect to its frequency response at the prescribed frequency to make it less sensitive to the frequency interference variation. Therefore, the design of the notch filter with controlled null width can be formulated as a constrained optimization problem, and then, by choosing an appropriate null matrix orthogonal to the constraint matrix, can be further transformed to an unconstrained problem.

In this paper, the M-band P-regular wavelet filters [3]-[4] are employed to constitute the null matrix. By using the subband decomposition capability of the wavelet filters, a reduced-rank wavelet-based method is presented to implement the FIR notch filter with controlled null width. Taking advantage of the vanishing moment property of the wavelet filters, the proposed approach controls the null width of the notch filter by adjusting the regularity of the wavelet filters. Simulation results show that the new method can offer comparable performance as those of the existing full-rank-based ones but with substantially reduced computational complexity, and thus provides a promising alternative to the existing works.

II. PROBLEM FORMULATION

Consider an N-tap FIR filter with weights \( w = [w_1, \cdots, w_N]^T \), in which the superscript \( T \) denotes the transpose of a matrix; then the frequency response of the filter is

\[
H(f) = \sum_{k=1}^{N} w_k e^{-j2\pi(f-k)Ts},
\]

(1)

where \( a(f) = [1, e^{j2\pi fT_s}, \cdots, e^{j2\pi f(N-1)T_s}] \), in which \( T_s \) denotes the intertap delay spacing of the filter; the superscript \( H \) denotes the transpose and complex conjugate operation. The basic idea behind the notch filter is to minimize the MMSE between the outputs of a unit-gain-all-pass filter and the notch filter over a frequency band of interest. In addition, the nullity of the notch filter, centered at a prescribed frequency \( f_0 \), is determined with the ability of the null width adjustment. By forcing the derivative terms of different orders to zeros for rendering the notch filter less sensitive to the frequency perturbation, the constrained optimization problem can be described as

\[
\min_{w} \frac{1}{f_u - f_l} \int_{f_l}^{f_u} \left| e^{-j2\pi f \tau_0} - H(f) \right|^2 \, df,
\]

subject to \( H(f_0) = 0, \frac{d}{df} H(f_0) = 0, \cdots, \frac{d^{P-1}}{df^{P-1}} H(f_0) = 0 \)

(2)

where \( f_l \) and \( f_u \) represent the upper and the lower cut off frequencies of the frequency band of interest, respectively, \( e^{-j2\pi f \tau_0} \) is the linear phase which can be optimized to achieve a better unity gain response outside the notch frequency band, and \( f_0 \) denotes the center frequency of the notch frequency band. For the sake of notational conciseness, also without loss of generality, we assume \( \tau_0 = 0 \), and \( f_0 = 0 \) in the following derivation since the design of notch filter centered at arbitrary \( f_0 \) can be easily generalized via a frequency translation operation. Substituting (1) into (2), the mean-square-error (MSE) of the notch filter with weights \( w = [w_1, \cdots, w_N]^T \) can be expressed as
\[
J(w) = \frac{1}{f_a - f_l} \int_{f_l}^{f_u} |e^{-j2\pi f \tau_0} - a^H(f)w|^2 df \quad (3)
\]
\[
= 1 - u^Hw - w^Hu + w^HQw \quad (4)
\]
where \( Q = [q_{m,n}] \) is an \( N \times N \) positive semi-definite matrix with \( q_{m,n} = e^{j\pi(m-n)(f_l+f_u)} \text{sinc}(f_u - f_l) (m-n) T_s \) in which the sinc function is defined as \( \text{sinc}(x) = \frac{\sin \pi x}{\pi x} \), and \( u = \begin{bmatrix} u_1, \ldots, u_N \end{bmatrix}^T \) is an \( N \times 1 \) vector with \( u_i = e^{j\pi(i-1)T_s(f_l+f_u)} \text{sinc}(f_u - f_l) (i-1) T_s \). With (3), (2) can be written as a quadratic optimization problem
\[
\min_w J(w) \quad \text{subject to } CW = 0 \quad (5)
\]
where \( C = \begin{bmatrix} 1_N, c_1, \ldots, c_{P-1} \end{bmatrix}^T \)
\[
\text{(6)}
\]
with
\[
c_p = \begin{bmatrix} 0, 1, \ldots, (N-1)^p \end{bmatrix}^T ; p = 1, \ldots, P - 1
\]
\[
\text{(7)}
\]
where \( 1_N \) denotes the all-one vector with dimension \( N \times 1 \).

### III. The Proposed Approach

In this section, we first derive the optimal solution of (5) in the null space of the constraint matrix \( C \), denoted as \( N(C) \), and then investigate the properties of the wavelet-based null matrix for the proposed notch filter design.

#### A. The subspace-based solution

From (5), due to the fact that the feasible weight vector \( w \) lies in \( N(C) \), the weight vector hence can be represented as a linear combination of the basis vectors of \( N(C) \) as
\[
w = Sw_a \quad (8)
\]
where \( S \) denotes the null matrix of size \( N \times (N-P) \) consisting of the basis vectors of \( N(C) \) as its column vectors. The vector \( w_a \) of size \( (N-P) \times 1 \) denotes the corresponding coefficient vector which is to be determined for minimizing \( J(w) \). Note that \( CS = 0 \). Using (8), (5) can be reformulated to be an unconstrained optimization problem as
\[
\min_{w_a} 1 - u^HSw_a - w_a^HS^H u + w_a^HS^HQsw_a \quad (9)
\]
and the optimal solution of (9) is readily solved as
\[
w_a^{opt} = (S^HQS)^{-1} Su \quad (10)
\]
Substituting (8) and (10) into (3) yields
\[
J_{min} = 1 - u^HS(S^HQS)^{-1} S^Hu \quad (11)
\]
which can be used to assess the performance of the notch filters based on different \( S \), especially for the reduced-rank cases with the null matrix being selected as a subset of the column space of \( S \). A well-defined reduced-rank null matrix not only offers similar performance as does the full-rank null matrix \( S \), which is obtained by methods such as the QR decomposition and the singular value decomposition (SVD), but also substantially mitigates the computational complexity in the calculation of the weight vector \( w_a^{opt} \).

#### B. The regular \( M \)-band wavelets

The wavelet is a set of basis functions, with dilation and translation structures, of the space of the square integrable functions, \( L^2(R) \). The \( M \)-band orthonormal wavelet can be generated via the dilation and translation of a set of prototype wavelets \( \{w_m(t) ; m = 1, \ldots, M-1\} \) as \( \{M^2w_m(M^2t - k) ; i, k \in Z\} \). Under the framework of multiresolution analysis, the construction of the regular \( M \)-band wavelet functions can be carried out by first determining the scaling function \( s(t) \) which satisfies the following equation [5]
\[
s(t) = \sqrt{M} \sum_{k=0}^{L} \phi(k) s(Mt - k) \quad (12)
\]
where \( [\phi(0), \ldots, \phi(L)] \) represents the weights of the scaling filter, which associates with a low pass filter. The prototype wavelets \( \{w_m(t)\} \) can accordingly be decided by the following equation
\[
w_m(t) = \sqrt{M} \sum_{k=0}^{L-1} \psi_m(k) w_m(Mt - k) , m = 1, \ldots, M-1 \quad (13)
\]
where \( [\psi_m(0), \ldots, \psi_m(L-1)] \) denotes the weights of the \( m \)th wavelet filter and satisfies [6]
\[
\sum_{k=0}^{L-1} \psi_m(k + Ml) \psi_n(k) = \delta(l) \delta(m-n) \quad (13)
\]
where \( \delta(l) \) is the Kronecker delta function defined as \( \delta(0) = 1 \) and \( \delta(l) = 0 \), if \( l \neq 0 \). Contrary to the scaling filter, each wavelet filter corresponds to a bandpass/ high pass filter carrying out the subband decomposition of the input signal.

#### C. The wavelet-based null matrix

Before constructing the wavelet-based null matrix, we first introduce the vanishing moment property of the wavelets, which is inherited from the regularity constraint during their generation process and is particularly useful to our study. Consider a set of \( M \)-band, \( P \)-regular wavelet filters with coefficients \( [\psi_m(0), \ldots, \psi_m(MP-1)] \), \( m = 1, \ldots, M-1 \), the regularity condition imposed on the generation of the wavelets ensures that the \( M \)-band, \( P \)-regular wavelet filters have \( P \) vanishing moments expressed as
\[
\sum_{k=0}^{L-1} k^r \psi_m(k) = 0 , m = 1, \ldots, M-1 \quad (14)
\]
where \( r = 0, \ldots, P - 1 \). By using the binomial formula, \( (k + k_0)^r = \sum_{n=0}^{r} \binom{r}{n} k^{r-n} k_0^n \), apparently (14) can be generalized as
where \( k_0 \) is an integer. Based on a set of \( M \)-band \( P \)-regular wavelet filters, we can constitute the null matrix as

\[
\mathbf{S}_{\text{wav}} = \left[ \Psi_1, \ldots, \Psi_{M-1} \right]_{N \times \left( \left\lfloor \frac{N-MP}{M} \right\rfloor +1 \right)(M-1)} \tag{16}
\]

with

\[
\Psi_m = \begin{bmatrix}
\psi_m(0) & 0_M & \cdots & 0_M \\
\psi_m(L-1) & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \psi_m(L-1)
\end{bmatrix},
\]

where \( m = 1, \ldots, M-1 \), and \( \lfloor \alpha \rfloor \) represents the largest integer less than \( \alpha \). \( \Psi_m \) is of size \( N \times \left( \left\lfloor \frac{N-MP}{M} \right\rfloor +1 \right) \). The wavelet-based null matrix \( \mathbf{S}_{\text{wav}} \) has two important properties: 1) using (13), we can infer that every column of \( \Psi_m \) and \( \Psi_n \) are orthonormal. It follows that \( \Psi_m \Psi_n^T = \delta(m-n)I_{\left( \left\lfloor \frac{N-L}{M} \right\rfloor +1 \right) \times N} \) and \( \mathbf{S}_{\text{wav}} \mathbf{S}_{\text{wav}}^T = I_{\left( \left\lfloor \frac{N-L}{M} \right\rfloor +1 \right)(M-1) \times N} \). 2) Taking advantage of the vanishing moment property in Eq. (15), it is obvious that the inner product of the \( i \)-th row of \( \Psi \) and the \( j \)-th column of \( \Psi_m \) is \( \sum_{k=0}^{MP-1} (k - j - 1)M^{-1} \psi_m(k) = 0 \), \( m = 1, \ldots, M-1 \), which implies that the matrix \( \mathbf{S}_{\text{wav}} \) is orthogonal to the constraint matrix \( \mathbf{C} \) given in (6), i.e., \( \mathbf{C} \mathbf{S}_{\text{wav}} = 0 \).

Fig. 1 illustrates the structure of the proposed wavelet-based adaptive notch filter. As shown in the figure, the proposed wavelet-based approach actually decomposes the notch filter as a linear combination of the wavelet filters. The null matrix in (16) is represented as a filter bank which consists of the regular \( M \)-band wavelet filters. The output of the filter bank is then summed up with coefficients given in (10) to achieve the MMSE output of the notch filter. Note, the length of the weight vector in (10) is reduced to \( \left( \left\lfloor \frac{N-MP}{M} \right\rfloor +1 \right)(M-1) \), and accordingly the weight vector of the wavelet-based notch filter can be expressed as

\[
\mathbf{w}_{\text{wav}} = \mathbf{S}_{\text{wav}} \left( \mathbf{S}_{\text{wav}}^H \mathbf{Q} \mathbf{S}_{\text{wav}} \right)^{-1} \mathbf{S}_{\text{wav}} \mathbf{u}. \tag{18}
\]

The computational complexity of the proposed approach in calculating the weight vector is reduced to \( O \left( \left\lfloor \frac{N-MP}{M} \right\rfloor +1 \right)^3 (M-1)^3 \), compared to \( O \left( (N-P)^3 \right) \) of the full-rank techniques such as SVD and QR. The proposed approach is not restricted to deal with the filter with a single notch. Filters with multiple notches can be achieved by cascading several one-notch filters each corresponding to a specific notch frequency. In addition, different null width requirement can be imposed on each individual notch frequency by simply adjusting the parameters \( M \) and \( P \) of the associated wavelet filter. In that way, the null matrix of the notch filter with \( D \) notches can be generalize as

\[
\mathbf{S}_{\text{wav}}^{(d)} = \begin{bmatrix}
\mathbf{S}_{\text{wav}}^{(1)} & \cdots & \mathbf{S}_{\text{wav}}^{(D)}
\end{bmatrix}, \tag{19}
\]

with

\[
\mathbf{S}_{\text{wav}}^{(d)} = \text{diag} \left\{ 1, \ldots, e^{-j2\pi f_d(N-1)} \right\} \mathbf{S}_{\text{wav}}, \tag{20}
\]

where \( f_d \) denotes the \( d \)-th notch frequency, and \( d = 1, \ldots, D \). The size of \( \mathbf{S}_{\text{wav}} \) is \( N \times \sum_{d=1}^{D} \left( \left\lfloor \frac{N-MP_d}{M_d} \right\rfloor +1 \right)(M_d-1) \), where \( M_d \) represents the number of bands of the wavelet filter associated with the \( d \)-th notch frequency, and \( P_d \) denotes the corresponding regularity. Apparently, for a fixed filter length \( N \), the maximum number of notches \( D \) is limited by \( \sum_{d=1}^{D} \left( \left\lfloor \frac{N-MP_d}{M_d} \right\rfloor +1 \right)(M_d-1) < N \), otherwise the term \( \mathbf{S}^H \mathbf{Q} \mathbf{S} \) becomes singular.

D. The LMS algorithm

With the output of the null matrix, the least mean square (LMS) can be exploited to adaptively find the optimal weight vector \( \mathbf{w}_{\text{wav}} \) as it is illustrated in Fig. 1. Accordingly, the adaptation of the \( (n+1)^{\text{th}} \) weight vector can be obtained as

\[
\mathbf{w}_{\text{wav}}(n+1) = \mathbf{w}_{\text{wav}}(n) + \mu \cdot e(n) \mathbf{u}(n),
\]

where \( \mu \) denotes the step size of the adaptation, \( \mathbf{u}(n) = \mathbf{S}^H \mathbf{x}(n) \) is the output of the null matrix, and \( e(n) = x(n) - (\mathbf{w}_{\text{wav}}(n))^H \mathbf{u}(n) \) represents the error signal.

IV. SIMULATIONS AND DISCUSSIONS

Consider a FIR notch filter with \( N = 30 \) weights. The intertap delay interval of the notch filter is set as \( T_s = 1 \) ms. The overall frequency band of the experiment is normalized to unity with respect to the band width of the observation window. The normalized frequency band of interest of the FIR notch filter is set as \( f_1 = 0 \), and \( f_a = 0.5 \), respectively. A set of wavelet filters with fixed \( M = 5 \), and various values of \( P = 1, 2, 3 \), are selected to construct the null matrix as given in (16), respectively. Fig. 2 shows the frequency responses of the notch filter with a center frequency \( f_0 = 0.2 \). As shown in the figure, the null width of the notch filter is dominated by the regularity of the employed wavelet filters, and a broadened null width can be achieved as the value of \( P \) increases. On the other hand, to explore the effect of the number of bands \( M \), Fig. 2-3 illustrates the frequency responses and the learning curves of the notch filter corresponding to a fixed \( P = 2 \), and various choices of \( M = 4, 5 \), and 6, respectively. It is apparent that, for a fixed regularity \( P \), the null widths of the notch filter are almost unchanged with respect to various selection of \( M \). We thus infer that the only factor on which the null width depends is the regularity of the wavelet filters. To further investigate the effect of the parameters of the wavelet filters (i.e., \( M \) and \( P \)), Fig. 4-5 demonstrates the resulting learning curves and the frequency responses of the notch filters corresponding to different selections of \( P \). As shown in the figure, for a fixed \( P \), an increasing \( M \) results in a decreasing MMSE, while for a fixed \( M \), a larger regularity \( P \) gives rise to a bigger MMSE on
account of the broadened null width generated by the wavelet filters with high regularity.

V. Conclusions

A wavelet-based design on the notch filter with adjustable null width is proposed in this paper. With the vanishing moment properties, the $M$-band, $P$-regular wavelet filters are exploited to construct the null matrix of the notch filter. In addition to having less computational complexity, the wavelet-based notch filter demonstrates similar frequency response as that of the notch filter with full-rank null matrix. The null width of the notch filter is controlled by the regularity $P$, whereas the associated MMSE is dominated by the number of bands $M$ of the employed wavelet filters.

REFERENCES


Fig. 5: The amplitude spectra of the notch filters corresponding to $P=2$, $M=4$, 5, and 6, respectively.