# A Simple Algorithm for Solving the Power Domination Problem on Grid Graphs

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## Abstract

*The power domination problem is a variant of the classical domination problem in graphs and is defined as follows. Given an undirected graph $G = (V, E)$, the problem is to find a minimum vertex set $P \subseteq V$, called the power dominating set of $G$, such that all vertices in $G$ are observed by the vertices of $P$. Herein, a vertex observes itself and all its neighbors, and if an observed vertex has all but one of its neighbors observed, then the remaining neighbor becomes observed as well. The minimum cardinality of a power dominating set of a graph is its power domination number. In* [1], *Dorfling and Henning determined the power domination number of an $n \times m$ grid graph. Their proof provides an algorithm for solving such a problem on grid graphs. In this paper, we present a simpler algorithm for solving the same problem.*

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## 1. Introduction

Electric power companies need to continually monitor their system's state as defined by a set of state variables (for example, the voltage magnitude at loads and the machine phase angle at generators [3]). One method of monitoring these variables is to place phase measurement units (abbreviated PMUs) at selected locations in the system. Because of the high cost of a PMU, it is desirable to minimize their number while maintaining the ability to monitor (observe) the entire system. A system is said to be *observed* if all of the state variables of the system can be determined from a set of measurements (e.g., voltages and currents).

Let $G = (V, E)$ be a graph representing an electric power system, where a vertex represents an electrical node (a substation bus where transmission lines, loads, and generators are connected) and an edge represents a transmission line joining two electrical nodes. The problem of locating a smallest set of PMUs to monitor the entire system is a graph theory problem closely related to the well-known vertex covering and domination problems. A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed and its incident edges and their end vertices (these vertices and edges are said to be observed.) The other observation rules are as follows:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of $k > 1$ edges and if $k-1$ of these edges are observed, then all $k$ of these edges are observed.

In [3], the power system monitoring problem was first studied as a variation of the well-known dominating set problem (see also [4,5]). A set $P \subseteq V$ is a *dominating set* in $G$ if every vertex in $V - P$ has at least one neighbor in $P$. The cardinality of a minimum dominating set of G is the *domination number* $\gamma(G)$. Considering the power system monitoring problem as a variation of the dominating set problem, we define a set $P$ to be a *power dominating set* (abbreviated PDS) if every vertex and every edge in G is observed by $P$. The *power domination number* $\gamma_P(G)$ is the minimum cardinality of a PDS of $G$. A PDS of G with the minimum cardinality is called a $\gamma_p(G)$-set. Since any dominating set is a power dominating set, $1 \leqslant \gamma_P(G) \leqslant \gamma(G)$ for all graphs $G$. The power domination problem is widely studied in [1–3, 6–9].

In [1], Dorfling and Henning determined the power domination number of an $n \times m$ grid graph. Their proof provides an algorithm for solving the power domination problem on grid graphs. There are five cases of construction rules in their algorithm. In this paper, we present a more simple algorithm with respect to their result.

## 2. Preliminaries

All graphs considered here are undirected and simple (i.e., finite, loopless, and without multiple edges). Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The *open neighborhood* $N(v)$ of a vertex $v$ is the set consisting of vertices adjacent to $v$, i.e., $N(v) = \{u \in V \mid (u, v) \in E\}$, and the *closed neighborhood* of $v$ is $N[v] = v \cup N(v)$. Also, we shall use the following notations. For $m \geqslant n \geqslant 1$, let $G_{n,m}$ be an $n \times m$ grid graph with vertex set $V(G_{n,m}) = \{(x, y) \mid 1 \leqslant x \leqslant n, 1 \leqslant y \leqslant m\}$ and two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. Further, we let $R(x_1 \cdots x_2, y)$ be the set consisting of vertices $(x_1, y), (x_1 + 1, y), \cdots, (x_2, y)$ where $x_1 \leqslant x_2$, and let $C(x, y)$ be the set consisting of vertices $(x, 1), (x, 2), \cdots, (x, y)$.

In [6], Kneis et al. found a way to simplify the problem description by using a smaller set of rules equivalent to the four original rules mentioned above.

Observation Rule 1 (OR1):
A vertex in the power domination set observes itself and all its neighbors.

Observation Rule 2 (OR2):
If an observed vertex $v$ of degree $d \geqslant 2$ is adjacent to $d - 1$ observed vertices, then the remaining unobserved vertex becomes observed as well.

We note that OR1 is just the rule in the definition of dominating set problem. A vertex may dominate vertices at arbitrary distance when certain conditions are fulfilled by OR2 (for example, if a PMU is put in the endvertex of a path, then it can dominate all the other vertices).

Theorem 1 and Algorithm A are the results in [1].

**Theorem 1** (Dorfling and Henning [1]).

$$\gamma_p(G_{n,m}) = \begin{cases} \lceil (n+1)/4 \rceil, & \text{if } n \equiv 4 (\text{mod } 8) \\ \lceil n/4 \rceil, & \text{otherwise.} \end{cases}$$

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**Algorithm** A (Dorfling and Henning's method)

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**Input:** An $n \times m$ grid graph with $m \geqslant n \geqslant 1$.
**Output:** A minimum PDS $P$ of $G$.

1. $k \leftarrow \lfloor n/8 \rfloor$
   $j \leftarrow n - k \times 8$
2. $P' = \bigcup_{i=0}^{k-1} \{(8 \times i + 3, 2), (8 \times i + 5, 3)\}$
3. **If** $j = 0$ **then** $P \leftarrow P'$
   **If** $j \in \{1, 2\}$ **then** $P \leftarrow P' \cup \{(n, 1)\}$
   **If** $j = 3$ **then** $P \leftarrow P' \cup \{(n-1, 1)\}$
   **If** $j = 4$ **then**
   $P \leftarrow P' \cup \{(n-2, 1), (n-1, 1)\}$
   **If** $j \in \{5, 6, 7\}$ **then**
   $P \leftarrow P' \cup \{(n+3-j, 2), (n+5-j, 3)\}$

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## 3. Main Result

There are five cases in algorithm A for determining a $\gamma_p(G)$-set. We now give another method to put PMUs in a grid graph. As a consequence, our method relies on a simple rule and is easier than the previous one. We first calculate $\gamma_p$ by Theorem 1. Next, we find a suitable position $(x_0, y_0)$ in the grid to set the first PMU. Finally, we generate all other positions to put PMUs according to $(x_0, y_0)$. The following is the detail.

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**Algorithm** B

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**Input:** An $n \times m$ grid graph with $m \geqslant n \geqslant 1$.
**Output:** A minimum PDS $P$ of $G$.

1. $\gamma_p \leftarrow \lceil n/4 \rceil$
   **If** $n$ mod $8 = 4$ **then** $\gamma_p \leftarrow \gamma_p + 1$
2. $y_0 \leftarrow \lfloor \gamma_p/2 \rfloor + 1$
   $x_0 \leftarrow y_0 + 1$
   **If** $n \in \{1, 4\}$ **then** $x_0 \leftarrow x_0 - 1$
   $P = \{(x_0, y_0)\}$
3. **For** $i = 1$ **to** $\gamma_p - 1$ **do**
   $P \leftarrow P \cup \{(x_0 + i \times 2, y_0 + i)\}$

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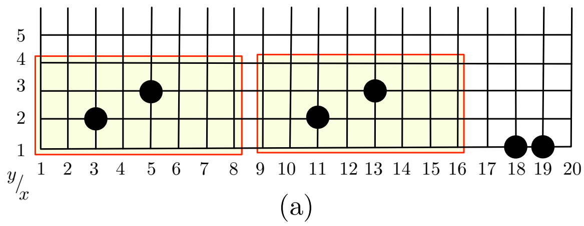
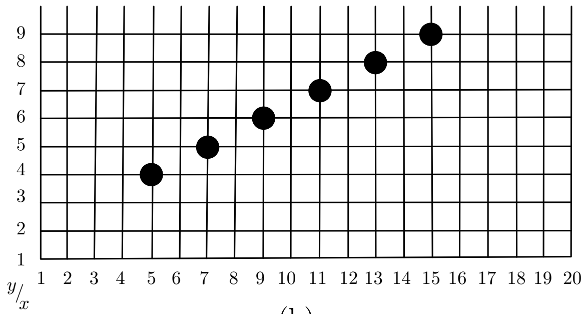
Note that the resulting $\gamma_p(G)$-set produced by Algorithms A and B are, in general, different. Fig. 1(a) and 1(b) show the vertices in the $\gamma_p(G_{20,m})$-set for Algorithms A and B, respectively.



(a)



(b)

Figure 1: An example of $20 \times m$ grid: (a) the minimum PDS output from Algorithm A; (b) the minimum PDS output from Algorithm B.

We now verify the correctness of Algorithm B.

**Lemma 2** *If every vertex in the first row is observed, then $G_{n,m}$ is observed.*

**Proof.** Suppose that every vertex in the first row is observed. Since the observed vertex $(1, 1)$ has degree 2 and its neighbor $(2, 1)$ is observed, by OR2, the other neighbor $(1, 2)$ becomes observed. By symmetry, we can show that $(n, 2)$ is also observed. Similarly, for $i = 2, 3, \ldots, n-1$, the observed vertex $(i, 1)$ has degree 3 and its neighbors $(i-1, 1)$ and

$(i+1,1)$ are observed, again by OR2, the remaining unobserved neighbor $(i,2)$ becomes observed. Thus, all vertices in the second row are observed. In general, if all vertices in two consecutive rows, say $j$ and $j+1$, are observed, using a reasoning as above we can show that every vertex in the $(j+2)$-th row is observed. Consequently, all vertices in the entire grid $G_{n,m}$ are observed. □

**Lemma 3** *Every vertex in the first row is observed by Algorithm B.*

**Proof.** For $1 \leqslant n \leqslant 3$, we have $\gamma_p = 1$ and the position of PMU is $(1,1)$ for $n=1$ and $(2,1)$ for $n=2,3$. For $n=4$, we have $\gamma_p = 2$ and the two positions to put PMU are $(2,2)$ and $(4,3)$. Applying OR1 and/or OR2, we can see that every vertex in the first row is observed.

For $n > 4$, we assume that the $(i+1)$-th PMU is located at the position $(x_i, y_i)$, where $x_i = x_0 + i \times 2$ and $y_i = y_0 + i$ by the setting of Algorithm B. Thus, $x_i = x_{i-1} + 2$ and $y_i = y_{i-1} + 1$ for $1 \leqslant i \leqslant \gamma_p - 1$. By OR1, all neighbors of each vertex $(x_i, y_i)$, $0 \leqslant i \leqslant \gamma_p - 1$, become observed. We now consider the following six cases in series.

**Case** 1: We first consider the vertex $(x_{\gamma_p-2}+1, y_{\gamma_p-2})$. Since it is adjacent to $(x_{\gamma_p-2}, y_{\gamma_p-2})$ (i.e., the position to put the $(\gamma_p-1)$-th PMU), the vertex is observed. Moreover, since it has degree 4 and there are three observed neighbors (i.e., one is $(x_{\gamma_p-2}, y_{\gamma_p-2})$ and the other two are the neighbors of $(x_{\gamma_p-1}, y_{\gamma_p-1})$), the remaining unobserved neighbor $(x_{\gamma_p-2}+1, y_{\gamma_p-2}-1)$ becomes observed by OR2. Thus, $R(x_{\gamma_p-3} \cdots x_{\gamma_p-3}+3, y_{\gamma_p-3})$ is observed. By the same way, we can show that $R(x_i \cdots x_i + \gamma_p - i, y_i)$ is observed for each $i = 0, 1, \ldots, \gamma_p - 3$.

**Case** 2: Next, we consider the vertex $(x_0 + \gamma_p - 1, y_0)$. From Case 1, we know that it is observed. Moreover, since it has degree 4 and there are three observed neighbors, the remaining unobserved neighbor $(x_0+\gamma_p-1, y_0-1)$ becomes observed by OR2. By the same way, we can show that every vertex in $R(x_0 \cdots x_0 + \gamma_p - 1, y_0 - 1)$ becomes observed by OR2. Furthermore, we can also prove that $R(x_0 + i - 1 \cdots x_0 + \gamma_p - i, y_0 - i)$ for all $i = 1, 2, \ldots, \lfloor \gamma_p/2 \rfloor$ are observed by a similar argument.

**Case** 3: In succession, we consider the vertex $(x_0 + \gamma_p - \lfloor \gamma_p/2 \rfloor, 1)$. From Case 2, we know that it is observed. Moreover, since it has degree 3 and there are two observed neighbors, the remaining unobserved neighbor $(x_0 + \gamma_p - \lfloor \gamma_p/2 \rfloor + 1, 1)$ becomes observed by OR2. Thus, $C(x_0 + \gamma_p - \lfloor \gamma_p/2 \rfloor + 1, 1)$ is observed. By the same way, we can show that $C(x_0 + \gamma_p - \lfloor \gamma_p/2 \rfloor + i, i)$ is observed for each $i = 1, 2, \ldots, \gamma_p + \lfloor \gamma_p/2 \rfloor - 1$.

**Case** 4: In this case, we start to consider the vertex $(x_0 + 2 \times \gamma_p - 1, 1)$. A proof similar to Case 3 can show that $C(x_0 + 2 \times \gamma_p + i - 1, \gamma_p + \lfloor \gamma_p/2 \rfloor - i)$ becomes observed for each $i = 1, 2, \ldots, \min\{\gamma_p + \lfloor \gamma_p/2 \rfloor - 1, n - 2 \times \gamma_p - \lfloor \gamma_p/2 \rfloor - 1\}$.

**Case** 5: In this case, we start to consider the vertex $(x_0 + \lfloor \gamma_p/2 \rfloor - 1, 1)$. A proof similar to Case 3 can show that $C(x_0 + \lfloor \gamma_p/2 \rfloor - i - 1, i)$ becomes observed for each $i = 1, 2, \ldots, \lfloor \gamma_p/2 \rfloor$.

**Case** 6: In this case, we start to consider the vertex $(x_0 - 1, 1)$. A proof similar to Case 3 can show that $C(x_0 - i - 1, \lfloor \gamma_p/2 \rfloor - i + 1)$ becomes observed for each $i = 1, 2, \ldots, \lfloor \gamma_p/2 \rfloor$.

Therefore, every vertex in the first row is observed. □

For example, we consider a $20 \times m$ grid graph with $m \geq 20$. As shown in Fig. 2, every vertex in the $\gamma_p(G_{20,m})$-set is represented by a dark vertex and all its neighbors are represented by gray vertices. According to the proof of Lemma 3, the observed vertices described in each case are labeled by the number $i = 1, 2, \ldots, 6$ as indicator. As a result, every

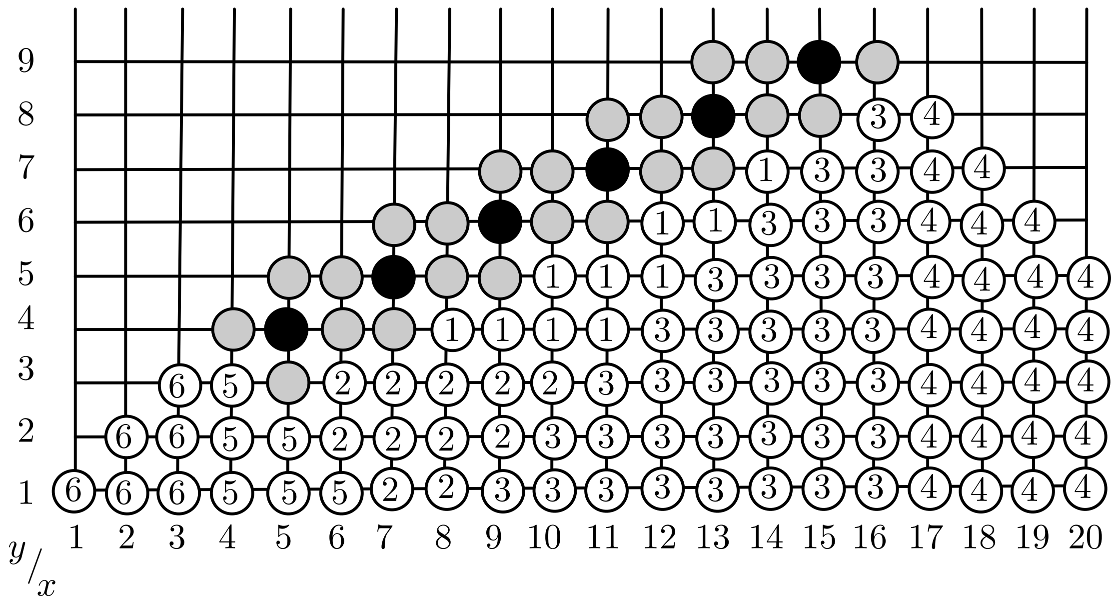



Figure 2: An example to illustrate the observed vertices in each case of the proof in Lemma 3.

vertex in the first row is observed.

**Theorem 4** *For $m \geqslant n \geqslant 1$, Algorithm B produces a $\gamma_p(G_{n,m})$-set.*

**Proof.** The number of PMUs is calculated by Theorem 1. Also, by Lemma 2 and Lemma 3, $G_{n,m}$ is observed and therefore the output of Algorithm B is a $\gamma_p(G_{n,m})$-set. □

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