Bounded Approaches in Radio Labeling Square Grids

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For my dear parents, Kumar and Viji Ananda, whose loving encouragement and affection have made my dreams possible throughout my life.
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Abstract

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by Dev Ananda

Let $d(u,v)$ denote the distance between two vertices $u$ and $v$ on a graph $G$ and let $diam(G)$ denote the diameter of such a graph $G$. A connected graph $G$ has a radio labeling $f$ if, for all vertices $u$, $v$ of $G$,

$$d(u,v) + |f(u) - f(v)| \geq diam(G) + 1.$$  (1)

The span of the labeling function $f$ is the maximum integer assigned by $f$. The radio number of a graph $G$, $rn(G)$, is the minimum possible span obtained over all possible radio labelings of the graph $G$. A path graph $P_n$ has $n$ consecutive vertices along $n-1$ consecutive edges. A grid graph is defined as the Cartesian product of two path graphs, and a square grid graph is obtained by taking the product of identical path graphs $P_n \square P_n$. In this paper, the radio number of all even grid graphs is determined using bounding techniques alone, while establishing fundamental guidelines for odd grids and distance-maximizing labelings, in general.
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I. INTRODUCTION

This paper investigates methods of determining the radio number for certain graphs and grids. We discuss the radio condition and the way in which the structure of an abstract graph affects certain integer-valued properties of the graph. For example, the diameter of a graph or its inter-connectivity may fundamentally affect the radio condition for non-adjacent or non-consecutively ordered vertices. Ultimately, determining the radio number of any family of graphs requires consideration of its subtle nuances. In this paper, we establish upper and lower bounds for the radio number of square grids in Sections 2 and 3 and close the gap between bounds in Section 4, The Bump, in order to determine the radio number.

In Section 2, An Upper Bound, we establish a maximum span for an optimal radio labeling. The upper bound is constructed by creating a labeling for the grid which provides a schematic by which the radio condition can govern. The ordering and labeling of the grid is structured such that distances between pairs of consecutively labeled vertices are maximized and an upper bound emerges from satisfying the various constraints. We explore the nature of constructing bounds by constructing labelings and consider how optimal radio labelings figure into the scope of integer mappings and graph space. We also review T-S. A. Jiang’s proof of the radio number of all grids and suggest insights about the significance of graph structure within the realm of algebraic optimization.

Section 3, A Lower Bound provides an account of distance maximization, as well as the application of labeling techniques in analyzing graphs. The lower bound is created by exploring the natural characteristics of grids and examining how those characteristics can be pulled apart and tied back together to illustrate the overall coherence of grids and other graphs. By establishing a lower bound, we bridge a natural “gap” between the mathematical ideal of a continuous spectrum of vertices and rigorous physical constraints that give graphs formal structure.

In Section 4, The Bump, we close the gap between the lower and upper bounds of odd and even grids. If the lower bound admits sufficient incrementation, then the radio number of square grids can be determined to be the upper bound. By accounting for all pairs of vertices within the fundamental structure of the grid, the radio number is determined by showing that the span of any viable radio labeling must be “bumped up” to the level of the upper bound and a distortion in span arises from the intricate balance between algebraic constraint and physical order. Finally, the radio number of all even grids \( r(n(P_{2k} \square P_{2k})) \) is determined by bounded approaches, while other key revelations are made regarding bounding techniques and their efficacy in radio labeling and graph structure analysis.
1.1. Useful Definitions.

A graph \( G \) is comprised of two main elements: \( V \), a set of vertices, and \( E \), a set of an unordered pairs of vertices, also called edges. For example, in the following path graph \( P_5 \), the vertices are denoted from left to right as \( v_1 \) through \( v_5 \). Edge \( e_2 = (v_2, v_3) \) may be visualized as the edge connecting \( v_2 \) and \( v_3 \).

![Graph P5](image)

A path is a sequence of distinct edges between vertices of a graph. A path may have the same vertex as its first and last, in which case it is defined more specifically as a cycle. A path graph \( P_n \), like the one illustrated above, is then a path on \( n \) vertices.

The distance \( d(v_i, v_j) \) between two vertices \( v_i \) and \( v_j \) is the length of the shortest path connecting \( v_i \) and \( v_j \). In \( P_5 \) above, the shortest path between \( v_2 \) and \( v_5 \) consists of 3 edges, namely \( e_2, e_3, \) and \( e_4 \). Thus, \( d(v_2, v_5) = 3 \).

The diameter of a graph is the maximum distance taken over all pairs of vertices in the graph. For example, \( \text{diam}(P_5) = 4 \) because, if we take the distances between all pairs of vertices of \( P_5 \), we see that the greatest of those distances lies between \( v_1 \) and \( v_5 \), which are separated by 4 edges.

A connected graph is a graph in which there exists a path from any vertex to any other vertex. For example, \( P_5 \) is a connected graph since it is possible to reach any vertex from any other vertex of the graph.

A labeling of a graph is an assignment of labels to vertices. All labels in this paper can be assumed to be integers.

Let \( f \) be a labeling of a graph \( G \). Define \( \{x_1, x_2, \ldots, x_n\} \) to be an ordering of the \( n \) vertices of \( G \) such that

\[
f(x_i) < f(x_{i+1}) \quad \text{for } i = 1, 2, \ldots, n - 1.
\]

(2)

where \( x_i \) is the \( i^{th} \) vertex in the established order of the labeling, i.e., \( x_1 \) is the first vertex labeled by \( f \) and \( x_n \) the last. Note that this ordering depends on the labeling \( f \); the choice of labeling will be clear in context.

Now, in a given labeling, define a 1\(^{st}\)-order pair of vertices as any pair of consecutively labeled vertices \( \{x_i, x_{i+1}\} \), a 2\(^{nd}\)-order pair of vertices as any pair of vertices \( \{x_i, x_{i+2}\} \), and an \( n^{th}\)-order pair of vertices as any pair of vertices \( \{x_i, x_{i+n}\} \).

A radio labeling \( f \) is a labeling for which, for any distinct vertices \( u \) and \( v \) in a graph \( G \),

\[
d(u, v) + |f(u) - f(v)| \geq \text{diam}(G) + 1.
\]

This inequality is called “the radio condition.”

The span of a labeling is the difference between the highest and lowest labels assigned to vertices under the given labeling. Our convention always assigns, for any graph and labeling, the first vertex the label 0, i.e. \( f(x_1) = 0 \), where \( x_1 \) is the first vertex labeled under \( f \). Thus, we can re-define the span of a labeling in this paper to be the highest label assigned to a vertex under that
graph labeling. If a graph $G$ with labeling $f$ has an ordering $\{x_1, x_2, ..., x_n\}$ of vertices corresponding to (2), then $\text{span}(f) = f(x_n)$.

The **radio number** of a graph $G$, $\text{rn}(G)$, is then the minimum possible span across all radio labelings of $G$. In other words, if all radio labelings of a graph are considered, then the radio number is the lowest span for that graph under any of those radio labelings.
1.2. Paths.

A path graph \( P_n \) is a connected set of \( n \) vertices connected by \( n - 1 \) edges, in which the terminal vertices have degree 1 and all others degree two. Below are examples of path graphs \( P_1 \), \( P_2 \), and \( P_4 \).

![Figure 1. Top to bottom: Path graphs \( P_1 \), \( P_2 \) and \( P_4 \)](image)

Figure 1. [top to bottom: Path graphs \( P_1 \), \( P_2 \) and \( P_4 \)]

Figure 2 illustrates radio labelings \( f_1 \) and \( f_2 \), respectively, of \( P_4 \). Note that the radio labeling \( f_2 \) of \( P_4 \) is non-optimal – that is, \( \text{span}(f_2) > \text{rn}(P_4) \). Radio labeling \( f_1 \), however, is optimal – that is, \( \text{span}(f_1) = \text{rn}(P_4) \), as discussed later in Section 1.4.

![Figure 2. Radio labelings of \( P_4 \)](image)

Figure 2. Radio labelings of \( P_4 \)
[ top: radio labeling \( f_1 \); bottom: optimal radio labeling \( f_2 \) ]
1.3. Grids.

A grid graph is the Cartesian product of two path graphs. The grids that are addressed in this paper are composed of two identical paths that construct a square grid graph. So, henceforth, for this paper, a grid will refer to a square grid, $P_n \square P_n$. An odd grid $P_{2k+1} \square P_{2k+1}$ is constructed from odd paths, and an even grid $P_{2k} \square P_{2k}$ from even paths. Odd grid $P_3 \square P_3$ is depicted below, first, as a set of vertices and edges, and subsequently, as a set of vertices in an array of boxes. In the latter case, the boxes represent vertices, and two vertices are considered adjacent if they share an edge.

An optimal radio labeling for $G = P_3 \square P_3$ is illustrated below. Note that the highest vertex label represents the radio number. In this case, we see that $\text{diam}(G) = 4$. Thus, for this graph, any radio labeling $f$ must satisfy the following inequality: $d(u, v) + |f(u) - f(v)| \geq 4 + 1 = 5$ for all distinct vertices $u$ and $v$. In the radio labeling $f^*$ below, $v_4$ is the last vertex in the ordering and $f^*(v_4) = 17$. Thus, $\text{span}(f^*) = 17$ (as denoted in blue). Since there is no radio labeling of $P_3 \square P_3$ that produces a lower span, $\text{rn}(P_3 \square P_3) = 17$.

Figure 3. A grid as a set of labeled boxes denoting vertices

Figure 4. A radio labeling of $3 \times 3$ grid $P_3 \square P_3$
1.4. Past Notable Work.

Radio labeling is a realm of graph theory motivated by a problem faced in the FCC’s regulation of FM radio frequencies. Radio stations are categorized into station classes. Stations are assigned channels based on certain factors. These factors include antenna height and signal power. In addition, the distances between stations also affect what channel is assigned to a radio station of a given station class. Radio stations assigned the same channel, for example, must be a minimum distance apart in order to satisfy FCC regulations. Similarly, stations of different channels must be some smaller distance apart. In general, stations that are in closer proximity to one another must be channels that are farther apart, after considering station class. This is called the channel assignment problem. Graph theory has been used to address the channel assignment problem for several decades. More specifically, radio labeling is the realm of graph theory that abstracts this problem, and its solution is thus aptly termed the radio number.

In [2], Chartrand, Erwin, and Zhang determined some bounds for the radio number of cycle graphs \( C_n \), as well as the radio numbers of \( C_6 \), \( C_7 \), and \( C_8 \) in *Radio Labelings of Graphs*. They also determined a number of key results for abstract families of graphs, in general. One of these is the assertion that, for \( n \geq 6 \), a radio labeling of \( C_n \) cannot contain 3 consecutive labels from \( \{1, 2, ..., n\} \). This result was particularly notable as it was the first to establish concrete features of any three consecutive vertices in a radio labeling. In Section 4, *The Bump*, we show that, for three consecutively labeled vertices in a grid, a radio labeling must abide by certain restrictions.

In [3], *A Graph Labeling Problem Suggested by FM Channel Restrictions*, Chartrand, Erwin, and Zhang provided key results with respect to the radio labeling of graphs. They found an upper bound for the radio number of all path graphs, \( r_n(P_n) \). In addition, they prove that \( r_n(P_5) = 11 \). These findings, described below, contribute inspiration to a considerable portion of the work in this paper. First, they showed how to systematically eliminate possibilities among labels for an abstract graph, but even more importantly, how to develop and refine strategies that efficiently model the growth of integer label values in optimized labelings of large abstract graphs.

\[
\begin{align*}
r_n(P_n) \leq \begin{cases} 
\binom{n-1}{2} + \frac{n}{2} + 1, & \text{if } n \text{ is even} \\
\binom{n}{2} + 1, & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

In [5], *Multilevel Distance Labelings for Paths and Cycles*, Liu and Zhu determined the radio number of all paths and all cycles (cycles not shown):

\[
\begin{align*}
r_n(P_n) &= \begin{cases} 
2n^2 + 2, & \text{if } n \text{ is even} \\
2n(n - 1) + 1, & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]

The aforementioned result in [5] was a breakthrough finding in radio labeling. While the radio number of numerous families of graphs had been discovered, Liu and Zhu’s approach was novel and has importance in the scope of this paper. They presented and proved sufficient conditions for a distance labeling. The radio number is then related back as the minimum span associated with a distance labeling.

After providing a labeling for each of \( P_{2k} \) and \( P_{2k+1} \), Liu and Zhu identified an upper bound for the radio number of all path graphs. By analyzing distance labeling, they constructed a lower bound for all path graphs. The key feature in their work was the use of a distance-maximizing strategy. Later in this paper, we will see that minimum spans result from maximizing the sum of distances between consecutively labeled pairs of vertices. Also, while distances are maximized, certain distance values disappear depending on the ordering chosen. Upon proving that only
certain orderings resulted in a distance labeling, Liu and Zhu determined $rn(P_n)$. The relevance of Liu and Zhu’s work to this paper is then two-fold: the conclusions (1) that a distance-maximization strategy used is efficient and of tremendous value for graphs of large order, such as grids, and (2) that the unique structure of specific kinds of graphs may exhibit characteristics, such as a “bump” value which excludes some distance labelings from being radio labelings.

In 2009, Calles and Gomez described the general approach for constructing upper and lower bounds for grids. Their efforts resonate here in terms of the basic approach in determining a viable lower bound, as well as the methodology used for computing an upper bound. By analyzing the sum of all distances and arranging ordered vertices to maximize the distances between pairs of consecutive vertices, Calles and Gomez minimized the span of any such graph, thus determining the lower bound for all grids. By using their own labeling, they constructed an upper bound that gives rise to the necessary “bump” value that closes the gap. Below, their findings are stated for even and odd grids, respectively:

$$n^3 - n^2 - 2n + 4 \leq rn(P_{2k} \square P_{2k}) \leq n^3 - n^2 + 1$$

$$n^3 - n^2 - n + 1 \leq rn(P_{2k+1} \square P_{2k+1}) \leq n^3 - n^2 - \frac{n-1}{2} + 1$$

Here, we extend the work of Calles and Gomez with regards to the lower bound by also examining sets and arrangements of vertices in the distance-maximizing summation by considering lost label values, the labels associated with vertices ordered first or last in the summation of distances between consecutive pairs of vertices and construct a more accurate lower bound for both odd and even grids.

In terms of distance-maximization in the labelings of similar graphs and Cartesian products, Wyels and Tomova prove, in The Radio Numbers of All Graphs of Order $n$ and Diameter $n-2$, the radio number of spires. Spires are variations on path graphs. They are composed of a path and a steeple, an edge that branches off from a non-terminal vertex. The account of distance-maximization over the scope of Cartesian products relates a valuable result in interpreting other graphs that have unifying features like long series of connected edges or a spine that disconnects easily. In doing so, they characterize graphs whose diameter relate so closely to order and provide a strong basis by which investigators can link distances and spacing to the innate structure of connected graphs with unique components. A similar account is provided in The Radio Number of Gear Graphs, in which radio labeling techniques applied to wheel graphs are abstracted to gear graphs, a natural extension of wheel graphs. As in our account of grids, specific labelings are eliminated from contention based on the validity of index values under the radio condition. Ultimately, we see that an upper bound can be determined when a lowest possible span coincides with the exhaustive maintenance of some set of algebraic constraints.

The examples cited in this section define key methodologies in investigating various families of graphs, as well as abstract graphs, in general. The novelty of these prior approaches emerges throughout the course of this investigation of grids and, as a whole, helps formulate the thought process and mathematical intuitions that succeed here.
2. An Upper Bound

In [2], Chartrand, Erwin, and Zhang analyze the radio condition (1) for paths and cycles. By considering the radio condition, we establish an upper bound by seeking to minimize sums of distances between consecutively labeled pairs of vertices. Then, the span of any such labeling is really just an upper bound for the radio number of that graph. In section 2.1, we discuss the general approach for constructing the upper bound for the radio number of odd grids $P_{2k+1} \square P_{2k+1}$ and determine an upper bound in Theorem 2.1.

We extend the discussion into the realms of even grids, in which the work of T-S. A. Jiang [4] and Calles and Gomez [1] is analyzed. In section 2.2, Jiang’s labeling is defined and the labeling itself analyzed. We see that, while Jiang’s labeling does satisfy the radio condition for all pairs of consecutively labeled vertices, it fails to maintain (1) for what is defined here as 2nd-order pairs of vertices. In section 2.3, we describe the methodology of Calles and Gomez and construct their upper bound for the radio number of even grids, which is formalized here as Theorem 2.2.

2.1. Upper Bound For Odd Grids $P_{2k+1} \square P_{2k+1}$

Here, we define an ordering and construct labeling $f$ as defined by Calles and Gomez in [1]. After proving that $f$ is a radio labeling, we calculate its span and determine and upper bound for the radio number of all odd grids. The formalized proof is supplied below.

Theorem 2.1 (The Upper Bound for the Radio Number of Odd Grids).

\[
\text{rn}(P_{2k+1} \square P_{2k+1}) \leq 8k^3 + 8k^2 + k.
\]

Proof. Let $G = P_{2k+1} \square P_{2k+1}$. We can substitute the diameter $diam(G) = 4k$ into (1) and rewrite the inequality to reflect the sum of label differences and distances between consecutively labeled vertices:

\[
d(x_i, x_{i+1}) + |f(x_i) - f(x_{i+1})| \geq 4k + 1. \tag{3}
\]

Note that the inequality does not necessarily satisfy the conditions of a radio labeling since only consecutively labeled vertices are considered. First, we outline a “diagonal method” for ordering vertices of $P_{2k+1} \square P_{2k+1}$. The ordering proceeds in a cyclical diamond-like pattern a la Figure 5. Each successive cycle of the labeling occurs just once and augments the diamond from the center outwards, while also ordering the corner vertices in a diagonal pattern. The inner diamond and outer diamond proceed through several cycles until they meet each other within the grid. Thus, the labeling function $f$ described above will be presented here as a series of labeling cycles that are organized in Table 1.

Initial Cycle 0 (1 vertex): $v_{k+1,k+1}$

The first vertex in the labeling order is assigned: it is always the center vertex of the graph.

Cycle 1: (8 vertices): $v_{1,2k+1}, v_{k+2,k+1}, v_{1,1}, v_{k+1,k+2}, v_{2k+1,1}, v_{k,k+1}, v_{2k+1,2k+1}, v_{k+1,k}$

The pattern enumerated above, in terms of alternately labeling vertices in a center diamond while labeling vertices towards the outer corners, continues into the remaining cycles of the ordering.
Cycle 1 contains 8 distinct parts, with one vertex labeled during each part. For Cycles 2 through $j$, the labeling continues in these 8 distinct parts with the number of vertices labeled within each part increasing incrementally. We see that $j$ vertices are labeled in each of the 8 parts of labeling Cycle $j$. It was alluded to previously that there are two distinct patterns that emerge during any given labeling Cycle $j \geq 2$. The vertices are ordered and labeled within each of these two patterns alternately, and thus we will refer to the alternating patterns of labeling as Stages 1 and 2. Stage 1 is the diagonal set of vertices labeled around the center of the grid and Stage 2 is the diagonal set of vertices labeled around the corners of the grid. The labeling hops back and forth between Stage 1 and Stage 2 in each of four diagonally opposite directions. Thus, we really have 4 distinct subcycles that comprise each of Stage 1 and Stage 2. The 4 subcycles will be called Cycles $j_1, j_2, j_3, \text{ and } j_4$. Subcycle 21 is provided for the sake of clarity:
Cycle 2: (16 vertices):

Subcycle 21: $v_{1,2k}$ (Stage 1), $v_{k+2,k}$ (Stage 2), $v_{2,2k+1}$ (Stage 1), $v_{k+3,k+1}$ (Stage 2)

\[ \vdots \]

Subcycle 24: alternating Stages 1 & 2

Notice that, during Cycle 21, two vertices are labeled in each stage. The vertices labeled in Stage 1 are diametrically opposite about the center vertex to those labeled in Stage 2 of subcycle 21. Another 3 subcycles occur within Cycle 2 at which point Cycle 3 (as well as any subcycle) commences, if necessary. Table 1 defines the general $j^{th}$ labeling cycle progression on the grid. Figure 6 illustrates Cycles 0 - 2 on $P_5 \square P_5$. 

**Figure 6.** labeling cycles for $P_5 \square P_5$
Table 1. Ordering of vertices labeled during Cycle $j$

<table>
<thead>
<tr>
<th>Subcycle</th>
<th>stage 1</th>
<th>stage 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_{1,j}$, $v_{2k+1,j}$</td>
<td>$v_{k+1,j}$, $v_{2k+1,j}$</td>
</tr>
<tr>
<td>2</td>
<td>$v_{j,1}$, $v_{2k,j}$</td>
<td>$v_{k+1,j}$, $v_{2k+1,j}$</td>
</tr>
<tr>
<td>3</td>
<td>$v_{2k+1,j}$, $v_{2k+1,j}$</td>
<td>$v_{k+1,j}$, $v_{2k+1,j}$</td>
</tr>
<tr>
<td>4</td>
<td>$v_{2k+1,j}$, $v_{2k+1,j}$</td>
<td>$v_{k+1,j}$, $v_{2k+1,j}$</td>
</tr>
</tbody>
</table>

We should note that any grid $P_{2k+1} \times P_{2k+1}$ will have a total of $k$ cycles under this labeling. Define $x_i$, $i = 1, 2, ..., n^2$, to be the $i^{th}$ vertex labeled in this ordering. Now, let us define the labeling itself:

$$f(x_i) = \begin{cases} 
0 & i = 1 \\
 f(x_{i-1}) + 2k & x_{i-1} \text{ and } x_i \text{ in the same cycle} \\
 f(x_{i-1} + 2k + 1) & x_{i-1} \text{ in Cycle } j-1 \text{ and } x_i \text{ in Cycle } j.
\end{cases}$$

We have shown that, if the distances between consecutive vertex pairs determine their respective label values, then we have constructed a labeling $f$ that abides by (1) for all 1st-order pairs of vertices. We need only show now that higher-order pairs of vertices also satisfy (1). We see that, for all consecutively labeled pairs of vertices in an odd grid, $max d(x_i, x_{i+1}) = 2k + 1$ occurs for pairs of vertices within the same cycle. Thus, we can check the lowest label differences amongst 2nd-order pairs of vertices as follows:

$$f(x_{i+1}) - f(x_i) \geq 2k$$

Adding these inequalities gives us:

$$f(x_{i+2}) - f(x_i) \geq 4k.$$
Considering that, for all pairs of distinct vertices \( u \) and \( v \), \( d(u,v) \geq 1 \),

\[
|f(x_{i+2}) - f(x_i)| + d(x_{i+2}, x_i) \geq 4k + 1
\]

Thus, (1) is satisfied for all 2\(^{nd}\)-order pairs of vertices in the grid. As \( f(x_{i+3}) - f(x_i) > f(x_{i+2}) - f(x_i) \), we have confirmed that radio condition (1) is satisfied for 3\(^{rd}\)-order pairs of vertices, and thus all distinct pairs of vertices in the grid. Therefore, \( f \) must be a radio labeling. Below, we formulate the span of the labeling by considering labeling differences between consecutively labeled vertices both \textit{within cycles} and \textit{between cycles}.

The distance between each pair of consecutive vertices \textit{within Cycle} \( j \) is \( 2k + 1 \). For all cycles \( j = 2, 3, \ldots, k \), the distance between the first vertex in Cycle \( j \) and the final vertex labeled in Cycle \( j - 1 \) is \( 2k \). The distance between each pair of consecutive vertices \textit{within Cycle} \( j \) is \( 2k + 1 \). Now, we ascertain the value added to the span due to these two unique kinds of pairs of consecutively labeled vertices under labeling \( f \): (i) the value added to the span as result of all pairs of consecutively labeled vertices \textit{within the same cycle} and (ii) the value added to the span associated with consecutively labeled vertices that are in \textit{different} cycles. Since the diameter of an odd grid \( P_{2k+1} \times P_{2k+1} \) is \( 4k + 1 \), pairs of consecutively labeled vertices that are part of \textit{different} labeling cycles must add \( 4k + 1 - 2k = 2k + 1 \) to the overall span and pairs of consecutively labeled vertices that are part of the \textit{same} cycle add \( 4k + 1 - (2k + 1) = 2k \) to the overall span. The following table gives the calculation of the span across all the cycles of an odd grid. These cyclic spans are summed, culminating in a span for the graph itself.

<table>
<thead>
<tr>
<th>labeling schema for ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cycle #</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>( k )</td>
</tr>
</tbody>
</table>

Table 2. How vertices labeled in Cycle \( k \) add to span

The span is computed by summing all the label differences between consecutive vertex pairs. Since there are a total of \( j = k \) cycles, we have \( 8k - 1 \) pairs of vertices within the same cycle that add a total of \( 2k(8k - 1) \) to \textit{span}(\( f \)) and \( j = k \) pairs of vertices in different (but consecutive) cycles that add \( k(2k + 1) \) to \textit{span}(\( f \)). The span is thus:

\[
\text{span}(\( f \)) = f(x_{(2k+1)^2}) \geq \sum_{j=1}^{k} [2k + 1 + 2k(8j - 1)]
\]

(4)

We find an aggregate span by considering label sums as a consequence of each cycle and its structure. Multiplying out and simplifying (4) yields

\[
f(x_{(2k+1)^2}) = \sum_{j=1}^{k} [2k(8j) + 1] = k + \sum_{j=1}^{k} 2k(8j)
\]
Using a summation formula to replace our sigma notation, we get

\[ f(x_{(2k+1)^2}) = k + 16k \sum_{j=1}^{k} j \]

Since \( f \) is a radio labeling and \( f(x_1) = 0 \), (5) constitutes an upper bound for the radio number of \( P_{2k+1} \square P_{2k+1} \). As a result, the span of \( f \) is really just the upper bound:

\[ \text{rn}(P_{2k+1} \square P_{2k+1}) \leq 8k^3 + 8k^2 + k. \]

2.2. Upper Bound For Even Grids \( P_{2k} \square P_{2k} \).

In *The Radio Number of Grid Graphs* [4], Jiang both comprehensively describes a labeling whose span is the radio number of square grids as well as determines the radio number of all grids. In Section 2.2, we both define the labeling mathematically and disprove Jiang’s radio number for even grids. However, we see that Jiang’s labeling can be used to generate the radio number of even grids. Jiang’s methods bypass reconciling the gap between bounds by employing a trace number analysis of all grids. Here, we focus on the pitfalls of radio labeling techniques that fail to ensure that the radio condition is uniformly satisfied for all possible pairs of vertices. Jiang manages to avoid failing (1) by changing the labeling rules when necessary mathematically, but does not show this algebraically. In addition, as alluded to before, Jiang’s labeling produces a span that exceeds his radio number. We address those concerns in Section 2.2.2.

2.2.1. Jiang’s Labeling and its Span.

**Theorem 2.2** (The Upper Bound for the Radio Number of Even Grids).

\[ \text{rn}(P_{2k} \square P_{2k}) \leq 8k^3 - 4k^2 - 4k + 6. \]

**Proof.** We reconstruct the upper bound associated with Jiang’s labeling algebraically.

The ordering Jiang creates for even grids bears some resemblance to the labeling we use for odd ones. We maximize distances by ensuring that consecutively labeled vertices are on opposite sides of the grid. To ensure that computation of the associated span is organized and accurate, we distinguish between labeling two halves of the grid. Define **Phase 1** as the ordering for the bottom-left and top-right quadrants, and **Phase 2** as the equivalent for the top-left and bottom-right quadrants.
The Phase 1 ordering is comprised of $2k - 2$ cycles, as depicted in Figure 7. In Cycle 1, the first 4 vertices are ordered.

**Cycle 1 (4 vertices):**

$v_{k,k+1}$, $v_{2k,1}$, $v_{1,2k}$, $v_{k+1,k}$

Each stage after Cycle 1 consists of 2 stages, or subcycles, that take place in quadrants opposite to one another. We see that these labeling cycles continue by hopping between diagonal quadrants and labeling corresponding vertices in each respective quadrant within Stages 1 and 2. Cycle 2 orders vertices in the diagonals adjacent to vertices labeled in the previous cycle. Generally, the cycles label the vertices along diagonals within each of the diagonal quadrants while spanning out to cover the rest of the vertices in those quadrants. This labeling of the grid also features a snake-like progression between cycles that preserves diagonalized ordering. It is of use also to recognize that Cycle $k$ orders vertices in the center diagonals of each of the quadrants as we will build our computation of the upper bound around this cycle. We outline the labeling of the vertices during each cycle below (stage in parentheses):

**Cycle 2: (4 vertices)**

$v_{2,2k}$ (Stage 1), $v_{k+2,k}$ (Stage 2), $v_{1,2k-1}$ (1), $v_{k+1,k-1}$ (2)

**Cycle 3: (6 vertices)**

$v_{1,2k-2}$ (1), $v_{k+1,k-2}$ (2), $v_{2,2k-1}$ (1), $v_{k+2,k-1}$ (2), $v_{3,2k}$ (1), $v_{k+3,k}$ (2)

::
Cycle $k - 1$: (2$k - 2$ vertices)
$v_{k-1,k}$ (1), $v_{k-1,k}$ (2), $v_{k-2,k-1}$ (1), $v_{k-2,k-1}$ (2), ... , $v_{1,k+2}$ (1), $v_{k,2}$ (2)

Cycle $k$: (2$k$ vertices)
$v_{1,k+1}$ (1), $v_{k+1,1}$ (2), $v_{2,k+2}$ (1), $v_{k+2,2}$ (2), ... , $v_{k,2k}$ (1), $v_{2k,k}$ (2)

Cycle $k + 1$: (2$k - 2$ vertices)
$v_{k,2k-1}$ (1), $v_{2k,k-1}$ (2), $v_{k-1,2k-2}$ (1), $v_{2k-1,k-2}$ (2), ... , $v_{2,k+1}$ (1), $v_{k+2,1}$ (2)

Cycle $2k - 2$: (4 vertices)
$v_{k,k+2}$ (Stage 1), $v_{2k,2}$ (Stage 2), $v_{k-1,k+1}$ (1), $v_{2k-1,1}$ (2)

<table>
<thead>
<tr>
<th>Cycle $j$</th>
<th>Subcycle</th>
<th>stage 1</th>
<th>stage 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_{1,2k+1-(j-1)}$</td>
<td>$v_{k+2,k+1-(j-1)}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$v_{2,2k+1-(j-2)}$</td>
<td>$v_{k+3,k+1-(j-2)}$</td>
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<td>...</td>
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<tr>
<td></td>
<td>$v_{j,2k+1-(j-j)}$</td>
<td>$v_{k+1+j,k+1-(j-j)}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$v_{j,1}$</td>
<td>$v_{k+1+(j-1),k+2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$v_{j-1,2}$</td>
<td>$v_{k+1+(j-2),k+3}$</td>
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<td>...</td>
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<tr>
<td></td>
<td>$v_{1,j}$</td>
<td>$v_{k+1+(j-j),k+1+j}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$v_{2k+1,j}$</td>
<td>$v_{k+1+(j-1),k}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$v_{2k,j}$</td>
<td>$v_{k+1+(j-2),k-1}$</td>
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<tr>
<td></td>
<td>$v_{2k+1-(j-1),1}$</td>
<td>$v_{1,k+1+(j-j)}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$v_{2k+1-(j-1),2k+1}$</td>
<td>$v_{k+1-(j-1),k}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$v_{2k+1-(j-2),2k}$</td>
<td>$v_{k+1-(j-2),k-1}$</td>
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<td></td>
<td>...</td>
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<tr>
<td></td>
<td>$v_{2k+1,2k+2-j}$</td>
<td>$v_{k+1-(j-j),1}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Ordering of vertices labeled during Cycle $j$

Phase 1 concludes with the ordering of vertex $v_{2k-1,1}$. Once the bottom-left and the top-right quadrants have been ordered, ordering begins on the other pair of diagonal quadrants. Within Phase 2, we will again refer to cycles and stages as defined above. For computational ease, we will refer to the next labeling cycle as Cycle 1, i.e. Cycle 1 of Phase 2. Cycle 1 commences with the labeling of vertices in each of the new quadrants.

Cycle 1 (2 vertices):
$v_{k+1,2k}$ (Stage 1), $v_{1,k}$ (Stage 2)
Once Cycle 1 of Phase 2 is complete, we see that ordering continues in the same manner, but opposite in direction to that of Phase 1. Also, we again see the snake-like progression between labeling cycles, as well as the diagonal hopping.

**Cycle 2:** (4 vertices)

\[ v_{2k,2k-1} \text{ (Stage 1), } v_{k,k-1} \text{ (Stage 2), } v_{2k-1,2k} \text{ (1), } v_{k-1,k} \text{ (2)} \]

**Cycle 3:** (6 vertices)

\[ v_{2k-2,2k} \text{ (1), } v_{k-2,k} \text{ (2), } v_{2k-1,2k-1} \text{ (1), } v_{k-1,k-1} \text{ (2), } v_{2k,2k-2} \text{ (1), } v_{k,k-2} \text{ (2)} \]

\[
\vdots
\]

**Cycle \( k - 1 \): (\( 2k - 2 \) vertices)

\[ v_{2k,k+2} \text{ (1), } v_{k,2} \text{ (2), } v_{2k-1,k+3} \text{ (1), } v_{k-1,k+3} \text{ (2), } \ldots, v_{k+2,2k} \text{ (1), } v_{2,k} \text{ (2)} \]

Note here that the Cycle \( k \) labeling will not begin at the uppermost vertex in the diagonal because those vertices were already labeled in Cycle 1, and thus we have only \( 2k - 2 \) vertices in Cycle \( k \).

**Cycle \( k \): (2k − 2 vertices)

\[ v_{k+2,2k-1} \text{ (1), } v_{2,k-1} \text{ (2), } v_{k+3,2k-2} \text{ (1), } v_{3,k-2} \text{ (2), } \ldots, v_{2k,k+1} \text{ (1), } v_{k,1} \text{ (2)} \]

**Cycle \( k + 1 \): (2k − 2 vertices)

\[ v_{2k-1,k+1} \text{ (1), } v_{k-1,1} \text{ (2), } v_{2k-2,k+2} \text{ (1), } v_{k-2,2} \text{ (2), } \ldots, v_{k+1,2k-1} \text{ (1), } v_{1,k-1} \text{ (2)} \]

\[
\vdots
\]

**Cycle \( 2k - 2 \): (4 vertices)

\[ v_{k+2,k+1} \text{ (Stage 1), } v_{2,1} \text{ (Stage 2), } v_{k+1,k+2} \text{ (1), } v_{1,2} \text{ (2)} \]

**Cycle \( 2k - 1 \) (4 vertices):

\[ v_{k+1,k+1}, v_{1,1}, v_{2k,2k}, v_{k,k} \]
<table>
<thead>
<tr>
<th>Subcycle</th>
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<th>stage 2</th>
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<tr>
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</tr>
<tr>
<td></td>
<td>$v_{2,2k+1-(j-2)}$</td>
<td>$v_{k+3,k+1-(j-2)}$</td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td></td>
<td>$v_{j,2k+1-(j-j)}$</td>
<td>$v_{k+1+j,k+1-(j-j)}$</td>
</tr>
<tr>
<td>2</td>
<td>$v_{i,1}$</td>
<td>$v_{k+1+(j-1),k+2}$</td>
</tr>
<tr>
<td></td>
<td>$v_{j-1,2}$</td>
<td>$v_{k+1+(j-2),k+3}$</td>
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<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
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<tr>
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<td>$v_{1,j}$</td>
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</tr>
<tr>
<td>3</td>
<td>$v_{2k+1,j}$</td>
<td>$v_{k,k+1-(j-1)}$</td>
</tr>
<tr>
<td></td>
<td>$v_{2k,j-1}$</td>
<td>$v_{k-1,k+1-(j-2)}$</td>
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<td>$\vdots$</td>
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<tr>
<td></td>
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<td>$v_{k+1-(j-1),k}$</td>
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<td>$v_{2k+1-(j-2),2k}$</td>
<td>$v_{k+1-(j-2),k-1}$</td>
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<tr>
<td></td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td></td>
<td>$v_{2k+1,2k+2-j}$</td>
<td>$v_{k+1-(j-j),1}$</td>
</tr>
</tbody>
</table>

Table 4. Ordering of vertices labeled during Cycle $j$

With the ordering of vertices established, we must ensure that our labeling is indeed a radio labeling. We formally define Jiang’s labeling $f$ so as to eliminate the risk of failing to satisfy the radio condition for some pair of vertices. We make sure that the radio condition is satisfied for all 1st-order pairs of vertices by rearranging it to form our labeling function $f$. Therefore, define the following for all $i = 1, 2, \ldots, 4k^2$:

$$f(x_i) = \begin{cases} 
0 & i = 1 \\
|f(x_i) - f(x_{i-1})| + d(x_{i-1}, x_i) + 4k - 1 & i \neq 3, 4k^2 - 2 \\
|f(x_i) - f(x_{i-1})| + d(x_{i-1}, x_i) + 4k & i = 3, 4k^2 - 2
\end{cases}$$

Since Jiang’s labeling $f$ is constructed from restructuring the radio condition around pairs of consecutive vertices, Remark 1 below will help in determining whether (1) is satisfied for all $n$th-order pairs of vertices.

**Remark 1.** Recall that, to be a radio labeling, $f$ must satisfy (1) for all distinct vertices of $G = P_{2k} \square P_{2k}$. Therefore, $|f(u) - f(v)| + d(u, v) \geq \text{diam}(G) + 1$, for all distinct vertices $u$ and $v$ of $G$. More specifically, after considering the diameter, the vertices $u$ and $v$ satisfy (1) whenever $|f(u) - f(v)| \geq 4k - 2$.

Labeling $f$, as defined above, guarantees that the labeling satisfies the radio condition for all 1st-order, or consecutive, pairs of vertices. However, we must check at all steps of the labeling that we have satisfied (1) for non-consecutive pairs as well. This is discussed in each section of the
computation of $\text{span}(f)$ below. Computation is done by adding (i) the overall label value added to the span within cycles and (ii) the overall label value added to the span between cycles (while making the transition from one cycle to the next).

Within Cycle 1, we see that $d(v_{k,k+1}, v_{2k,1}) = 2k$, $d(v_{2k,1}, v_{1,2k}) = 4k - 2$, and $d(v_{1,2k}, v_{k+1,1}) = 2k$. Considering the radio condition carefully and noting that $f(x_1) = f(v_{k,k+1}) = 0$, we have $f(x_2) = 2k - 1$, $f(x_3) = 4k - 2 - (4k - 2) + 4k = 2k + 1$, and $f(x_4) = 4k$. Therefore, the label values added to the span are $2k - 1$, $2$, and $2k - 1$, respectively. Summing these gives us an added span value in Cycle 1 equal to $4k$. Note that normally we always choose the lowest label value that satisfies (1) for consecutively labeled pairs of vertices, but that our labeling makes an algebraic exception for $x_3$. Looking carefully at the radio condition for the first $2^{nd}$-order pair of vertices $(x_1, x_3)$, we see that $d(x_1, x_3) + |f(x_1) - f(x_3)| \geq 4k - 1$ under our modified labeling. Had $f$ been defined as a “naive labeling”, $f(x_3)$ would be 1 too low to satisfy (1). We later generalize this bump occurrence by defining a “diagonal triple” following Proposition 1 in Section 4. It is clear that labeling $f$ satisfies (1) for $(x_1, x_3)$. We also see that $d(x_2, x_4) + |f(x_2) - f(x_4)| = 2k + 2k - 1 \geq 4k - 1$ so (1) is also satisfied for $(x_2, x_4)$ under $f$. After computing the added span value associated with the rest of Phase 1 labeling, we show that $f$ satisfies (1) for all other $n^{th}$-order pairs of vertices in Phase 1.

Within Cycles 2 through $k - 1$ and Cycles $k + 1$ through $2k - 2$, we see that the distance between each pair of consecutively labeled vertices is $2k$. As a result, each pair of vertices contributes $2k - 1$ to the overall span of the grid. Each Cycle $j$ in this part of the labeling has $2j - 1$ such consecutive pairs. Since Cycles 2 through $k - 1$ and Cycles $k + 1$ through $2k - 2$ are labeled in an identical fashion, doubling the value for one of these intervals in the below summation will suffice for both. We compute the total span value added during Cycles 2 through $k - 1$ and Cycles $k + 1$ through $2k - 2$:

\[
\{\text{added span value}\} = 2 \cdot \sum_{j=2}^{k-1} (2j - 1)(2k - 1) = 2(2k - 1) \cdot \sum_{j=2}^{k-1} (2j - 1)
\]

\[
= (4k - 2) \cdot \left[ \sum_{j=2}^{k-1} 2j - \sum_{j=2}^{k-1} 1 \right]
\]

\[
= (4k - 2) \cdot \left[ 2 \sum_{j=2}^{k-1} j - (k - 1 - 1) \right]
\]

\[
= (4k - 2) \cdot \left[ 2 \left( \frac{k(k - 1)}{2} - 1 \right) - (k - 2) \right]
\]

\[
= (4k - 2) \cdot [k^2 - k - k] = (4k - 2)(k^2 - 2k)
\]

\[
= 4k^3 - 16k^2 + 4k.
\] (7)

Within Cycle $k$, we have $2k - 1$ pairs of consecutively labeled vertices. Each such pair of consecutive vertices is distance $2k$ from one another. Therefore, $2k - 1$ is added to the span for each of $2k - 1$ pairs. This total value added to the span is easily computed as
(2k - 1)^2 = 4k^2 - 4k + 1. It is sensible to note that this computation can very easily be added to the above summation of label values in Cycles 2 through k - 1 and Cycles k + 1 through 2k - 2. However, we calculate this separately for the purposes of continuity since our Phase 2 computation of added span value in Cycle k will not contain 2k - 1 pairs, but instead 2k - 3 pairs. Thus,

\[ [4k] + [4k^3 - 10k^2 + 4k] + [4k^2 - 4k + 1] = 4k^3 - 6k^2 + 4k + 1 \] (8)

is associated with Phase 1 labeling within cycles.

Now, we must compute the total span value associated with Phase 1 labeling between cycles. This is defined as the sum of span values added while labeling the first vertex in the pending cycle after labeling the final vertex in the just-concluded cycle.

**Between each of consecutive Cycles 1 through 2k - 2, we have distance 2k - 1. For example, the last vertex labeled in Cycle 1, v_{k+1,k}, is distance 2k - 1 from the first vertex labeled in Cycle 2, v_{2,2k}, and thus adds 2k to the span value of our labeling. Since there are 2k - 2 - 1 such transitions between cycles, the total span value added between cycles in Phase 1 is (2k - 3)(2k) = 4k^2 - 6k.**

The total span value added in Phase 1 is the sum of the overall label value added to the span (i) within cycles and (ii) between cycles. Thus,

\[ f(x_{2k^2}) = [4k^3 - 6k^2 + 4k + 1] + [4k^2 - 6k] \]
\[ f(x_{2k^2}) = 4k^3 - 2k^2 - 2k + 1 \] (9)

Before computing the span associated with Phase 2, we must show that f satisfies (1) for all n^{th}-order pairs of vertices within Phase 1. **Within Cycles 2 through 2k - 2, we have either |f(x_{i+2} - f(x_i))| + d(x_{i+2}, x_i) = 2k - 1 + 2k + 1 or |f(x_{i+2} - f(x_i))| + d(x_{i+2}, x_i) = 2k - 1 + 2k - 1 + 2** depending on index i. In either case, |f(x_{i+2} - f(x_i))| + d(x_{i+2}, x_i) ≥ 4k - 1. Therefore, f satisfies (1) for all 2^{nd}-order pair of vertices such that i ≤ 2k^2 - 2. Taking note of Remark 1, since |f(x_{i+3}) - f(x_i)| ≥ 4k - 2 for all i in the ordering of f, all 3^{rd}-order pairs of vertices, and thus, all n^{th}-order pairs of vertices within Phase 1 labeling also satisfy (1). We will confirm the same for Phase 2 labeling before computing a final span for the grid.

Below, we outline the computation of the span of Phase 2 along with the summing of spans for Phases 1 and 2 to produce the upper bound. It should be noted that Phase 2 contains 2k - 1 cycles as a result of the brief labeling cycle that commences Phase 2. Thus, it is also necessary to compute span values associated with Cycle 2k - 1 as well when considering the total span value arising from Phase 2. Lastly, we will ensure that f is a radio labeling by verifying that (1) is satisfied for all n^{th}-order pairs of vertices in Phase 2.

**Within Cycle 1, vertices v_{k+1,2k} and v_{1,k} are labeled.** Yet, since Phase 1 concluded, we will define our labeling of v_{k+1,2k} as part of “between cycles” (between phases, to be more specific). Therefore, the labeling of v_{1,k}, which is of distance 2k from v_{k+1,2k}, adds 2k - 1 to the span value.
Within Cycles 2 through \( k - 1 \) and Cycles \( k + 1 \) through \( 2k - 2 \), the labeling proceeds in an identical fashion, at least algebraically, to that of Phase 1, so the span value added here is the same as in Phase 1: \( 4k^3 - 10k^2 + 4k \).

Within Cycle \( k \), as alluded to before, there are \( 2k - 3 \) pairs of consecutive vertices since labeling in Cycle 1 reduces this number by 1 in each quadrant. This is illustrated in Figure 7. Therefore, \( 2k - 1 \) is added to the span for each of \( 2k - 3 \) pairs. The corresponding span value added \((2k - 1)(2k - 3) = 4k^2 - 8k + 3\).

Within Cycle \( 2k - 1 \), the final 4 vertices are labeled. As with Cycle 1 of Phase 1, the span value added is \( 4k \).

Thus, the total span value added from labeling within cycles during Phase 2 is:
\[
[2k - 1] + [4k^3 - 10k^2 + 4k] + [4k^2 - 8k + 3] + [4k] = 4k^3 - 6k^2 + 2k + 2.
\]

We now find the span value added by Phase 2 is to find the value added to the span between cycles of Phase 2.

Between Cycle \( 2k - 1 \) of Phase 1 and Cycle 1 of Phase 2,
\[
d(v_{2k-1,1}, v_{1,2k}) = 2k - 1 + k - 2 = 3k - 3. \text{ The span value added is } 4k - 1 - (3k - 3) = k + 2.
\]

Between Cycle 1 and Cycle 2, \( d(v_{1,k}, v_{2k,2k-1}) = 2k - 1 + k - 1 = 3k - 2. \text{ The span value added is } 4k - 1 - (3k - 2) = k + 1.\)

Between each of consecutive Cycles 2 through \( 2k - 1 \), there is distance \( 2k - 1 \), so the span value added is \((2k - 1 - 2)(2k - 1) = 4k^2 - 6k\).

Consequently, the total span value added between cycles in Phase 2 is
\[
[k + 2] + [k + 1] + [4k^2 - 6k] = 4k^2 - 4k + 3.
\]

Thus, the total span value added in Phase 2 is obtained by summing the individual span values already determined: \([4k^3 - 6k^2 + 2k + 2] + [4k^2 - 4k + 3] = 4k^3 - 2k^2 - 2k + 5\).

Before establishing a final upper bound for \( r_n(P_{2k} \square P_{2k}) \) by summing the added span values of Phase 1 and Phase 2 labeling, we must show that \( f \) satisfies (1) for all \( n^{th} \)-order pairs of vertices of Phase 2. First, for the \( 2^{nd} \)-order pair of vertices \((x_{2k^2-1}, x_{2k^2+1})\),
\[
|f(x_{2k^2+1}) - f(x_{2k^2-1})| + d(x_{2k^2+1}, x_{2k^2-1}) = 4k - 2 + 2k - 2 = 6k - 4 \geq 4k - 1 \text{ for } k > 1.
\]
Therefore, (1) is satisfied under \( f \) for \((x_{2k^2-1}, x_{2k^2+1})\).

Within Cycles 1 through \( 2k - 2 \) of Phase 2 labeling, we have the following 4 cases:

(a) \(|f(x_{i+2} - f(x_i)) + d(x_{i+2}, x_i) = 2k - 1 + 2k - 1 + 2k = 6k - 2,\)
Figure 8. Jiang’s Solution [left: the ordering; right: radio labeling f]

(b) \(|f(x_{i+2} - f(x_i)) + d(x_{i+2}, x_i)\) = 2k - 1 + k + 1 + k,
(c) \(|f(x_{i+2} - f(x_i)) + d(x_{i+2}, x_i)\) = 2k - 1 + 2k - 1 + 2
(d) \(|f(x_{i+2} - f(x_i)) + d(x_{i+2}, x_i)\) = 2k - 1 + 2k + 1

depending on index i. In any of the cases, \(|f(x_{i+2} - f(x_i)) + d(x_{i+2}, x_i)\) ≥ 4k - 1. Therefore, f satisfies (1) for all 2nd-order pairs of vertices for 2k^2 - 1 ≤ i ≤ 4k^2 - 4. Finally, as with Cycle 1 of Phase 1, looking at the last 2nd-order pairs of vertices in Cycle 2k - 1, (x_{4k^2-3}, x_{4k^2-1}) and (x_{4k^2-2}, x_{4k^2}), we see that:

\(|f(x_{4k^2-3} - f(x_{4k^2-1})) + d(x_{4k^2-3}, x_{4k^2-1})\) = 2k - 1 + 2 + 2k - 2 ≤ 4k - 1
\(|f(x_{4k^2-2} - f(x_{4k^2})) + d(x_{4k^2-2}, x_{4k^2})\) = 2 + 2k - 1 + 2k - 2 ≤ 4k - 1.

Therefore, all 2nd-order pair of vertices in P_{2k} \bigtriangleup P_{2k} satisfy (1) under f. Again considering Remark 1, and since \(|f(x_{i+3} - f(x_i))| ≥ 4k - 2 - 2 for all i in the ordering of f, all 3rd-order pairs of vertices in Phase 2 labeling, and thus, all nth-order pairs of vertices within Phase 2 labeling also satisfy (1). Therefore, all pairs of vertices satisfy (1) under f and f must be a radio labeling.

Finally, we sum the values added in Phases 1 and 2 to establish an upper bound:

\(f(x_{4k^2}) = [4k^3 - 2k^2 - 2k + 1] + [4k^3 - 2k^2 - 2k + 5]\)
\(= 8k^3 - 4k^2 - 4k + 6.\)

Thus, for odd k,

\(rn(P_{2k} \bigtriangleup P_{2k}) ≤ 8k^3 - 4k^2 - 4k + 6.\) \hspace{1cm} (10)

Now, we lay out the computation of the upper bound for the radio number of P_{2k} \bigtriangleup P_{2k} for even k. As mentioned before, most all of the aspects of the labeling and computation for even k.
remain the same, including the snake-like progression of labeling cycles. Any differences will be mentioned in the following discussion. We begin with Phase 1 ordering, which orders the first half of the grid. The most notable difference is the change in diagonals present in each quadrant. As a result, the \(k^{th}\) labeling cycle begins in the lower right hand corner working its way up rather than moving down from the top-left corner. This can be seen in the listing of \(k^{th}\) Cycle vertices below.

**Cycle 1 (4 vertices):**

\(v_{k,k+1}, v_{2k,1}, v_{1,2k}, v_{k+1,k}\)

**Cycle 2: (4 vertices)**

\(v_{2,2k}\) (Stage 1), \(v_{k+2,k}\) (Stage 2), \(v_{1,2k-1}\) (1), \(v_{k+1,k-1}\) (2)

**Cycle 3: (6 vertices)**

\(v_{1,2k-2}\) (1), \(v_{k+1,k-2}\) (2), \(v_{2,2k-1}\) (1), \(v_{k+2,k-1}\) (2), \(v_{3,2k}\) (1), \(v_{k+3,k}\) (2)

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Cycle $k$: $(2k - 2$ vertices$angle$
\[ v_{2k,k+1} \ (1), \ v_{k,1} \ (2), \ v_{2k-1,k+2} \ (1), \ v_{k-1,2} \ (2), \ ... \ , \ v_{k+2,2k-1} \ (1), \ v_{2,k-1} \ (2) \]

Cycle $k + 1$: $(2k - 2$ vertices$angle$
\[ v_{k+1,2k-1} \ (1), \ v_{1,k-1} \ (2), \ v_{k+2,2k-2} \ (1), \ v_{2,k-2} \ (2), \ ... \ , \ v_{2k-1,k+1} \ (1), \ v_{k-1,1} \ (2) \]

Cycle $2k - 2$: (4 vertices)
\[ v_{k+2,k+1} \ (\text{Stage 1}), \ v_{2,1} \ (\text{Stage 2}), \ v_{k+1,k+2} \ (1), \ v_{1,2} \ (2) \]

Cycle $2k - 1$ (4 vertices):
\[ v_{k+1,k+1}, \ v_{1,1}, \ v_{2,k+2}, \ v_{k,k} \]

The computation for the upper bound of $P_{2k} \square P_{2k}$ for even $k$ is constructed in almost the same way as it is for odd $k$. Labeling for even $k$ proceeds in an identical fashion algebraically. While the physical labeling of the even-$k$ square grid emerges in a slightly different manner (visually, in particular), it is apparent from the algebraic computation that all facets of the computation mimic that for odd $k$. That is, we have the summation of all label value additions to the span within Cycles 1, 2 through $k - 1$, $k$, and $k + 1$ through $2k - 2$ algebraically under the same limits and considerations. Between cycles, the same added span value is obtained given the identical algebraic progression of labeling cycles. Because the progression is algebraically equivalent, we can assume safely that $f$ satisfies (1) for even $k$ as it did in the previous case. In other words, $f$ is also a radio labeling for even $k$. Consequently, it is foregone that the span value associated will be equal algebraically to that of the $P_{2k} \square P_{2k}$ for the odd-$k$ incarnation, and thus, summing (i) the overall label value added to the span within cycles and (ii) between cycles yields the same result as with even $k$:

\[ f(x_{4k}) = [4k^3 - 2k^2 - 2k + 1] + [4k^3 - 2k^2 - 2k + 5], \]
\[ = 8k^3 - 4k^2 - 4k + 6. \]

Thus, for even $k$,
\[ \text{rn}(P_{2k} \square P_{2k}) \leq 8k^3 - 4k^2 - 4k + 6. \]  

Finally, we have that, by (10) and (11),
\[ \text{rn}(P_{2k} \square P_{2k}) \leq 8k^3 - 4k^2 - 4k + 6. \]

2.2.2. Comments about Jiang’s Proof.

Before moving on to Section 3, A Lower Bound, we include some thoughts about Jiang’s proof for $P_{2k} \square P_{2k}$, in general [4]. Looking closely at Jiang’s solution (shown in Figure 8), we see that $\text{span}(f) = 174$ for $k = 3$. Jiang insinuates that this ordering and subsequent labeling yields the eventual radio number. Substituting $a = 6$ and $b = 6$ into Jiang’s formula, we get $\text{rn}(G_{a,b}) = \frac{a^2b + b^2a}{2} - ab - a - b + 4 = \frac{6^2*6+6^2*6}{2} - 6(6) - 6 - 6 + 4 = 172$. 


We can assume a few things from this inconsistency. Either Jiang’s solution $s^*$ is not an optimal ordering given the demands of (1) or Jiang has neglected to necessarily include bumps in his computation of the span. That Jiang’s radio number for all (including non-square) grids does not coincide with the span of his optimal labeling of even square grids may, in fact, be a typo since Jiang shows that there is an ordering which requires 2 bumps to satisfy the radio condition on somewhat different grounds. He also shows, after considering all possible radio labelings, that there is at least 1 bump for every 2 corners in the grid. Including the 2 bumps from his proof in the computation of the radio number may be the only neglected step. We discuss bumps at greater length in Section 4.

Jiang’s proof demonstrates that it is possible to characterize large, abstract graphs without considering bounds on the radio number, which he accurately concludes is a simpler, more palatable journey than “dealing with distances $d_i$, bumps $b_i$, and labels $f_i$ simultaneously” [4]. That said, Jiang also avoids directly proving that his labeling is indeed a radio labeling, which may have obscured his results given that (1) must also be satisfied by “higher-order,” or non-consecutive, pairs of vertices in the ordering. Our paper instead tends to focus on the cyclic progression of label values under an explicitly constructed and defined labeling. This affords us the relative luxury of only having to verify that distances are maximized – not that the ordering is optimal. The contrast in styles provides an interesting account of analyzing distance inequalities, optimization problems, and abstract mappings, in general.
3. A Lower Bound

Here, we establish the lower bound for the radio number of all square grids by considering (1) and identifying the aspects that determine a lowest possible span for labelings of grids. Specifically, the proofs of the lower bounds for odd and even grids are outlined in subsections 3.1 and 3.2, respectively. Originally, Liu and Zhu established the methodology that is formalized here. First, one maximizes distances between consecutively labeled pairs of vertices in a hypothetical labeling that satisfies the radio condition for at least these pairs. We will call this a “naive labeling”: note that there is no assumption of satisfying the radio condition by all pairs of vertices. Maximizing these distances minimizes the span of this naive labeling. Its span provides a lower bound for the span of all radio labelings and, thus, a lower bound for the radio number of the graph [5]. In 2009, Calles and Gomez applied this methodology to grids [1]. To carry out the distance-maximization technique on odd grids, the distances between consecutively ordered pairs of vertices are divided up into horizontal and vertical components. These components distinguish characteristics between consecutively labeled pairs of vertices that allow us to analyze the labeling possibilities of the graph.

Let \( f \) be a radio labeling of \( G = P_n \Box P_n \) that defines an ordering of the vertices of \( G \):

\[
 f(x_i) < f(x_{i+1}) \text{ for all } i = 1, 2, \ldots, n^2 - 1.
\]

Now, the ordering of the vertices is used to construct a labeling. We start with (1):

\[
d(x_i, x_{i+1}) + |f(x_i) - f(x_{i+1})| \geq \text{diam}(G) + 1.
\]

Since we are interested in determining a lower bound for the radio number of \( P_n \Box P_n \), we rearrange the above inequality. By determining the lowest possible span amongst the spans of all radio labelings of \( P_n \Box P_n \), we obtain a lower bound for \( rn(G) \). Since each successive vertex label value \( f(x_{i+1}) \) increases, we discard the absolute value sign in (1) and reverse the order of the label difference. We obtain an expression representing the progression of label values in a naive labeling — one in which the radio condition is satisfied for all 1st-order pairs of the labeling, but that does not take into account label differences for 2nd-order or higher pairs, e.g. \( |f(x_{i+1}) - f(x_{i-1})| \):

\[
f(x_{i+1}) \geq f(x_i) + \text{diam}(G) + 1 - d(x_i, x_{i+1}) \text{ for } i = 1, 2, \ldots, n. \tag{12}
\]

By definition, it is assumed that \( f(x_1) \geq 0 \). Substituting the diameter into (12) gives

\[
f(x_{i+1}) \geq f(x_i) + 2n - 2 + 1 - d(x_i, x_{i+1}) \text{ for } i = 1, 2, \ldots, n.
\]

The inequality is recursive, so we generate our lowest possible span by summing along consecutive vertices across all indices.

\[
f(x_{i+1}) \geq \sum_{i=1}^{n^2-1} [2n - 1 - d(x_i, x_{i+1})] = (n^2 - 1)(2n - 1) - \sum_{i=1}^{n^2-1} d(x_i, x_{i+1}). \tag{13}
\]

Upon showing that distance must be maximized for a minimized span, we complete our lower bound proof by computing distance summation value. Calles and Gomez noticed that we can only interpret distances between vertices on a grid by developing a system by which we can deconstruct the distances between any two vertices into discrete value. Clearly, distances between vertices organized in a grid-like fashion are comprised of horizontal and vertical components. To this end, assume \( x_i = \sigma(i), \tau(i) \), where \( \sigma(i) \) defines the row index of \( x_i \) and \( \tau(i) \) the column index of \( x_i \). Furthermore, for the sake of clarity, assume \( \sigma_1 \) and \( \tau_1 \) represent \( \sigma(i) \) and \( \tau(i) \), respectively. Then

\[
d(x_i, x_{i+1}) = |\sigma_i - \sigma_{i+1}| + |\tau_i - \tau_{i+1}|. \tag{14}
\]
Expanding (14) across all consecutive pairs of vertices in any particular labeling, a long summation accounting for all indices of all vertices is achieved.

\[
\sum_{i=1}^{n^2-1} d(x_i, x_{i+1}) = |\sigma_1 - \sigma_2| + |\tau_1 - \tau_2| \\
+ |\sigma_2 - \sigma_3| + |\tau_2 - \tau_3| \\
\vdots \\
+ |\sigma_{n^2-1} - \sigma_{n^2}| + |\tau_{n^2-1} - \tau_{n^2}|
\]

(15)

Thus, the span of the labeling is constructed by combining (13) and (15) into one long inequality:

\[
f(x_{n^2}) \geq (n^2 - 1)(2n - 1) - (|\sigma_1 - \sigma_2| + |\tau_1 - \tau_2| + \\
\vdots \\
+ |\sigma_{n^2-1} - \sigma_{n^2}| + |\tau_{n^2-1} - \tau_{n^2}|).
\]

(16)

We refer to the above expression as the summation of distances. Each vertex contributes 4 elements to the summation of distances. However, since \(x_1\) and \(x_{n^2}\) are the first and last vertices to be labeled, respectively, the terms \(\sigma_1, \sigma_{n^2}, \tau_1,\) and \(\tau_{n^2}\) each only appear once in the summation of distances. For example, it is clear from the expansion (15) that \(x_2\) contributes a total of 4 elements to the summation, but \(x_1\) contributes only 2 elements, as a result of appearing at the start of the summation. Henceforth, the value associated with a specific arrangement will be referred to as lost label sum \(\lambda\). Therefore, 4 total lost label values \(-\sigma_1, \tau_1, \sigma_{n^2}, \) and \(-\tau_{n^2}\) will vanish as a result of corresponding to either \(x_1\) or \(x_{n^2}\). Of these 4 lost label values, two are positive in the above summation and two are negative. This must be considered when incorporating these absences from (15). Below, the process by which the lower bounds for the radio number of both odd and even square grids is set forth.

We devise a strategy by which to maximize the sum of distances in order to minimize the span. Calles and Gomez’s work demonstrates that it is possible to arrange index values of vertices such that the lowest values are being subtracted and the highest added. In essence, we express the sum of distances outlined above as a difference of the sum of all positive elements and that of all negative elements:

\[
D = \sum_{i=1}^{n^2-1} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n^2-1} P - \sum_{i=1}^{n^2-1} N,
\]

(17)

where \(P\) is the set of the highest row and column indices and \(N\) is the set of lowest row and column indices for a grid \(P_n \sqcap P_n\). When \(n = 2k\), \(P = \{k+1, k+2, \ldots, 2k\}\), while \(N = \{1, 2, \ldots, k\}\). Similarly, for odd \(n = 2k+1\), \(P = \{k+2, k+3, \ldots, 2k+1\}\), while \(N = \{1, 2, \ldots, k\}\). The index \(k+1\) does not yet appear in either set. We safely assume that all \(k+1\) index elements can be divided evenly amongst the sets and thus cancel one another out completely.

Using the notation \(D\) to represent the sum of distances of all consecutive pairs, we can express a primitive incarnation (not yet having incorporated the effect of lost label values in \(\lambda\)) for the maximized sum of distances with respect to odd (18) and even (19) grids as follows:
Recall that lost label sums are the outcome of values that do not occur in the maximized summation of distances as a result of corresponding to the indices of vertices falling first or last in the ordering of \( f \). Now, the lower bounds for even and odd grids are established by incorporating vanishing sum \( \lambda \) into (18) and (19), which do not yet consider it. Thus, \( \lambda \) is simply subtracted from the summation to express a totally maximized sum of distances for any given labeling of the grid:

\[
D = \sum_{i=1}^{(2k+1)^2-1} d(x_i, x_{i+1}) = 4 \left[ \sum_{i=k+2}^{2k+1} i - \sum_{i=1}^{k} i \right] \tag{18}
\]

\[
D = \sum_{i=1}^{(2k)^2-1} d(x_i, x_{i+1}) = 4 \left[ \sum_{i=k+2}^{2k} i - \sum_{i=1}^{k} i \right] \tag{19}
\]

For the purposes of discussing lost label sums that now appear in (20) and (21), let \( O \) refer to the set \( \{\sigma_1, \tau_1, \sigma_{n^2}, \tau_{n^2}\} \) in the proofs for both odd and even grids in subsections 3.1 and 3.2, respectively. We now extend Calles and Gomez’s groundwork for deconstructing grids to incorporate the consequence of lost label values within the summation of distances.

### 3.1. Lower Bound For Odd Grids \( P_{2k+1} \square P_{2k+1} \).

**Theorem 3.1 (Lower Bound of Odd Grids).** For odd grid graphs,

\[
\rho n(P_{2k+1} \square P_{2k+1}) \geq 8k^3 + 8k^2 - 1.
\]

**Proof.** For odd grids, there are 3 distinct cases of possibilities uniquely arranging lost label values in \( O = \{\sigma_1, \tau_1, \sigma_{n^2}, \tau_{n^2}\} \). As mentioned in the previous section, cases emerge based on what vertices in the mapping function are labeled first and last – that is, which vertices in the grid are \( x_1 \) and \( x_{n^2} \) in the labeling order. We rigorously examine all ways in which vertices can be ordered. In order to distinguish between possibilities, we look at each of the following three cases, which contain unique compositions of \( O \) derived from how many distinct values are inherent to the indices. The three cases are as follows:

**Case 1.** \( O \) is comprised solely of distinct elements

e.g., \( x_1 = v_{1,2} \) and \( x_{n^2} = v_{3,4} \)

If we proceed by employing the original strategy of ordering low row and column indices negatively and high indices positively, two scenarios in which four distinct label values are lost can be identified: \( O_1 = \{k, k + 1, k + 2, k + 3\} \) or \( O_2 = \{k - 1, k, k + 1, k + 2\} \). For set \( O_1 \), the lost label sum is \( (k + 3) + (k + 2) - (k + 1) - k = 4 \). Either scenario gives rise to \( \lambda = 4 \). Therefore, **Case (1)** would give rise to an updated form of (20) if it represented the lowest possible \( \lambda \):
Case 2. $O$ is comprised of exactly 3 distinct elements
e.g., $x_1 = v_{1,2}$ and $x_{n^2} = v_{3,2}$

In Case (2), there are three distinct elements in $O$. Thus, two distinct values are lost once each and the other twice. Recall that low values should be negative, while high values should be positive. Only two possible sets emerge:

$O_1 = \{k, k + 1, k + 1, k + 2\}$ or $O_2 = \{k - 1, k, k, k + 1\}$. Again, in each case, one middle value is defined to be positive, the other negative. In either scenario, $\lambda = 2$.

Case 3. $O$ is comprised of exactly 2 distinct elements
There are precisely two possible subcases that arise:

Subcase 3.A. Each distinct element is repeated once
e.g., $x_1 = v_{1,1}$ and $x_{n^2} = v_{3,3}$

Considering that $k + 1$ is the undistributed value for an odd grid, it is necessary to place $k + 1$ in $O$. We have two possibilities: $O_1 = \{k, k, k + 1, k + 1\}$ and $O_2 = \{k + 1, k + 1, k + 2, k + 2\}$. Adding up negative and positive values yields $\lambda = 2$ for each set.

Subcase 3.B. 1 element is repeated 3 times, the other element just once
e.g., $x_1 = v_{2,2}$ and $x_{n^2} = v_{2,4}$

We can arrange indices so that $k$ disappears three times or so that $k + 1$ disappears three times. From these two options arise three distinct sets: $O_1 = \{k, k, k, k + 1\}$, $O_2 = \{k, k + 1, k + 1, k + 1\}$, and $O_3 = \{k + 1, k + 1, k + 1, k + 2\}$. After making two high elements positive and two low elements negative, it is clear that each set results in $\lambda = 1$.

It is worthwhile to take note that $O$ cannot consist of just one distinct element as this would require labeling the same vertex twice. Thus, it is now possible to definitively choose an arrangement in which the lost label value $\lambda$ is minimized in order to maximize $\sum d(x_i, x_{i+1})$. This occurs in Case 3.B. We now obtain an updated form of (20) for the maximum sum of distances:

$$\sum_{i=1}^{(2k+1)^2-1} d(x_i, x_{i+1}) = 4 \left[ \sum_{i=k+2}^{2k+1} i - \sum_{i=1}^{k} i \right] - 4$$  \hspace{1cm} (22)
Using summation formulas, the inequality is reduced to:

\[
\begin{align*}
f(x_{4k^2-1}) &\leq 4k(2k+1)(4k+1) - 4k(2k+1)(k+1) - 1 \\
&= (k+1)(4k)(2k) - 1 \\
&= 8k^2(k+1) - 1 \\
f(x_{4k^2-1}) &\geq 8k^3 + 8k^2 - 1 \quad (24)
\end{align*}
\]

Thus, the lower bound is attained:

\[
\text{rn}(P_{2k+1} \Box P_{2k+1}) \geq 8k^3 + 8k^2 - 1.
\]

3.2. Lower Bound For Even Grids \(P_{2k} \Box P_{2k}\).

**Theorem 3.2 (Lower Bound of Even Grids).** For even grids,

\[
\text{rn}(P_{2k} \Box P_{2k}) \geq 8k^2 - 4k^2 - 4k + 2.
\]

**Proof.** The proof is analogous to that of the odd case.

**Case 1.** \(O\) is comprised solely of distinct elements
e.g., \(x_1 = v_{1,2}\) and \(x_n = v_{3,4}\)

\(O\) becomes \(\{k - 1, k, k + 1, k + 2\}\). When the elements of \(O\) are summed (again remembering to use low indices negatively and high indices positively), \(\lambda = 4\), as in the odd case.

**Case 2.** \(O\) is comprised of exactly 3 distinct elements
e.g., \(x_1 = v_{1,2}\) and \(x_n = v_{3,2}\)

Then \(O_1 = \{k - 1, k, k, k + 1\}\) or \(O_2 = \{k - 1, k, k + 1, k + 1\}\). In either case, \(\lambda = 2\).

**Case 3.** \(O\) is comprised of exactly 2 distinct elements

Again, precisely two possible subcases emerge:

**Subcase 3.A.** Each distinct element is repeated once
e.g., \(x_1 = v_{1,1}\) and \(x_n = v_{3,3}\)

\(O_1 = \{k, k, k + 1, k + 1\}\). Adding up negative and positive values results in a lost label sum \(\lambda = 2\).

**Subcase 3.B.** 1 element is repeated 3 times; the other element just once
e.g., \(x_1 = v_{2,2}\) and \(x_n = v_{2,4}\)

\(O_1 = \{k, k, k + 1\}\) or \(O_2 = \{k, k + 1, k + 1, k + 1\}\). In either case, \(\lambda = 1\).

Of the three cases for even grids, **Case 3.B** provides the lowest lost label sum. We represent the maximized sum of distances for even grids by updating (21) with \(\lambda = 1\):
The effective span, \( f(x_n) \), is then computed for an even grid with a maximum sum of distances by substituting the above into (13) and simplifying. Again, a multiplicative factor of \( 2k \) must be included to represent the emergence of row and column indices over the \( 2k \) rows (or columns) of the grid.

Using summation formulas gives us

\[
\begin{align*}
f(x_{4k^2}) &\geq \left[ (2k)^2 - 1 \right] (4k - 1) - 4(2k) \left( \sum_{i=k+1}^{2k} i - \sum_{i=1}^{k} i \right) - 1.
\end{align*}
\]

Since \( \text{span}(f) = f(x_{4k^2}) \),

\[
\text{span}(f) \geq 8k^3 - 4k^2 - 4k + 2,
\]

and the lower bound is attained:

\[
\text{rn}(P_{2k} \Box P_{2k}) \geq 8k^3 - 4k^2 - 4k + 2. \tag{26}
\]
For the remainder of analysis, let the following definitions and assumptions apply.

- Let $G = P_{2k} \Box P_{2k}$.
- Let $f$ be some labeling of a graph $H$. Define $\{x_1, x_2, ..., x_n\}$ to be an ordering of the vertices of $H$ such that $f(x_i) < f(x_{i+1})$ for $i = 1, 2, ..., n - 1$.
- Define vertex label $x_i = v_{\sigma(i), \tau(i)}$, where $\sigma(i)$ defines the row index of $x_i$ and $\tau(i)$ the column index of $x_i$. Furthermore, for the sake of clarity, assume $\sigma_i$ and $\tau_i$ represent $\sigma(i)$ and $\tau(i)$, respectively.
- Without loss of generality, assume $f(x_1) = 0$.
- Define $f$ to be a naive labeling if $f(x_i) = f(x_{i-1}) + \text{diam}(G) + 1 - d(x_i, x_{i-1})$.
- The term “bump” will be used to refer to the increase in the span of a naive labeling required to ensure that the labeling satisfies the radio condition (1) for some $n^{th}$-order pair of vertices.
- An even grid can be subdivided into four quadrants, defined as $Q_1$, $Q_2$, $Q_3$, and $Q_4$ as labeled in Figure 1. Define diagonally opposite quadrants as one of the following quadrants pairs: $(Q_1, Q_3)$ or $(Q_2, Q_4)$ – e.g., $Q_1$ and $Q_3$ are diagonally opposite quadrants. Define adjacent quadrants as a pair of quadrants having the characteristic of not being diagonally opposite to one another – e.g., $Q_1$ and $Q_2$ are adjacent quadrants (see Figure 8).

**Proposition 1** (Diagonal Triple). Let $G$ be an even grid $P_{2k} \Box P_{2k}$. Let $f$ be a naive labeling of $G$ with ordering $\{x_1, x_2, ..., x_n\}$ defined by $f$. Let $f^*$ be any radio labeling inducing the same ordering $\{x_1, x_2, ..., x_n\}$ on the vertices of $G$. Say $x_{i-1}$ and $x_{i+1}$ are vertices in a diagonally opposite quadrant. Then,

$$f^*(x_{i+1}) - f^*(x_{i-1}) \geq f(x_{i+1}) - f(x_{i-1}) + 1,$$  \hspace{1cm} (27)
for any 3 consecutively labeled vertices $x_{i-1}$, $x_i$, and $x_{i+1}$ as defined above.

**Proof.** Let $G$, $f^*$, and $f$ be as hypothesized. By applying the definition of naive labeling $f$ and considering restrictions placed on $\tau$ and $\sigma$ values, we will show that the difference of label values between $2^{\text{nd}}$-order pairs of vertices requires a "bump" of +1 for $f$ to be a radio labeling. Our naive labeling conveys that $f(x_i) = f(x_{i-1}) + \text{diam}(G) + 1 - d(x_i, x_{i-1})$, so after inserting the diameter,

$$f(x_i) = f(x_{i-1}) + 4k - 1 - d(x_i, x_{i-1}).$$

Similarly,

$$f(x_{i+1}) = f(x_i) + 4k - 1 - d(x_{i+1}, x_i).$$

After substitution,

$$f(x_{i+1}) = f(x_{i-1}) + 8k - 2 - [d(x_i, x_{i-1}) + d(x_{i+1}, x_i)]. \quad (28)$$

It is important to establish conditions that will govern analysis of our designated set of vertices. Since we have a $2^{\text{nd}}$-order pair of vertices $x_{i-1}$ and $x_{i+1}$ in one quadrant and another vertex $x_j$ in a diagonally opposite quadrant, it is advantageous to relate their placement in the grid to their row and column indices. So, assume without loss of generality that the following conditions hold and are referred to as our **quadrant inequalities**:

$$\tau_{i-1}, \sigma_{i-1} \in P,$$

$$\tau_{i+1} \leq \tau_{i-1},$$

$$\sigma_{i+1} \leq \sigma_{i-1} \quad (29)$$

Set restrictions on relative row and column indices of vertices part of the same triple such that they abide by quadrant inequalities (29) as follows:

$$|\tau_{i+1} - \tau_i| \geq k \pm \Delta,$$

$$|\sigma_{i+1} - \sigma_i| \geq k \mp \Delta \quad (30)$$

where $\Delta$ is the value associated with distance between consecutive vertices that causes any deviation from a naive labeling.

Upon setting restrictions on $\tau$ and $\sigma$, we examine the sum of distances closely.

$$d(x_i, x_{i-1}) + d(x_{i+1}, x_i) = |\sigma_i - \sigma_{i-1}| + |\tau_i - \tau_{i-1}| + |\sigma_{i+1} - \sigma_i| + |\tau_{i+1} - \tau_i|$$

$$= \sigma_{i-1} - \sigma_i + \tau_{i-1} - \tau_i + \sigma_{i+1} - \sigma_i + \tau_{i+1} - \tau_i$$

$$= \sigma_{i-1} + \sigma_{i+1} - 2\sigma_i + \tau_{i-1} + \tau_{i+1} - 2\tau_i.$$  

We add $2\sigma_{i+1} - 2\sigma_i + 2\tau_{i+1} - 2\tau_i$ to rearrange the equality and substitute for expressions that permit us to characterize particular relationships that exist within the grid. Continuing the equality established with a more useful form of the right side, we have

$$d(x_i, x_{i-1}) + d(x_{i+1}, x_i) = 2(\sigma_{i-1} - \sigma_i + \tau_{i-1} - \tau_i) + [2\sigma_i - 2\sigma_i + 2\tau_i - 2\tau_i]$$

Noting carefully that $(\sigma_{i-1} - \sigma_i) + (\tau_{i-1} - \tau_i)$ is really just the distance between $2^{\text{nd}}$-order pairs of vertices $x_{i-1}$ and $x_{i+1}$, we can establish a final equality that captures the evolution of the
naive labeling at the outset into a distance expression that relates both a 1st-order pair as well as a 2nd-order pair of vertices. Substituting (30),

\[ d(x_i, x_{i-1}) + d(x_{i+1}, x_i) = 2(\sigma_{i+1} - \sigma_i) + 2(\tau_{i+1} - \tau_i) + d(x_{i-1}, x_{i+1}) \]

\[ \geq 4k + 2\Delta + 2\Delta + d(x_{i-1}, x_{i+1}) \]

Substituting the above into (28) yields an inequality characterizing 2nd-order pairs of vertices:

\[ f(x_{i+1}) \leq f(x_{i-1}) + 8k - 2 - 4k - d(x_{i-1}, x_{i+1}) \]

Reorganizing gives rise to an inequality reflecting the relationship between 2nd-order pairs of vertices constrained by quadrant inequalities (29):

\[ f(x_{i+1}) - f(x_{i-1}) + d(x_{i-1}, x_{i+1}) \leq 4k - 2 \]

(1) mandates that, for any pair of vertices u and v in \( P_{2k} \boxtimes P_{2k} \), \( |f(u) - f(v)| + d(u, v) \geq 4k - 1 \). Therefore, the result of Proposition 1 demonstrates that the value \( f(x_{i-1}) \) is too low for \( f \) to be a radio labeling of an even grid \( G \).

Henceforth, define any set of 3 consecutively labeled vertices in the above configuration to be a **diagonal triple**. Since we employ a naive labeling, the radio condition is satisfied for pairs in the above configuration, but is not necessarily fulfilled for 2nd-or-higher-order pairs of vertices. Thus, we have shown that a typical naive labeling for an even grid structure will not necessarily satisfy the radio condition. We will use this to show that (i) the lower bound for \( rn(P_{2k} \boxtimes P_{2k}) \) must be at least 2 higher, and (ii) that the lower bound may in fact also require several “bumps” of various kinds in order to define a radio labeling. Lemma 1 builds on the latter, detailing how the effect of this kind of vertex triplet in an even grid rules out many kinds of labelings and ultimately helps determine the lowest span of a viable radio labeling for an even grid \( G \).

Recall, from Section 3, that the distance-maximizing labeling used to construct the lower bound for \( rn(P_{2k} \boxtimes P_{2k}) \) is not necessarily a radio labeling. We show that radio labelings of grids are forms of “adjusted distance-maximizing labelings” that abide by radio condition (1) at all times. First, it is necessary re-define **distance-maximizing labeling** to clarify how optimal spans relate to naive labelings of even grids. In Section 3, we demonstrate that we can obtain maximized distances between consecutively labeled vertices by making higher \( \sigma \) and \( \tau \) index values positive and lower \( \sigma \) and \( \tau \) values negative. In Section 3.2, we see that there are many possible spans with maximized distances due to vanishing index values, and consequently, different \( \lambda \) values arise. Henceforth, refer to distance-maximizing labelings as an ordering of vertices with the characteristics specified in Section 3.2. Let \( S = 8k^3 - 4k^2 - 4k + 2 \). Thus, for any distance-maximizing labeling \( f \), \( \text{span}(f) \in \{S, S + 1, S + 3\} \) corresponding to \( \lambda \in \{1, 2, 4\} \), respectively.

For the purposes of Lemma 1 below, define \( A \) to be a set of **independent diagonal triples** if no two diagonal triples in \( A \) share more than one vertex.
Lemma 1 (α independent diagonal triples). Let G be an even grid. Let \( c_0 \) be a distance-maximizing and naive labeling with ordering \( \{x_1, x_2, \ldots, x_n\} \) defined by \( c_0 \) and let \( c^* \) be a radio labeling inducing the same ordering \( \{x_1, x_2, \ldots, x_n\} \) on the vertices of G.

\[
\text{span}(c^*) \geq \text{span}(c_0) + \alpha.
\]

where \( \alpha \) is the size of the largest set \( A \) associated with labeling \( c^* \) of G.

**Proof.** Say \( x_{i-1}, x_i, x_{i+1}, x_{i+2}, \) and \( x_{i+3} \) constitute 3 consecutively labeled diagonal triples in the ordering patterns of labeling \( c_0 \) and radio labeling \( c^* \) of an even grid G.

Then, by Proposition 1, \( c^*(x_{i+1}) - c^*(x_{i-1}) \geq c_0(x_{i+1}) - c_0(x_{i-1}) + 1 \).

Proposition 1 does not require that \( c^*(x_{i+2}) - c^*(x_i) \geq c_0(x_{i+2}) - c_0(x_i) + 1 \).

However, Proposition 1 does imply that \( c^*(x_{i+3}) - c^*(x_{i+1}) \geq c_0(x_{i+3}) - c_0(x_{i+1}) + 1 \).

Vertex triplets \( (x_{i-1}, x_i, x_{i+1}) \) and \( (x_{i+1}, x_{i+2}, x_{i+3}) \) are independent diagonal triples. Then, each element of the largest set \( A \) associated with a radio labeling \( c^* \) contributes a “bump” of +1 to the span of any radio labeling relative to that of a distance-maximizing labeling with the same ordering. Therefore,

\[
\text{span}(c^*) \geq \text{span}(c_0) + \alpha.
\]

Remark 2. Since, in any distance-maximizing and naive labeling, there are 2 sets of vertices that satisfy the conditions set forth in Lemma 1 and quadrant inequalities (29), we can infer, without loss of generality, that, up to permutations, there is at least 1 diagonal triple for each of diagonal quadrant pairs \( (Q1, Q3) \) and \( (Q2, Q4) \). Thus, we have at least a bump of +2 resulting from independent diagonal triples in a distance-maximizing, naive labeling – that is, \( \alpha \geq 2 \).

Remark 2 confirms that there are multiple bumps associated with a radio labeling of \( P_{2k} \square P_{2k} \). The following discussions regarding “adjacent flips” demonstrates a diagonal triple is not the only source of bumps. Before stating Lemma 2, we define the meaning of adjacent flip. An adjacent flip is the occurrence of any consecutively labeled pair of vertices in adjacent quadrants. Thus, an adjacent flip represents one of two possibilities: (a) \( x_i \) is in either \( Q1 \) or \( Q3 \) and \( x_{i+1} \) is in either \( Q2 \) or \( Q4 \) or (b) \( x_i \) is in \( Q2 \) or \( Q4 \) and \( x_{i+1} \) is in either \( Q1 \) or \( Q3 \).

Lemma 2 (β adjacent flips). Let G be an even grid. Let \( c_0 \) be a distance-maximizing and naive labeling with ordering \( \{x_1, x_2, \ldots, x_n\} \) defined by \( c_0 \) and let \( c^* \) be a radio labeling inducing the same ordering \( \{x_1, x_2, \ldots, x_n\} \) on the vertices of G. Then,

\[
\text{span}(c^*) \geq \text{span}(c_0) + 2 \cdot \beta,
\]

where \( \beta \) is the number of adjacent flips that occur in the complete labeling of G under \( c^* \).

**Proof.** Let G be an even grid and \( c_0 \) a distance-maximizing labeling of G. Define \( D_0, P_0, \) and \( N_0 \) as in (17) corresponding to labeling \( c_0 \). Thus, \( P_0 \) and \( N_0 \) represent the sets of largest and smallest row and column indices, respectively, in \( D_0 \), for a distance-maximizing labeling \( c_0 \). Also, define \( D^* \)
as the equivalent for radio labeling $c^*$. 

Consider exchanging the smallest element in $P_0$ with the largest element in $N_0$, while preserving all other elements of $P_0$ and $N_0$. This can be thought of as making the smallest possible concession in deviating from the distance-maximizing labeling $c_0$. As the largest element in $N_0$ is $k$ and the smallest in $P_0$ is $k + 1$, exchanging these elements results in replacing $+(k + 1) - k$ with $+k - (k + 1)$ in the maximized sum of distances for labeling $c^*$. Thus, each adjacent flip results in an increase of at least 2 in the span of the resulting labeling, so $D^* = D_0 - 2$. The statement follows directly.

**Remark 3.** Notice that any complete labeling of $P_2k \square P_2k$ must include at least one adjacent flip. Thus, $\beta \geq 1$. This necessitates an additional bump of at least $+2$ to any complete radio labeling of an even grid and, more specifically, in relation to the span of any distance-maximizing labeling that constructs a lower bound for the graph.

Finally, we associate a change in the span of any radio labeling corresponding to $\alpha$ diagonal triples and $\beta$ adjacent flips in Lemma 3 below.

**Lemma 3 (\(\alpha\) independent diagonal triples and \(\beta\) adjacent flips).** Let $G$ be an even grid. Let $c_0$ be a distance-maximizing and naive labeling with ordering \(\{x_1, x_2, \ldots, x_n\}\) defined by $c_0$ and let $c^*$ be a radio labeling inducing the same ordering \(\{x_1, x_2, \ldots, x_n\}\) on the vertices of $G$. If $c^*$ contains $\alpha$ independent diagonal triples and $\beta$ adjacent flips, then

\[
\text{span}(c^*) \geq \text{span}(c_0) + \alpha + 2 \cdot \beta. \tag{33}
\]

**Proof.** Given that any distance-maximizing and naive labeling of $G$ contains adjacent flips and independent diagonal triples, we note that independent diagonal triples and adjacent flips are independent of one another, and Lemma 3 follows directly from Lemmas 1 and 2.

We see in Section 3.2 that we can construct a lower bound for the radio number of even grids by considering all naive labelings. We also observe that distance-maximizing labelings have the lowest spans of all naive labelings. Since we choose only one of the distance-maximizing, naive labelings to produce a lowest possible span $S$ (there are numerous arrangements with $\lambda = 1$), it is necessary to show that all other distance-maximizing, naive labelings have spans that must be bumped up sufficiently in order to be radio labeling. Thus, we eliminate the spans arising from other distance-maximizing, naive labelings corresponding to unique values $\lambda \in \{1, 2, 4\}$, and show that any naive labeling must have its span bumped up sufficiently to be radio labeling.

**Corollary 1 (Minimum bump moves).** Let $G$ be an even grid. Let $c_0$ be a distance-maximizing and naive labeling with ordering \(\{x_1, x_2, \ldots, x_n\}\) defined by $c_0$ and let $c^*$ be a radio labeling inducing the same ordering \(\{x_1, x_2, \ldots, x_n\}\) on the vertices of $G$. Then

\[
\alpha + \beta \geq 3 \tag{34}
\]

**Proof.** Section 3.2 shows that we can have many possible naive labelings of $G$ given different $\lambda$ values. By contrast, Remarks 1 and 2 show that, regardless of labeling arrangement, any distance-maximizing and naive labeling of $G$ must contain at least 2 independent diagonal triples and 1 adjacent flip. Thus, $\alpha \geq 2$ and $\beta \geq 1$. The statement follows directly.
Corollary 2 (Minimum span of a distance-maximizing radio labeling of $P_{2k} \square P_{2k}$). Let $G$ be an even grid. Let $c_0$ be a distance-maximizing and naive labeling with ordering $\{x_1, x_2, ..., x_n\}$ defined by $c_0$ and let $c^*$ be any radio labeling inducing the same ordering $\{x_1, x_2, ..., x_n\}$ on the vertices of $G$. Then

$$\text{span}(c^*) \geq \text{span}(c_0) + 4. \quad (35)$$

Proof. The statement follows directly from Lemma 3 and Corollary 1.

With Corollaries 1 and 2 now in place, we determine the radio number of all even grids. We have shown that a distance-maximizing and naive labeling of an even grid requires numerous bumps in order to be a radio labeling. In Theorem 4, we eliminate all labelings whose span exceeds our upper bound for the radio number and thus determine the radio number to be the span of Jiang’s even grid labeling.

Theorem 4 (Radio number of even square grids). Let $G = P_{2k} \square P_{2k}$. Let $f^*$ be a naive labeling of $G$. Let $f$ be a radio labeling of $G$. The radio number of an even square grid is given by

$$\text{rn}(P_{2k} \square P_{2k}) = 8k^3 - 4k^2 - 4k + 6.$$ 

Proof. Let $S = 8k^3 - 4k^2 - 4k + 2$. Theorem 3.2 proves that the lowest possible span of any naive labeling is $S$. The distance-maximizing approach employed in Section 3 shows that distance-maximizing labelings have the lowest spans of all naive labelings. Suppose $f^*$ has a span in $\{S, S + 1, S + 3\}$. Corollary 2 proves that any radio labeling $f$ of $G$ must have a span of at least $S + 4$.

Theorem 2.2 proves an upper bound for the radio number:

$$\text{span}(f) \leq 8k^3 - 4k^2 - 4k + 6.$$ 

Closing bounds on $\text{rn}(P_{2k} \square P_{2k})$ gives us

$$S + 4 \leq \text{span}(f) \leq 8k^3 - 4k^2 - 4k + 6.$$ 

Thus, $\text{span}(f) = \text{rn}(G) = 8k^3 - 4k^2 - 4k + 6$, as required.
5. Observations and Future Research

An investigation of the radio number of grids brings to light many important results. First and foremost, it relates the value of certain techniques in exploring highly specialized Cartesian products. In other words, it is seen how the structure of any abstract graph can affect how that graph must abide by certain constraints and the ways in which it can even be analyzed.

Calles and Gomez lay the groundwork for applying bounded approaches in determining radio numbers of grids. Their inspiration becomes the focus of another endeavor entirely, namely a commentary on what shortcomings bounded approaches have in accomplishing this task. It is seen in [4] that labeling functions geared at capturing “radio characteristics” of specific graphs may be too indirect for some examples in the space of graph structure analysis. Jiang’s labeling itself echoes of the inherent bound-reconciling analysis that is required in Jiang’s overall approach, but reduces the problem to optimization. He shows that it is possible to circumvent conventional bounding techniques and avoid the pitfalls of balancing the various demands of distances, bumps, and labelings (or orderings) under (1). It is also clear that bump analysis is not inherently linked to bounded approaches. In this paper, however, it is shown that bump analysis does in fact parallel bound construction itself. Similarly, exhaustive fulfillment of (1) under algebraic constraints requires an eventual choice in analytical paths.

Those who are interested in pursuing other avenues in this field might consider applying the techniques illustrated here for the immediate purpose of finding $\text{rn}(P_{2k+1} \boxtimes P_{2k+1})$ using conventional bounding approaches. While the exact same strategy used here may not prove fruitful, the approach outlined in this paper provides a strong basis by which to examine square grids. When considering the fact that grids, like numerous other families of graphs, are really just products of simpler graphs, it is likely that one may extend the findings in this paper suitably to odd grids. An odd grid additionally contains an even grid and two more paths (one extra row and one extra column). This itself, may be a solid enough base with which to determine the radio number of odd grids by applying Lemma 3 and Proposition 1 rigorously. Likewise, it is a natural possibility to investigate Cartesian products of grids and other graphs while incorporating some of the abstract considerations established here, as well as in [4] and [5].

The methodologies employed within this paper intend to capture the nature of the sometimes nebulous field of radio labeling. Networks of radio transmitters tie in closely with the idea of graph theory because their arrangements transcend the rigid constraints of just a singular characteristic. Instead, investigating the radio number of abstract graph structures that possess numerous, and often disparate, structural components leads to a great deal of insight about how abstract graph families preserve overall form. We know that, in the real world, the study of networks and inter-connectivity is of profound significance and the interplay between overall structure and internal chaos is manifest in global systems. Hopefully, in addition to commenting successfully about the radio number of square grids, this paper will develop into a source of knowledge about applying analytical techniques for the exploration of all kinds of networks and structures around us. Although it may seem like a daunting task, graph properties and structure correspond so fundamentally to the radio labelings of graphs that linking properties such as diameter, adjacency, or chromatic number is both a rich source of both future progress in graph theory and combinatorics, as well as relatively unprobed territory with considerable potential in modern physics and applied mathematics.
REFERENCES


