

# The cosmological behavior of Bekenstein's modified theory of gravity

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We study the background cosmology governed by the Tensor-Vector-Scalar theory of gravity proposed by Bekenstein. We consider a broad family of potentials that lead to modified gravity and calculate the evolution of the field variables both numerically and analytically. We find a range of possible behaviors, from scaling to the late time domination of either the additional gravitational degrees of freedom or the background fluid.

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## I. INTRODUCTION

Bekenstein has proposed a covariant and relativistic theory of gravity [1] which is meant to provide for an alternative to dark matter. The theory is called TeVeS (which stands for Tensor-Vector-Scalar) and can have the Bekenstein-Milgrom Aquadratic Lagrangian [2] non-relativistic limit, which provides for Milgrom's Modified Newtonian Dynamics [3] as an explanation of galactic rotation curves. TeVeS incorporates a dynamical scalar field  $\phi$  and two metric tensors and unit-timelike vector field  $A^\alpha$ .

In the case of a homogeneous and isotropic space-time, the TeVeS field equations reduce to a form similar to Friedman-Robertson-Walker (FRW) form. There have been several studies [1, 4, 5, 6, 7] of the resulting background cosmology with different choices of the TeVeS free function. In this paper we extend the preliminary analysis of TeVeS cosmology to more general settings.

The paper is organized as follows. In Sec. 2 we give the general TeVeS equations, their applications to cosmology and the notation we will use throughout the paper. In Sec. 3 we will present our numerical results and we will discuss their behavior with some analytical insights. We conclude in Sec. 4.

## II. FUNDAMENTALS OF TEVES COSMOLOGY

### A. TeVeS action

Let us first describe the basics of TeVeS theory. We use the conventions of [8].

In TeVeS we have two tensor fields which are the two metrics  $g_{\mu\nu}$  and  $\tilde{g}_{\mu\nu}$  mentioned above, a scalar field  $\phi$  and a vector field  $A^\alpha$ . They can be linked through

$$g_{\mu\nu} = e^{-2\phi} (\tilde{g}_{\mu\nu} + A_\mu A_\nu) - e^{2\phi} A_\mu A_\nu \quad (1)$$

The total action of the system is  $S_{\text{sys}} = S_g + S_s + S_v + S_m$  where  $S_g$  is the traditional Einstein-Hilbert action

built with the *Einstein frame* metric  $\tilde{g}_{\mu\nu}$ ,  $S_s$  is the action for  $\phi$  and  $\mu$ ,  $S_v$  is the action for the vector field and  $S_m$  denotes the action of ordinary matter. Explicitly we have

$$\begin{aligned} S_g &= \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \tilde{R} \\ S_s &= -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} [\mu (\tilde{g}^{\mu\nu} - A^\mu A^\nu) \phi_{,\alpha} \phi_{,\beta} + V(\mu)] \\ S_v &= -\frac{1}{32\pi G} \int d^4x \sqrt{-\tilde{g}} [K_B F^{\alpha\beta} F_{\alpha\beta} - 2\lambda (A^\mu A_\mu + 1)] \\ S_m &= \int d^4x \sqrt{-g} L_m [g_{\mu\nu}, \chi^A, \nabla \chi^A] \end{aligned}$$

where  $G$  is the bare gravitational constant,  $K_B$  is a dimensionless constant,  $\lambda$  is a Lagrange multiplier imposing a unit-timelike constraint on the vector field,  $F_{\alpha\beta} = 2\nabla_{[\alpha} A_{\beta]}$  and  $\mu$  is a non-dynamical field with a free function  $V(\mu)$ . The matter Lagrangian  $L_m$  depends only on the *matter frame* metric  $g_{\mu\nu}$  and a generic collection of matter fields  $\chi^A$ .

Although the theory was designed to provide for a relativistic theory of MOND, taken at face value this need not be the case. It all lies within the choice of the function  $V(\mu)$ . The requirement that both exact MOND and exact Newtonian limits exist within the theory puts some constraint on the possible forms of  $V(\mu)$ . In particular these two requirements fix the derivative of  $V(\mu)$  to be of the form

$$V'(\mu) \equiv \frac{dV}{d\mu}(\mu) = -V_0 \frac{\mu^2}{(1 - \mu/\mu_0)^m} f(\mu) \quad (2)$$

where  $V_0$  and  $\mu_0$  are constants and  $f(\mu)$  is still an arbitrary function, not related to the MOND or Newtonian limits, whose only constraint is that  $f(0) = f_0$  is non-vanishing. The  $\mu^2$  in the numerator is what is precisely required to have an exact MOND limit (which is reached as  $\mu \rightarrow 0$ ), whereas the factor  $1 - \mu/\mu_0$  is what is required to have an exact Newtonian limit (which is reached as  $\mu \rightarrow \mu_0$ ). Bekenstein chooses the simplest case, namely  $m = 1$ , which we shall also do in this work.

What remains is to choose the (still free) function  $f(\mu)$ . It turns out that for cosmological models we must have  $V' \geq 0$ . Therefore in the  $m = 1$  case that we are considering, this cannot happen in the same branch as for a quasistatic system, i.e. we need  $\mu > \mu_0$  for cosmology. (There is a very clever alternative considered in [9] with its cosmology studied in [10]. We do not follow that approach here). In the case that there is a single extremum in  $V'$  for  $\mu > \mu_0$  one has a further choice imposed by the requirement that  $V'$  be single valued in the branch considered : either use the branch from  $\mu = \mu_0$  up to the position of the extremum or the branch from the extremum to infinity. Bekenstein makes the second choice, which is what we will also do in this work.

With the above considerations in mind, Bekenstein makes the choice  $f(\mu) = (\mu/\mu_0 - 2)^2$ . We will therefore generalize this choice as

$$f(\mu) = \sum_n c_n (\hat{\mu} - \mu_a)^n \quad (3)$$

where  $\hat{\mu} = \mu/\mu_0$ ,  $\mu_a$  is a constant and  $c_n$  a set of additional constants. We also let  $n$  run over negative values. The Bekenstein model is recovered as  $\mu_a = 2$ ,  $c_2 = 1$  and  $c_{n \neq 2} = 0$ .

Different functions have also been proposed especially by [9] and [11] and slight modifications to Bekenstein's toy model have been given by [7].

### B. Homogeneous and isotropic cosmology

For the purpose of this paper we will restrict ourselves to homogeneous and isotropic spacetimes (see [5, 8] for a detailed investigation of the inhomogeneous case). We adopt the Friedmann-Lemaitre-Robertson-Walker metric with the scale factor  $a$  for the matter frame metric and  $b = ae^\phi$  for the Einstein frame metric. We choose a coordinate system such that the components of the vector field take the form  $A^\alpha = (1, 0, 0, 0)$ . The modified Friedmann-equation becomes

$$3\tilde{H}^2 = 8\pi G e^{-2\phi} (\rho_\phi + \rho_X) \quad (4)$$

where  $\tilde{H} = \dot{b}/b$  is the Einstein-frame Hubble parameter with an overdot means a derivative with respect to the coordinate time chosen and where we have defined a field density  $\rho_\phi$  as

$$\rho_\phi = \frac{1}{16\pi G} e^{2\phi} (\mu V' + V) \quad (5)$$

The ordinary fluid energy density  $\rho_X$  evolves as usual as

$$\dot{\rho}_X = -3\frac{\dot{a}}{a}(1+w)\rho_X \quad (6)$$

where  $w$  is the equation of state parameter of the fluid. Relative densities  $\Omega_i$  can be defined as usual as

$$\Omega_i = \frac{\rho_i}{\rho_i + \rho_\phi} \quad (7)$$

Variation of the action with respect to the field  $\mu$  gives back the constraint

$$\dot{\phi}^2 = \frac{1}{2}V' \quad (8)$$

which fixes  $\mu$  as a function of  $\dot{\phi}$ .

Finally the scalar field evolves according to a system of two first order equations given by

$$\dot{\phi} = -\frac{1}{2\mu}\Gamma \quad (9)$$

and

$$\dot{\Gamma} + 3\tilde{H}\Gamma = 8\pi G e^{-2\phi} \rho_X (1 + 3w). \quad (10)$$

## III. NUMERICAL RESULTS

### A. Bekenstein's toy model

Let us first revise the results presented in [5], i.e. that  $\phi$  acts as a tracker field in the radiation, matter and  $\Lambda$  eras.

In this special case of the Bekenstein toy model, the function  $V'$  can be explicitly inverted analytically by setting  $q = 64\pi l_B^2 \dot{\phi}^2 / (3\mu_0)$ ,  $r = q\sqrt{q^2/12 + 64/81}$   $s = (27q^2 + 128)/54$ ,  $p_1 = (r+s)^{1/3}$ ,  $p_2 = \sqrt{9p_1 + 12 + 16/p_1}$ ,  $p_3 = (9p_1^2 - 24p_1 + 16)p_2 - 54qp_1$  and one obtains an explicit expression for the non-dynamical field  $\mu$ :

$$\mu = \frac{1}{6} \sqrt{-\frac{p_3}{p_1 p_2}} + \frac{1}{6} p_2 + 1 \quad (11)$$

In fig.1 we plot  $\Omega_\phi$  as a function of  $\log(a)$  for the different epochs in the Universe history: panel *a*) radiation era, panel *b*) matter era and panel *c*) the  $\Lambda$  era. As can be seen in the figure, for several different initial conditions,  $\phi$  has an attractor in the different epochs, which leads to the tracking behavior.

In panel (a) of fig.2 we plot the fractional energy densities  $\Omega_i$  as a function of  $\log(a)$ . As can be seen from the figure,  $\phi$  synchronizes its energy density with the dominant component of the Universe. As mentioned in [5], during the radiation era  $\Omega_\phi$  reaches  $\frac{3}{2\mu_0}$  and during the matter and  $\Lambda$  eras the limit is  $\Omega_\phi = \frac{1}{6\mu_0}$ . From panel (a) of fig.2 it is clear that the overall behavior of  $\phi$  is simply a sequence of the three tracking epochs. This behavior is similar to other general cosmological theories involving tracker fields [12].

### B. Generalized function

The behavior of  $\phi$  depends on the form of the function  $f(\mu)$  chosen. In this section we investigate the main features which result from a function given by equation (3).

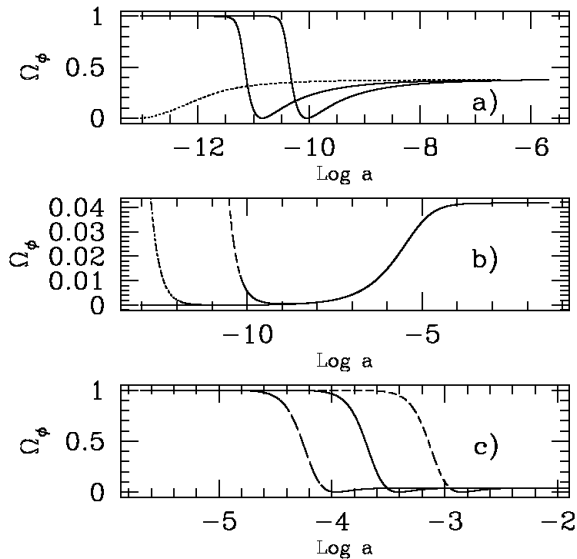


FIG. 1:  $\Omega_\phi$  as a function of  $\log(a)$ . The units for the scale factor are arbitrary.  $\phi$  has an attractor which correspond to tracking the three constituents independently of the initial conditions. We illustrate this here, taking  $\mu_0 = 4$  and plotting three different histories in: a) radiation era, b) matter era and c) the  $\Lambda$  era.

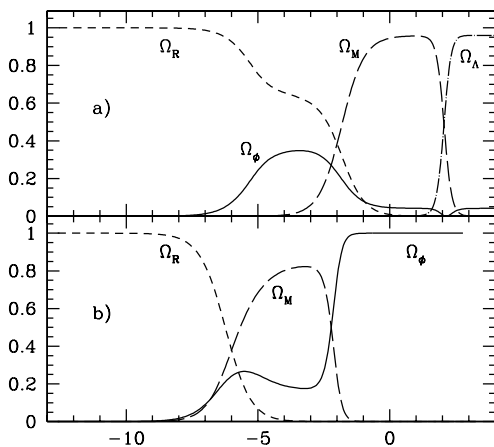


FIG. 2: Panel a) represents Bekenstein's toy model, panel b) is a model with  $c_2$ ,  $c_{-1}$  and  $c_{-2}$  as non vanishing. The long dashed line represents  $\Omega_R(t)$ , the dot-dashed one is  $\Omega_\Lambda(t)$ , the solid line is  $\Omega_\phi(t)$  and the short dashed line is for  $\Omega_M(t)$ .

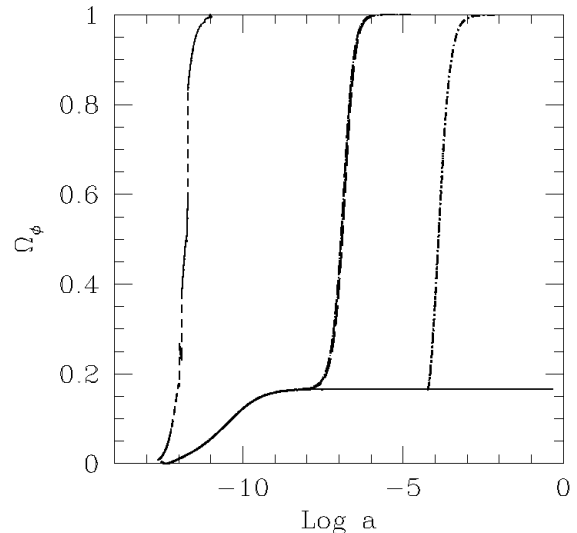


FIG. 3:  $\Omega_\phi$  vs  $\log a$  for the generalized function given by equation (3). The solid line corresponds to  $c_2 \neq 0$ . The dotted and dotted-short-dashed curves mixed together correspond to  $(c_2, c_1)$  and  $(c_2, c_1, c_0)$  as the only non vanishing coefficients. The long-dashed curve corresponds to  $(c_2 \neq 0, c_{-1} \neq 0)$  and the dotted-long-dashed one to  $(c_2 \neq 0, c_{-1} \neq 0, c_{-2} \neq 0)$ . Finally the short-dashed curve on the left corresponds to  $c_{-2} \neq 0$ .

In Fig.3 we plot the evolution of  $\Omega_\phi$  in the presence of non-relativistic matter. With only  $c_2$  as non-zero coefficient, we have an apparently stable attractor. If we choose in (3) a non-zero  $c_1$  (while keeping the non-zero  $c_2$ ) we find that the tracking is still possible but less stable than the one with  $c_2$  alone. Hence, we can have longer lived tracking by decreasing  $c_1$ . Adding  $c_0$  with suitably chosen coefficients (see subsection on analytical results below) does not change this behavior as can be seen in Fig.3. The curves corresponding to the two cases just discussed essentially overlap, and depart from tracking at  $\log a = -4$ . It seems that in general mixing cases of different  $n$  together will lead to the loss of tracking behavior, with at best having temporary trackers. While this turns out to be the most frequent behavior there are counter examples : a mixed case of  $n = 1$  and  $n = 3$  with  $\mu_a = 5/7$  and  $c_1/c_3 = -4/49$  also exhibits perfectly stable tracker behavior (see fig. 4). As we will see in the next section on analytical treatment, the existence of tracking behavior or not, can be understood in terms of a generic set of rules.

Tracking behavior disappears altogether when we add a negative power in  $(\hat{\mu} - \mu_a)$ . We can take either  $c_{-1} \neq 0$  or  $c_{-1} \neq 0$  and  $c_{-2} \neq 0$  and the effect will essentially be the same. The curves corresponding to these cases almost overlap in Fig.3 (by chance), and  $\Omega_\phi$  rapidly tends to one at  $\log a = -8$ . In both of these cases, the system

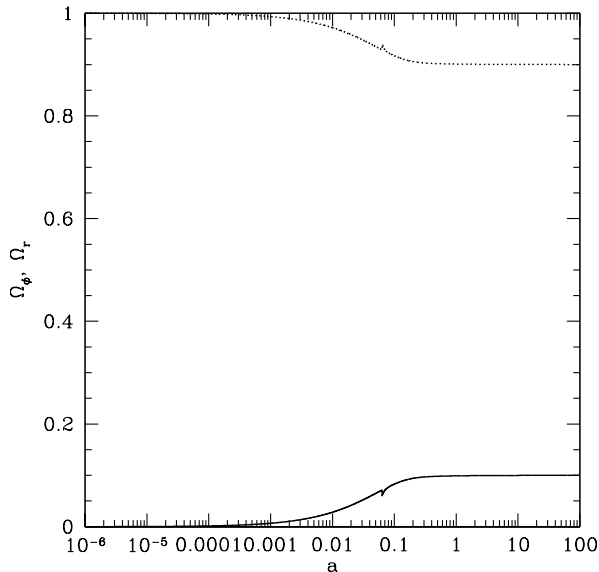


FIG. 4: A mixed case with  $n = 1$  and  $n = 3$  which exhibits a perfectly stable tracker, in the radiation era.

exits tracking quite quickly. The initial stage tracking is a result of the presence of  $c_2$ ; if we set  $c_2 = 0$  and keep  $c_{-2} \neq 0$ , for example, the leftmost curve of Fig.3 indicates that tracking is impossible. Higher powers of  $(\hat{\mu} - \mu_a)$  can suppress this instability temporarily.

What if we add more negative powers  $n$ ? The lower panel of Fig.5 shows the evolution of the relative densities  $\Omega_i$  in the case where only  $c_{-3} \neq 0$  and where only a cosmological term is present.

The middle panel of the same figure show that  $\Gamma/\tilde{H}$  is a constant during tracking while  $\phi$  tends to a constant in the infinite future. The upper panel shows that  $\tilde{H}$  also tends to a constant. In the case where only radiation and/or pressureless matter is present, we find that evolution has an early unstable tracking behavior just like the  $\Lambda$  case, but eventually evolution stops in finite time. Adding a cosmological term can remedy the situation by creating a second tracking era.

Considering even more negative terms, e.g. only  $c_{-4} \neq 0$  as in Fig.6 gives yet different behaviors. An early tracking era eventually comes to an end followed by a fluid dominated era ( $\Omega_X = 1$ ). During both eras we find that the field  $\Gamma$  always evolves linearly with  $\log t$  (if the background fluid is radiation).

Our numerical study of the dynamics of this system picks out four types of behavior: perfect tracking, tracking followed by domination of  $\Omega_\phi$ , no tracking and eventual singular behavior and finally initial tracking and eventual fluid domination. Given that we have considered a reasonable generalization of the function proposed by Bekenstein, we can see that these types of behavior will be widespread in TeVeS. We can now proceed to un-

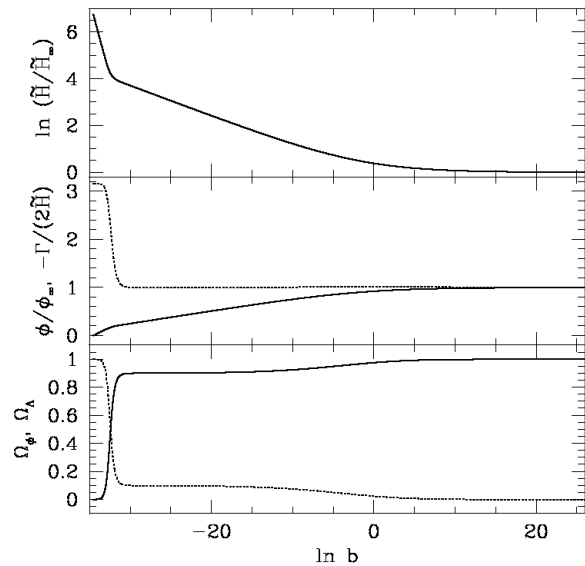


FIG. 5: A case with only  $c_{-3} \neq 0$  and a cosmological constant. Upper panel : Evolution of  $\ln(\tilde{H}/\tilde{H}_\infty)$  with  $\ln b$  (see section on the analytical treatment of the  $n = -3$  case for the definition of  $\tilde{H}_\infty$ ). Middle panel : Evolution of  $\phi/\phi_\infty$  (solid) and  $\Gamma/\tilde{H}$  (dotted) with  $\ln b$  (see section on the analytical treatment of the  $n = -3$  case for the definition of  $\phi_\infty$ ). Lower panel : Evolution of the relative densities  $\Omega_\phi$  (dotted) and  $\Omega_\Lambda$  (solid) showing an initial temporary tracking phase followed by  $\Lambda$ -domination.

derstand what lies behind these different regimes.

#### IV. ANALYTICAL RESULTS

In this section we investigate the existence and stability of tracker solutions and develop approximate analytical approximations to the evolution of the scalar field for the various choices of function and tracking regimes.

When does the field track? The numerical studies of the previous section showed that tracking occurs as  $V'$  tends to its zero point. It therefore makes sense to study solutions when  $\mu$  is close to this point, i.e. we let  $\hat{\mu} = \mu_a(1 + \epsilon)$  with  $\epsilon \ll 1$ . The impact of different terms in the function series is investigated below.

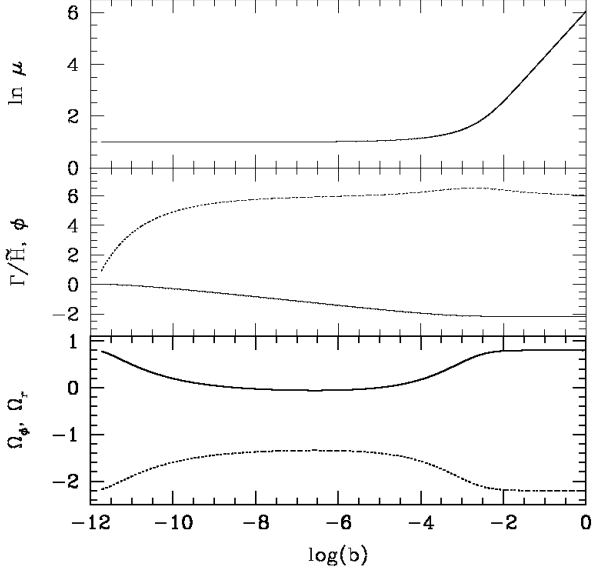


FIG. 6: A case with only  $c_{-4} \neq 0$ . Upper panel : Evolution of  $\ln \mu$  (solid) with  $\ln b$ . Middle panel : Evolution of  $\phi$  (solid) and  $\Gamma/\tilde{H}$  (dotted) with  $\ln b$ . Lower panel : Evolution of relative densities of radiation (solid) and scalar field (dotted) versus  $\ln b$  with only a radiation fluid present, showing a late radiation domination.

### A. Case $n \geq 1$

In this case the function  $V$  and its derivative  $V'$  are given by

$$V_{(n)} = \mu_0 V_0 \left[ I_n + \frac{n+2+\mu_a+(n+1)\hat{\mu}}{(n+1)(n+2)} (\hat{\mu} - \mu_a)^{n+1} + \sum_{m=1}^n \frac{(1-\mu_a)^{n-m}}{m} (\hat{\mu} - \mu_a)^m + (1-\mu_a)^n \ln |\hat{\mu} - 1| \right] \quad (12)$$

and

$$V'_{(n)} = \mu_0^2 V_0 \frac{\hat{\mu}^2}{\hat{\mu} - 1} (\hat{\mu} - \mu_a)^n \quad (13)$$

respectively, where we put a subscript  $(n)$  to denote the case considered, and where  $I_n$  is an integration constant (for the case  $n$ ).

Clearly  $V'_{(n)}(\mu_a) = 0$  for any  $n$ . In the case of  $V_{(n)}$  however, it depends on the choice of the integration constant  $I_n$ . We will discuss the relevance of this constant further below.

Let us choose for the moment the constant  $I_n$  such that  $V_{(n)}(\mu_a) = 0$ . We then expand the relevant quantities as a Taylor series about the point  $\hat{\mu} = \mu_a$ . We have

$$V = \frac{\mu_0 \mu_a \hat{V}_0}{n+1} \epsilon^{n+1} \quad (14)$$

and

$$V' = \hat{V}_0 \epsilon^n \quad (15)$$

where  $\hat{V}_0 = \mu_0^2 \mu_a^{2+n} V_0 / (\mu_a - 1)$ . The field density is therefore given by

$$\rho_\phi = \frac{1}{16\pi G} e^{2\phi} \mu_0 \mu_a \hat{V}_0 \epsilon^n \quad (16)$$

to lowest order in  $\epsilon$ .

Using the constraint equation which links  $\mu$  with  $\dot{\phi}$ , namely (8), and taking the square root, we obtain

$$\dot{\phi} = \beta \sqrt{\frac{\hat{V}_0}{2}} \epsilon^{n/2} \quad (17)$$

where  $\beta = \pm 1$  is an artifact of taking the square root.

The Taylor expansion of the auxiliary scalar field  $\Gamma$  is

$$\Gamma = -2\mu_a \mu_0 \beta \sqrt{\frac{\hat{V}_0}{2}} \epsilon^{n/2} \quad (18)$$

Since we are interested on finding possible tracking behavior, motivated by the numerical results, it is reasonable to make the ansatz  $\rho_\phi = A \rho_X$  with  $A$  a constant, giving

$$\epsilon^n = A \frac{16\pi G}{\mu_0 \mu_a \hat{V}_0} e^{-2\phi} \rho_X \quad (19)$$

Using the Friedmann equation we get

$$\tilde{H} = \sqrt{\frac{8\pi G(1+A)}{3}} e^{-2\phi} \rho_X, \quad (20)$$

Eliminating  $\epsilon^n$  from the auxiliary scalar field  $\Gamma$  we get

$$\Gamma = -2\beta \sqrt{8\pi G A \mu_a \mu_0} e^{-2\phi} \rho_X. \quad (21)$$

Using (10) along with the derivative of (21) we find

$$\Omega_\phi = \frac{A}{1+A} = \frac{(1+3w)^2}{3\mu_a \mu_0 (1-w)^2}. \quad (22)$$

More specifically in the three special eras considered above, namely radiation, matter and  $\Lambda$  eras, we get  $\Omega_\phi = \frac{1}{3\mu_a \mu_0}$  in matter and  $\Lambda$  eras, while  $\Omega_\phi = \frac{3}{\mu_a \mu_0}$  in the radiation era. Taking  $\mu_a = 2$  for the Bekenstein toy model we obtain the results previously found in [5].

Now we specify the constant  $\beta$  which tells whether  $\dot{\phi}$  is negative or positive. Changing the time variable to  $\ln a$  and using (20) and (17), one finds that  $\frac{d\phi}{d \ln a} = \phi_1$  is approximately a constant during tracking, given by

$$\phi_1 = \frac{1+3w}{\mu_a \mu_0 (1-w)\beta - (1+3w)} \quad (23)$$

This results to  $\phi = \phi_0 + \phi_1 \ln a$ . Introducing the above in (19) we find  $\epsilon \propto a^{-\frac{2\phi_1+3(1+w)}{n}}$  where  $\rho_0$  is the fluid

density at  $a = 1$ . Consequently  $\epsilon$  will be a decreasing function going to 0 asymptotically, and  $\epsilon \rightarrow 0$  if and only if tracking is stable, that's to say if  $2\phi_1 + 3(1+w) > 0$ . Using (23) this condition is equivalent to

$$\frac{-(1+3w)^2 + 3(1-w^2)\mu_a\mu_0\beta}{\mu_a\mu_0(1-w)\beta - (1+3w)} > 0 \quad (24)$$

The above inequality is satisfied if and only if either  $\beta = 1$  and  $w > -1/3$  or  $\beta = -1$  and  $w < -1/3$ . Therefore  $\beta$  is completely fixed by the equation of state of the background fluid. In particular we see that  $\dot{\phi}$  has to change sign when passing from the matter to  $\Lambda$  era which results to  $\rho_\phi$  going momentarily to zero [5].

Now let us discuss the role integration constant  $I_n$ . Its sole impact would be to add a constant in the Friedmann equation. What can motivate such a constant? For exact MOND limit we need  $V \rightarrow 0$  as  $\mu \rightarrow 0$ . It is in general impossible to have both  $V(0) = 0$  and  $V(\mu_a) = 0$ , except in a very special and unique case : the Bekenstein toy model, i.e.  $n = 2$  and  $\mu_a = 2$ . In every other case (not a mixed case with many  $c_n$  non-zero, which will be consider in another section), setting  $V(\mu_a) = 0$  would spoil the *exact* MOND limit. The impact of such a constant in the evolution equations however, is what precisely destroys the perfect tracker. In other words as long as  $I_n \ll 16\pi G e^{-2\phi} \rho_X$ , we would have the tracker solution found above. When eventually  $\rho_X$  decreases below this threshold, which is bound to happen always, tracking stops. In such case there are two possible behaviors after tracking stops. If  $V(\mu_a) < 0$  the requirement that the LHS of the Friedmann equation is positive, creates a pathological situation where  $\Omega_\phi$  decreases without bound *below* zero while the fluid relative density increases unbounded above one to counterbalance. In the opposite case where  $V(\mu_a) > 0$  the universe enters a de-Sitter phase in the Einstein frame where  $\dot{b}/b$  tends to a constant, and  $\Omega_\phi$  tends rapidly to one. This second case is what stops the temporary trackers found in the previous section numerically, in the mixed cases.

We end this case by noting that the Hubble parameter in our approximation will be slightly different from the one we obtain in General Relativity but rather if the background radiation temperature is  $T$ , it will be given by  $H \propto T^\gamma$  where  $\gamma = 2\phi_1 + \frac{3}{2}(1+w)$ . This can be potentially important when calculating nucleosynthesis constraints, as in the radiation era we would have  $\gamma = 2(1+\phi_1)$  rather than the canonical value  $\gamma = 2$ . The impact of this kind cosmologies on nucleosynthesis has been investigated in [13].

### 1. Case $n = 0$

In this case the function is given by

$$V_{(0)} = \mu_0^3 V_0 \left( I_0 + \frac{1}{2}\hat{\mu}^2 + \hat{\mu} + \ln|\hat{\mu} - 1| \right) \quad (25)$$

and

$$V'_{(0)} = \mu_0^2 V_0 \frac{\hat{\mu}^2}{\hat{\mu} - 1}. \quad (26)$$

The constraint can be inverted analytically to give

$$\hat{\mu} = \frac{\dot{\phi}^2 + \dot{\phi}\sqrt{\dot{\phi}^2 - 2\mu_0^2 V_0}}{\mu_0^2 V_0} \quad (27)$$

It is therefore clear that  $|\dot{\phi}|$  is bounded from below :  $|\dot{\phi}| \geq \mu_0\sqrt{2V_0}$ . At this value  $\hat{\mu}$  takes its minimum possible value given by  $\mu_c = 2$ . The upshot is that contrary to the case where  $n \geq 1$ , we have  $V'(\mu_c) = 4\mu_0^2 V_0$  i.e. it is not zero. We will show in this subsection that this creates a pathological situation.

First note that since  $\dot{\phi}$  cannot be zero, the two branches of positive and negative  $\dot{\phi}$  are disconnected and we can change variables from  $\dot{\phi}$  to  $\hat{\mu}$ . The equivalent of (10) is then given by

$$\frac{3\hat{\mu} - 4}{\hat{\mu} - 1}\dot{\hat{\mu}} + 6\tilde{H}\hat{\mu} + \frac{\beta(1+3w)}{\mu_0^2} \sqrt{\frac{2}{V_0}} \frac{\sqrt{\hat{\mu} - 1}}{\hat{\mu}} y = 0 \quad (28)$$

where  $y = 8\pi G e^{-2\phi} \rho_X$  and  $\beta$  is once again the sign of  $\dot{\phi}$ .

Suppose that the  $\hat{\mu} = 2$  point is reached at some moment in time  $t_c$ . Then  $V(\hat{\mu} = 2) = \mu_0^3 V_0 (I_0 + 12)$  giving

$$\dot{\hat{\mu}}_c = -\sqrt{6\mu_0^3 V_0 (I_0 + 12) + 12y_c} - \frac{\beta(1+3w)}{2\mu_0^2 \sqrt{2V_0}} y_c \quad (29)$$

where the subscript "c" on any variable denotes the value of that variable at time  $t_c$ , and where we have assumed that  $\dot{b}/b > 0$ . We consider again two subcases depending on the sign of  $\beta(1+3w)$ .

If  $\beta(1+3w) > 0$ , then  $\dot{\hat{\mu}}$  is always negative (including the point  $\hat{\mu}_c$ ) except the very special case where  $I_0 = -12$  and  $y_c = 0$ . Since however we are free to choose the initial condition for  $\mu$  and  $y$  independently, we will get that generically  $\dot{\hat{\mu}}_c < 0$ . Therefore the system of differential equations is bound to become ill-defined at some point  $t_c$ .

If  $\beta(1+3w) < 0$  then there are values of  $\hat{\mu}$  (and also  $\hat{\mu}_c$ ) which are positive given a large enough  $y$ , which means that  $\hat{\mu}$  will bounce off the point  $\hat{\mu} = 2$  and start increasing again. However as the equations are propagated further in time,  $y$  decreases even further until once again  $\dot{\hat{\mu}} < 0$  and  $\hat{\mu}$  will decrease back to  $\hat{\mu} = 2$ . This time though  $\dot{\hat{\mu}}_c < 0$ , and the system of differential equations becomes once more ill-defined.

### 2. Case $n = -1$

It turns out that for negative  $n$  we need to consider each case separately for the first few  $n$ 's. Lets start with the  $n = -1$  case.

In this case the function is given by

$$V_{(-1)} = \mu_0^3 V_0 \left( I_{-1} + \hat{\mu} + \frac{\mu_a^2}{\mu_a - 1} \ln |\hat{\mu} - \mu_a| - \frac{1}{\mu_a - 1} \ln |\hat{\mu} - 1| \right) \quad (30)$$

and

$$V'_{(-1)} = \mu_0^2 V_0 \frac{\hat{\mu}^2}{(\hat{\mu} - 1)(\hat{\mu} - \mu_a)}. \quad (31)$$

The constraint can be inverted analytically to give

$$\hat{\mu} = \frac{(\mu_a + 1)\dot{\phi}^2 + |\dot{\phi}| \sqrt{(\mu_a - 1)^2 \dot{\phi}^2 + 2\mu_0^2 V_0 \mu_a}}{2\dot{\phi}^2 - \mu_0^2 V_0} \quad (32)$$

Restricting the allowed range for  $\hat{\mu}$  to be greater than  $\mu_a$  (since we have a singularity in  $V'$  at  $\hat{\mu} = \mu_a$ ) we see that  $\dot{\phi}$  is once again bounded from below as  $|\dot{\phi}| \geq \mu_0 \sqrt{\frac{V_0}{2}}$ , similarly to the  $n = 0$  case. Since the two branches of positive and negative  $\dot{\phi}$  are once again disconnected, we can change variables from  $\dot{\phi}$  to  $\hat{\mu}$  as in the case  $n = 0$ .

The equivalent of (10) is then given by

$$\dot{\hat{\mu}} = -\frac{2(\hat{\mu} - 1)(\hat{\mu} - \mu_a)}{2\hat{\mu}^2 - 3(\mu_a + 1)\hat{\mu} + 4\mu_a} \times \left[ 3H\hat{\mu} + \frac{\beta(1 + 3w)}{\mu_0^2 \sqrt{2V_0}} \frac{\sqrt{(\hat{\mu} - 1)(\hat{\mu} - \mu_a)}}{\hat{\mu}} y \right] \quad (33)$$

where  $y = 8\pi G e^{-2\phi} \rho_X$  and  $\beta$  is once again the sign of  $\dot{\phi}$ . Now for  $\hat{\mu} \geq \mu_a$  we have that  $2\hat{\mu}^2 - 3(\mu_a + 1)\hat{\mu} + 4\mu_a = 0$  at a value  $\hat{\mu} = \mu_c = \frac{1}{4} \left[ 3(\mu_a + 1) + \sqrt{9\mu_a^2 - 14\mu_a + 9} \right]$ . In other words there the evolution of  $\dot{\hat{\mu}}$  reaches a singularity at  $\hat{\mu} = \mu_c$ . To make things worse, since  $\dot{\hat{\mu}}$  is negative for  $\hat{\mu} > \mu_c$  while being positive for  $\hat{\mu} < \mu_c$  then this singularity is always reached! This is true independently of the sign of  $\beta$  for the same reason as the case  $n = 0$ , i.e. for positive  $\tilde{H}$ , as  $y$  decreases below some threshold value the term proportional to  $y$  can be neglected.

### 3. Case $n = -2$

Let us now discuss the  $n = -2$  case.

In this case the function is given by

$$V_{(-2)} = \mu_0^3 V_0 \left( I_{-2} + \frac{\mu_a^2}{1 - \mu_a} \frac{1}{\hat{\mu} - \mu_a} + \frac{\mu_a(\mu_a - 2)}{(\mu_a - 1)^2} \ln |\hat{\mu} - \mu_a| + \frac{1}{(\mu_a - 1)^2} \ln |\hat{\mu} - 1| \right) \quad (34)$$

and

$$V'_{(-2)} = \mu_0^2 V_0 \frac{\hat{\mu}^2}{(\hat{\mu} - 1)(\hat{\mu} - \mu_a)^2}. \quad (35)$$

Contrary to the last case ( $n = -1$ ),  $\dot{\phi}$  is no longer bounded from below and can take all possible values. However the point  $\dot{\phi} = 0$  can only occur at  $\mu \rightarrow \infty$ , which means that once again the two sectors of positive and negative  $\dot{\phi}$  are disconnected, since the  $\mu \rightarrow \infty$  is a singular point (the Friedman equation blows up). We can therefore follow the approach we took in the  $n = -1$  case and change variables from  $\dot{\phi}$  to  $\mu$  giving

$$\dot{\mu} = -\frac{2(\hat{\mu} - 1)(\hat{\mu} - \mu_a)}{\hat{\mu}^2 - (3\mu_a + 2)\hat{\mu} + 4\mu_a} \times \left[ 3\mu_0 \hat{\mu} H + \frac{\beta(1 + 3w)(\hat{\mu} - \mu_a) \sqrt{\hat{\mu} - 1}}{\mu_0 \hat{\mu} \sqrt{2V_0}} y \right] \quad (36)$$

where  $y = 8\pi G e^{-2\phi} \rho_X$  and  $\beta$  is once again the sign of  $\dot{\phi}$ . We immediately see that the same problem as the  $n = -1$  case arises : there is a singularity in the evolution when  $\hat{\mu} = \mu_c = \frac{1}{2} \left[ 3\mu_a + 2 + \sqrt{9\mu_a^2 - 4\mu_a + 4} \right]$ . Since  $\mu_c > \mu_a$  always while at the same time  $\dot{\mu} < 0$  for  $\hat{\mu} > \mu_c$  and  $\dot{\mu} > 0$  for  $\hat{\mu} < \mu_c$ , then once again this singularity is always reached.

### 4. Case $n = -3$

We now turn to the  $n = -3$  case.

In this case the function is given by

$$V_{(-3)} = \mu_0^3 V_0 \left( I_{-3} + \frac{\mu_a^2}{2(1 - \mu_a)} \frac{1}{(\hat{\mu} - \mu_a)^2} - \frac{\mu_a(\mu_a - 2)}{(1 - \mu_a)^2} \frac{1}{\hat{\mu} - \mu_a} + \frac{1}{(1 - \mu_a)^3} \ln \left| \frac{\hat{\mu} - 1}{\hat{\mu} - \mu_a} \right| \right) \quad (37)$$

and

$$V'_{(-3)} = \mu_0^2 V_0 \frac{\hat{\mu}^2}{(\hat{\mu} - 1)(\hat{\mu} - \mu_a)^3}. \quad (38)$$

Contrary to the  $n = -1$  and  $n = -2$  cases, in this case,  $2V' + \mu V''$  never goes to zero for  $\hat{\mu} > \mu_a$ , hence no singularity of the same type as  $n = -1$  and  $n = -2$  occurs in this case. We consider two limiting cases of behavior.

The first case is when  $\hat{\mu}$  approaches  $\mu_a$ . In this limit we get that  $V'$  is of order  $\dot{\phi}^2$  while  $V$  is of lower order, giving  $\mu V' + V = 2\mu_0 \mu_a \dot{\phi}^2$ . Therefore in this limit the equations take the form

$$3\tilde{H}^2 = \mu_0 \mu_a \dot{\phi}^2 + 8\pi G e^{-2\phi} \rho_X \quad (39)$$

$$\ddot{\phi} + 3\tilde{H}\dot{\phi} + \frac{(1 + 3w)}{2\mu_0 \mu_a} 8\pi G e^{-2\phi} \rho_X = 0 \quad (40)$$

in other words the system behaves as a one with a canonical scalar field coupled to matter. In this limit we have

tracker solutions such that  $\dot{\phi} = -\frac{1+3w}{(1-w)\mu_0\mu_a}\tilde{H}$ , which gives  $\Omega_\phi = \frac{(1+3w)^2}{3(1-w)^2\mu_0\mu_a}$ . This tracker however is unstable (but can be long-lived depending on the fluid energy budget), i.e. as  $\mu$  becomes larger this limit stops to be valid.

The second case is when  $\mu \rightarrow \infty$ . We expand all functions in powers of  $\epsilon = 1/\hat{\mu}$ , giving  $V' = \mu_0^2 V_0 \epsilon^2 [1 + (3\mu_a + 1)\epsilon]$  resulting to lowest order in  $\epsilon$ ,  $\dot{\phi} = \beta\mu_0\sqrt{V_0/2}\epsilon$ , and  $\Gamma = -\beta\mu_0^2\sqrt{2V_0}[1 + (3\mu_a + 1)\epsilon/2]$ . It also turns out that although  $\mu V'$  is of  $O(\epsilon)$ , this order is precisely canceled from a term in  $V$ , resulting to  $\mu V' + V = \mu_0^3 V_0 (3\mu_a + 1)\epsilon^2/2$  to lowest order in  $\epsilon$ . The evolution equations become

$$3\tilde{H}^2 = \frac{1}{4}\mu_0^3 V_0 (3\mu_a + 1)\epsilon^2 + y \quad (41)$$

and

$$\frac{3\mu_a + 1}{2}\dot{\epsilon} + 3\left(1 + \frac{3\mu_a + 1}{2}\epsilon\right)\tilde{H} + \frac{(1+3w)\beta y}{\mu_0^2\sqrt{2V_0}} = 0 \quad (42)$$

First we consider the case when the fluid is a cosmological constant. Letting  $\epsilon$  and  $\dot{\epsilon}$  to zero, we get that the Hubble parameter tends to a constant  $\tilde{H} \rightarrow \tilde{H}_\infty = \mu_0^2\sqrt{2V_0}/2$ , while  $\phi \rightarrow \phi_\infty = \frac{1}{2}\ln\frac{8\pi G\rho_\Lambda}{3\tilde{H}_\infty^2}$  and  $\beta = 1$ . Perturbing  $\phi$  about  $\phi_\infty$  as  $\phi = \phi_\infty(1 + \delta)$  we get that  $\tilde{H}^2 = \tilde{H}_\infty^2(1 - 2\phi_\infty\delta)$ . Differentiating (42) once under these approximations and eliminating  $\delta$  via  $2\mu_0\phi_\infty\dot{\delta} = \tilde{H}_\infty\epsilon$  we get

$$\ddot{\epsilon} + 3\tilde{H}_\infty\dot{\epsilon} + \frac{6\tilde{H}_\infty^2}{(3\mu_a + 1)\mu_0}\epsilon = 0 \quad (43)$$

This has decaying solutions of the form  $\epsilon = \epsilon_1 e^{-A+t} + \epsilon_2 e^{-A-t}$  where

$$A_\pm = \frac{3}{2}\tilde{H}_\infty \left(1 \pm \sqrt{1 - \frac{8}{3(3\mu_a + 1)\mu_0}}\right) \quad (44)$$

provided that  $(3\mu_a + 1)\mu_0 \geq 8/3$ . The final stage of this particular case is a universe with  $\Omega_\Lambda = 1$ .

It turns out that when the fluid is not a cosmological constant then, the evolution reaches  $\epsilon \rightarrow 0$  in finite time. This is once again a singularity of a similar form to the  $n = 0$  case. The reason we have such a singularity is the same as the  $n = 0$  case : one can choose initial conditions such that  $\dot{\phi} = 0$  which corresponds to  $\mu \rightarrow \infty$ , at any point in time. However this is inconsistent with the equations of motion except in the special case of the presence of a cosmological constant. Therefore in this case, if the evolution is to be non-singular in the infinite future, one must include a cosmological constant.

### 5. Case $n \leq -4$

Finally we consider the  $n \leq -4$  case.

In this case the function is given by

$$V_{(-n)} = \mu_0^3 V_0 \left[ I_{-n} + \frac{\mu_a^2}{(n-1)(1-\mu_a)} \frac{1}{(\hat{\mu} - \mu_a)^{n-1}} - \frac{\mu_a(\mu_a - 2)}{(n-2)(1-\mu_a)^2} \frac{1}{(\hat{\mu} - \mu_a)^{n-2}} + \frac{1}{(1-\mu_a)^n} \sum_{m=1}^{n-3} \frac{(1-\mu_a)^m}{m(\hat{\mu} - \mu_a)^m} + \frac{1}{(1-\mu_a)^n} \ln \left| \frac{\hat{\mu} - 1}{\hat{\mu} - \mu_a} \right| \right] \quad (45)$$

and

$$V'_{(-n)} = \mu_0^2 V_0 \frac{\hat{\mu}^2}{(\hat{\mu} - 1)(\hat{\mu} - \mu_a)^n}. \quad (46)$$

It is easy to show that  $2V' + \mu V'' \neq 0$  for  $\hat{\mu} > \mu_a$  unless  $\hat{\mu} \rightarrow \infty$ . We therefore expect to have smooth evolution and furthermore we expect that  $\hat{\mu}$  will eventually increase to infinity.

Eventually  $\mu$  increases to large values where we can expand  $V$  and  $V'$  in powers of  $\epsilon = 1/\hat{\mu}$ . For  $V'$  we get

$$V' \approx \mu_0^2 V_0 \epsilon^{n-1} \quad (47)$$

giving

$$\Gamma \approx \beta\mu_0^2\sqrt{2V_0}\epsilon^{\frac{n-3}{2}} \quad (48)$$

where  $\beta$  is the sign of  $\Gamma$ .

It turns out that  $\mu V' + V$  is at least of  $O(\epsilon^2)$  for  $n = -4$  or higher for  $n < -4$  while  $\dot{\phi}$  is at least of  $O(\epsilon^{3/2})$  for  $n = -4$  or higher for  $n < -4$ . We can therefore assume that the universe evolves like Einstein gravity with  $\tilde{H} = H$  and we have the usual Friedmann equation :  $3H^2 = 8\pi G\rho_X$

We consider three separate subcases :  $w = -1$ ,  $w = 1$  and  $-1 < w < 1$ .

For the  $w = -1$  case we have that  $H$  will tend to a constant. The evolution equation for  $\Gamma$  reads

$$\dot{\Gamma} + 3H\Gamma + 6H^2 = 0 \quad (49)$$

The solution is

$$\Gamma = 2H(e^{-3Ht} - 1) \quad (50)$$

and therefore  $\Gamma \rightarrow 0$  as required by consistency. Feeding back to (48) we get that  $\beta = 1$ .

For the  $w = 1$  case we have that  $H = \frac{1}{t}$  which means that  $\Gamma$  evolves according to

$$\dot{\Gamma} + \frac{3}{t}\Gamma - \frac{12}{t^2} = 0 \quad (51)$$

The solution in this case is

$$\Gamma = \frac{6}{t} + \frac{\Gamma_0}{t^3} \quad (52)$$



where  $\Gamma_0$  is a constant. Once again  $\Gamma \rightarrow 0$  as required by consistency. Feeding back to (48) we get that  $\beta = 1$ .

The last case is when  $-1 < w < 1$ . In this case  $\Gamma$  evolves as

$$\dot{\Gamma} + 3H\Gamma - 3(1 + 3w)H^2 = 0 \quad (53)$$

which has solution

$$\Gamma = \frac{2(1 + 3w)}{1 - w}H = \frac{4(1 + 3w)}{3(1 - w^2)} \frac{1}{t} \quad (54)$$

and therefore  $\Gamma$  consistently goes to zero. Feeding back to (48) we get that  $\beta = 1$  for  $w > -1/3$  and  $\beta = -1$  for  $w < -1/3$ .

### B. Mixed cases

Mixing cases for  $n \geq 1$  can have two generic kinds of behavior depending on whether  $V(\mu_a) = 0$  or not.

If  $V(\mu_a) = 0$  then we have perfect trackers. If  $V(\mu_a) > 0$  there will be a temporary tracker until  $V(\mu_a) \sim 16\pi G e^{-2\phi} \rho_X$  after which the tracker is destroyed and we get  $\phi$  domination. The solution in the  $\phi$ -domination era is de-Sitter space (in the Einstein frame).

Now the condition of exact MOND means that only a very restricted set of function can have  $V(\mu_a) = 0$ . For the single  $n$  cases only the Bekenstein toy model ( $n = 2$  and  $\mu_a = 2$ ) has this feature. By mixing cases however, this feature can arise again for very specific choices of  $\mu_a$  and  $n$ 's. For example this is impossible for a mixed  $n = 1$  and  $n = 2$  case, but possible for a mixed  $n = 1$  and  $n = 3$  case with  $\mu_a = 2/7$  and  $c_1/c_3 = -4/49$ .

While individually  $n = -1$  and  $n = -2$  lead to singularities, mixing the two can for some choices of  $c_{-1}$  and  $c_{-2}$  lead to  $2V' + \mu V''$  being non-zero for all  $\hat{\mu} > \mu_a$ . This will lead to a better behaved evolution although problems can still persist as  $\mu \rightarrow \infty$  just like the  $n = -3$  case. This last problem could be removed by mixing in a positive power of  $n$ .

Mixing  $n = 0$  with positive  $n$  alone cannot remove the problem when  $\hat{\mu} \rightarrow \mu_a$ , i.e. the evolution will still become ill defined at that point. One can still however mix  $n = 0$  with both positive and negative powers; the effect of the negative power is to drive  $\hat{\mu}$  away from the  $\hat{\mu} = \mu_a$  point.

Finally the most general mixed case including both positive and negative powers with suitably chosen coefficients  $c_i$ , will give a function  $V'$  which goes to zero not at  $\hat{\mu} = \mu_a$  but at a shifted point  $\hat{\mu} = \mu'_a$ . The large  $\mu$  behavior will be dominated by the positive powers of  $n$  while the small  $\mu$  behavior by the negative powers with the function still being singular at  $\hat{\mu} = \mu_a$ . If we restrict the evolution for values  $\hat{\mu} > \mu'_a$ , then we have a situation similar to the positive  $n$  cases. We therefore expect to have the same behavior as if we had a function expanded about  $\hat{\mu} = \mu'_a$  with positive powers only, i.e. we expect to get stable trackers or in the case of a non-zero integration constant in  $V$  eventual  $\phi$ -domination.

## V. CONCLUSIONS

In this paper we have tried to extract general properties of the cosmology of TeVeS gravity. We have restricted ourselves to analyzing the background equations for a homogeneous and isotropic universe. We have considered a general potential for  $\mu$  which we believe encompasses most of the possible regimes one could encounter in this theory.

We find that Bekenstein's choice of  $V(\mu)$  naturally leads to stable tracking behavior. This confirms the results of [5]. Generalizing the Bekenstein function by including arbitrary strictly positive powers of  $(\hat{\mu} - \mu_a)$  we either get exactly the same type of tracking or temporary tracking followed by  $\phi$ -domination. What controls this kind of behavior is the value of the integration constant in  $V(\mu)$ . If  $V(\mu_0\mu_a) = 0$  then the tracking behavior is retained to the infinite future, otherwise the tracking period is temporary followed by  $\phi$ -domination.

We find that the  $n = 0$ ,  $n = -1$  and  $n = -2$  cases alone lead to future singularities in the evolution. The same happens in the  $n = -3$  case unless the background fluid is a cosmological constant. For  $n \leq 4$  the evolution is non-singular and eventually leads to fluid domination with  $\mu \rightarrow \infty$ .

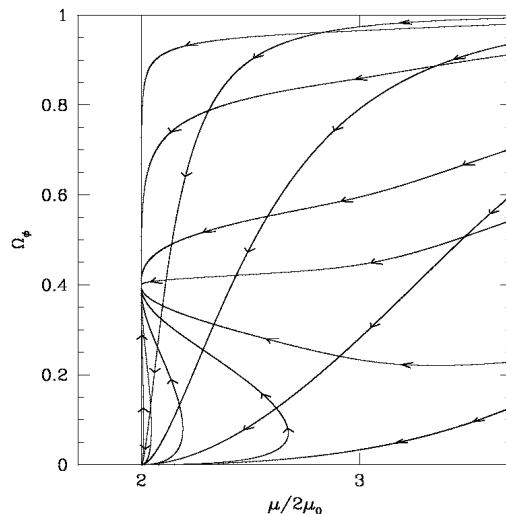


FIG. 7: Phase plot for the Bekenstein's model. All the curves join the tracker solution ( $\Omega_\phi=0.4$ ) and tracking remains stable.

Finally, the singularities can be avoided by mixing cases together. If positive and negative cases are mixed, it turns out that this is equivalent to a positive-only mixture but with a different expansion parameter  $\mu'_a$  rather than  $\mu_a$ , i.e. the function is equivalent to expanding in terms of positive powers of  $(\hat{\mu} - \mu'_a)$  in which case we again get the tracking behavior discussed above.

A useful way to visualise the results of this paper is through the two phase plots in Figs.7, and 8. For each

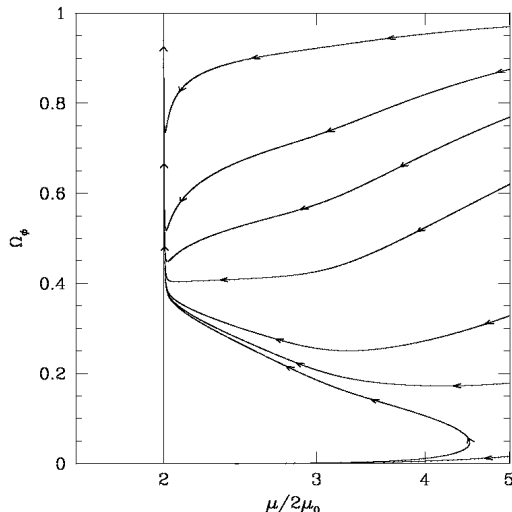


FIG. 8: Phase plot for the  $n = 2$  and  $n = -1$  model. The upper curves don't join the tracker solution ( $\Omega_\phi=0.4$ ) and tracking becomes unstable eventually leading to the singularity mentioned in the text (in this case with a  $\phi$  dominated Universe). We used  $c_2c_{-1} < 0$ .

plot we pick a range initial conditions and we look at the function  $\Omega_\phi(\hat{\mu})$  for the choice of  $\mu_0$  such that  $\Omega_\phi = 0.4$  in the tracker limit. In Fig.5 (Bekenstein's model) we see that all the curves join the tracker solution. In Fig.8 we plot a model with  $n = 2$  and  $n = -1$  where with some choices of initial conditions we have a temporary tracker due to the  $n = 2$  factor and eventually rolling to the singularity mentioned in the section on the  $n = -1$  case, while with other choices of initial conditions, the tracker is avoided all together.

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