A generalization of Dirac’s theorem on cycles through \( k \) vertices in \( k \)-connected graphs

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Abstract

Let \( X \) be a subset of the vertex set of a graph \( G \). We denote by \( \kappa(X) \) the smallest number of vertices separating two vertices of \( X \) if \( X \) does not induce a complete subgraph of \( G \), otherwise we put \( \kappa(X) = |X| - 1 \) if \( |X| \geq 2 \) and \( \kappa(X) = 1 \) if \( |X| = 1 \). We prove that if \( \kappa(X) \geq 2 \) then every set of at most \( \kappa(X) \) vertices of \( X \) is contained in a cycle of \( G \). Thus, we generalize a similar result of Dirac. Applying this theorem we improve our previous result involving an Ore-type condition and give another proof of a slightly improved version of a theorem of Broersma et al.

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1. Introduction

Throughout this article we will consider only undirected, finite and simple graphs. Let \( G \) be a connected graph and let \( S \) be a proper subset of \( V(G) \). \( S \) is called a vertex cut of \( G \) if the graph \( G - S \) (i.e., the graph obtained by removing all vertices of \( S \) from \( G \)) is not connected. Let \( S \) be a vertex cut of \( G \) and \( Y, Z \) two connected components of \( G - S \). If \( x \in V(Y) \) and \( y \in V(Z) \), we say that the vertex cut \( S \) separates \( x \) and \( y \). Observe that only two nonadjacent vertices can be separated and the vertex cut \( S \) that separates \( x \) and \( y \) contains neither \( x \) nor \( y \).

Let \( X \) be a subset of the vertex set of a graph \( G \) such that \( |X| \geq 2 \) and \( X \) does not induce a complete subgraph of \( G \). The connectivity of \( X \) in \( G \), denoted by \( \kappa(X) \), is the smallest number \( k \) such that there exists in \( G \) a vertex cut of \( k \) vertices that separates two vertices of \( X \). If \( X \) is a clique in \( G \), then, by definition, \( \kappa(X) = |X| - 1 \) for \( |X| \geq 2 \) and \( \kappa(X) = 1 \) if \( |X| = 1 \). The number \( \kappa(V(G)) = \kappa(G) \) is called the connectivity of the graph \( G \). Thus, for the complete graph \( K_n \) we have \( \kappa(K_n) = n - 1, n \geq 2 \), and \( \kappa(K_1) = 1 \). Note that some authors (see for example [4]) define \( \kappa(G) = \infty \) if \( X \) is a clique. By \( \omega(X) \) we denote the maximum number of pairwise nonadjacent vertices in the subgraph of \( G \) induced by \( X \). We say that \( X \) is cyclable in \( G \) if \( G \) has a cycle containing all vertices of \( X \).

The investigation on cycles passing through a given set of vertices in \( k \)-connected graphs was initiated by Dirac [7].

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Theorem 1. Let $G$ be a $k$-connected graph, where $k \geq 2$, and let $X$ be a set of $k$ vertices of $G$. Then there is a cycle in $G$ containing every vertex of $X$.

There are many improvements of the last theorem. For example, Egawa et al. [8] proved the common generalizations of Theorem 1 and the classical Dirac’s theorem [6] on the existence of hamiltonian cycles in graphs.

Theorem 2. Let $G$ be a $k$-connected graph, where $k \geq 2$, and let $X$ be a set of $k$ vertices of $G$. Then $G$ contains either a cycle of length at least $\frac{2}{\delta(G)}$ including every vertex of $X$ or a hamiltonian cycle.

Recently Häggkvist and Mader [12] showed that every set of $k + \lfloor \frac{1}{3} \sqrt{k} \rfloor$ vertices in a $k$-connected $k$-regular graph belongs to some cycle.

Bollobás and Brightwell [1] and Shi [18] obtained an extension of Dirac’s theorem on hamiltonian graphs.

Theorem 3. Let $G$ be a 2-connected graph of order $n$ and let $X$ be a set of vertices of $G$. If $d_G(x) \geq \frac{n}{2}$ for each $x \in X$, then $X$ is cyclable.

Shi [18] improved both Ore’s theorem [16] and the previous one in the following way.

Theorem 4. Let $G$ be a graph of order $n$ and let $X$ be a subset of its vertex set such that $\kappa(X) \geq 2$. If $d_G(x) + d_G(y) \geq n$ for each pair $x, y$ of nonadjacent vertices of $X$, then $X$ is cyclable in $G$.

This theorem was proved under the assumption that the graph is 2-connected. However, the presented version follows easily from a theorem due to Ota [17] that we give in Section 7.

The main result of this paper is the following generalization of Theorem 1 involving the notion of the connectivity of a set of vertices.

Theorem 5. Let $G$ be a graph and $Y$ a subset of $V(G)$ with $\kappa(Y) \geq 2$. Let $X$ be a subset of $Y$ with $|X| \leq \kappa(Y)$. Then $X$ is cyclable in $G$.

Broersma et al. [4] studied cyclability of sets of vertices of graphs satisfying a local Chvátal–Erdős-type condition that involves the defined above parameters. They obtained a generalization of a result of Fournier [10] and of Chvátal–Erdős theorem under the assumption that the graph is 2-connected. The first application of Theorem 5 is an alternative proof of their theorem (there is a gap in the original proof).

Theorem 6. Let $G$ be a graph and let $X \subset V(G)$ with $\kappa(X) \geq 2$. If $\alpha(X) \leq \kappa(X)$, then $X$ is cyclable in $G$.

The second one is the following extension of a result of Flandrin et al. [9].

Theorem 7. Let $G = (V, E)$ be a graph of order $n$. Let $X_1, X_2, \ldots, X_q$ be subsets of the vertex set $V$ such that the union $X = X_1 \cup X_2 \cup \cdots \cup X_q$ satisfies $2 \leq q \leq \kappa(X)$. If for each $i, i = 1, 2, \ldots, q$, and for any pair of nonadjacent vertices $x, y \in X_i$, we have
\[d(x) + d(y) \geq n,\]
then $X$ is cyclable in $G$.

The condition of the last theorem is weaker than that of Shi and is called the regional Ore’s condition.

As an immediate consequence of Theorem 7 we get the following generalization of Theorem 5.

Theorem 8. Let $G = (V, E)$ be a graph, $k \geq 2$. For every set of $k$ cliques $X_1, X_2, \ldots, X_k$ of $G$, such that $\kappa(X_1 \cup X_2 \cup \cdots \cup X_k) \geq k \geq 2$ there exists a cycle of $G$ containing all vertices of these cliques. □

Therefore, for any set of $k$ cliques in a $k$-connected graph $G$ there is a cycle of $G$ containing all vertices of these cliques.

Theorem 9. Let $G = (V, E)$ be a graph, $k \geq 2$. For every set of $k$ cliques $X_1, X_2, \ldots, X_k$ of $G$, such that $\kappa(X_1 \cup X_2 \cup \cdots \cup X_k) \geq k \geq 2$ there exists a cycle of $G$ containing all vertices of these cliques.

Let us recall the notion of $k$-closure of a graph which was introduced in the classical paper due to Bondy and Chvátal [2]. Namely, given an integer $k$, we will call the $k$-closure of $G$ the graph obtained by recursively joining pairs $x, y$ of
nonadjacent vertices such that \( d(x) + d(y) \geq k \) until no such pair remains. It will be denoted by \( Cl_k(G) \). It is known that \( k \) closure is well defined, that is, if no order of inserting the edges is specified, this operation gives always the same graph.

Čada et al. [5] studied the property of cyclability under several closure concepts. For the Bondy–Chvátal closure they proved the following result.

**Theorem 9.** Let \( G \) be a graph of order \( n \) and let \( X \subseteq V(G) \), \( X \neq \emptyset \). Then \( X \) is cyclable in \( G \) if and only if \( X \) is cyclable in \( Cl_n(G) \).

In [13] Harant investigated cyclability of sets \( X \) satisfying the condition \( |X| = \kappa(X) + 1 \) and obtained a generalization of a result due to Watkins and Mesner [19]. His main result is seemingly stronger than Theorem 5, but its proof is based on Theorem 6 where the necessary correction is provided by Theorem 5. There are other interesting results due to Gerlach et al. [11] that give sufficient conditions for the existence of cycles through specified vertices. These results involve the notion of local toughness of a set of vertices and the \( A \)-separator of a graph (for definitions see [11]). Note that Harant [14] proved independently Theorem 5.

The proofs of Theorems 5–7 are given in Sections 4–6, respectively. In Section 3 we present several properties of the notion of connectivity of a set of vertices based on Menger’s theorem.

### 2. Notation

We use the book of Bondy and Murty [3] for terminology and notation not defined here. If \( A \) is a subgraph of \( G \) (or a subset of \( V(G) \)), \( |A| \) is the number of vertices in \( A \).

Let \( C \) be a cycle of \( G \) and \( a \) a vertex of \( C \). We shall denote by \( \overrightarrow{C} \) the cycle \( C \) with a given orientation, by \( a^+ \) the successor of \( a \) on \( \overrightarrow{C} \) and by \( a^- \) its predecessor.

Let \( a \) and \( b \) be two vertices of \( C \). The segment of \( \overrightarrow{C} \) from \( a \) to \( b \), denoted by \( a \overrightarrow{C} b \), is the (ordered) set of consecutive vertices of \( C \) from \( a \) to \( b \) (\( a \) and \( b \) included) in the direction specified by the orientation of \( C \) while \( b \overrightarrow{C} a \) denotes the same set but in opposite order. Clearly, when \( a = b \) the symbol \( a \overrightarrow{C} b \) means the one-vertex subset \( \{a\} \) of \( V(C) \). A similar notation is used for paths. Throughout the paper the indices of a cycle \( C = x_1, x_2, \ldots, x_p \) are to be taken modulo \( p \).

### 3. Connectivity of a set of vertices

Consider now a set \( \mathcal{P} = \{P_1, P_2, \ldots, P_s\} \) of paths of a graph \( G \). These paths are internally disjoint if no two have an internal common vertex. The relation between the presented notions is given in the well-known Menger’s theorem [15].

**Theorem 10.** If \( x \) and \( y \) are two nonadjacent vertices of a graph \( G \), then the maximum number of internally disjoint \( x \)–\( y \) paths is equal to the minimum number of vertices in a vertex cut separating \( x \) and \( y \).

**Lemma 1.** Let \( k \geq 1 \) be an integer and let \( X \) be a subset of the vertex set of a graph \( G \) such that \( X \) is not a clique and \( |X| > k \). The following two statements are equivalent:

(i) \( \kappa(X) \geq k \);

(ii) any two vertices of \( X \) are connected by at least \( k \) internally disjoint paths (in \( G \)).

**Proof.** For \( k = 1 \) our assertion is trivial. Suppose \( k \geq 2 \) and let \( X \) verify the assumption of the lemma. Clearly, if (ii) holds then we cannot separate two vertices of \( X \) by a vertex cut of at most \( k - 1 \) vertices, so (i) is true. Assume that \( \kappa(X) \geq k \) and let \( x \) and \( y \) be two vertices of \( X \). If \( x \) and \( y \) are not adjacent, we can apply Menger’s theorem and we are done. Assume that \( xy \in E(G) \) and denote by \( G' \) the graph obtained from \( G \) by deleting the edge \( xy \). Suppose there is a vertex cut \( S \) separating \( x \) and \( y \) in \( G' \) and such that \( |S| < k - 2 \). Denote by \( U \) and \( V \) the connected components of \( G' - S \) containing \( x \) and \( y \), resp. Notice that no other connected component of \( G' - S \) contains a vertex of \( X \) since otherwise
S would separate two vertices of X in G. Therefore, because \(|X| > k \geq 2\), U or V contains at least two vertices of X. Suppose for instance that \(z \in U \cap X\) and \(z \neq x\). Then \(S \cup \{x\}\) separates \(z\) and \(y\) in the graph \(G\) and the cardinality of \(S \cup \{x\}\) is \(k - 1\), which is a contradiction. Therefore, every vertex cut that separates \(x\) and \(y\) in \(G'\) has at least \(k - 1\) vertices. It follows by Menger’s theorem that there are at least \(k - 1\) internally disjoint \(x-y\) paths in \(G'\). Adding the edge \(xy\) we get the desired system of \(k\) internally disjoint \(x-y\) paths in \(G\).

This lemma is best possible. Indeed, take three vertex-disjoint graphs: \(K_s, \overline{K_t}\) and \(\overline{K_2}\) where \(t, s \geq 2\) and \(V(\overline{K_2}) = \{x_1, x_2\}\). Denote by \(G\) the graph obtained by joining every vertex \(x_i, i = 1, 2\), to each vertex of \(K_s\) and \(\overline{K_t}\) and let \(X = V(K_s) \cup \{x_1, x_2\}\). Clearly, \(\kappa(X) = s + t > s + 1\), \(|X| = s + 2 \leq \kappa(X)\) and no two vertices of the subgraph induced by \(V(K_s)\) are connected by more than \(s + 1\) internally disjoint paths.

The well-known condition for a graph to be \(k\)-connected is an easy corollary of Lemma 1.

**Corollary 1.** A graph \(G\) on at least two vertices is \(k\)-connected \((k \geq 0)\) if and only if any two vertices of \(G\) are connected by at least \(k\) internally disjoint paths.

Using the same method as in the proof of Lemma 1 we get the following.

**Corollary 2.** Let \(X, |X| \geq 2\), be a subset of the vertex set of a graph \(G\). Then any two vertices of \(X\) are connected by at least \(\min(|X| - 1, \kappa(X))\) internally disjoint paths (in \(G\)).

There exist several versions of Menger’s theorem. The following result, usually called the Fan Lemma, is a very useful tool in studying problems related to the connectivity of graphs. We present a version of this lemma involving the connectivity of a set of vertices.

**Lemma 2.** Let \(G\) be a graph and let \(X\) be a subset of \(V(G)\) with \(\kappa(X) \geq 1\). Let \(\{x_1, x_2, \ldots, x_q\}\) be a subset of \(X\) with \(q \leq \kappa(X)\), and let \(Y \subset V(G)\) be another set of vertices such that \(\{x_1, x_2, \ldots, x_q\} \subset Y\) and \(x \notin Y\).

Then there are different vertices \(y_1, y_2, \ldots, y_q\) in \(Y\) and \(q\) internally disjoint paths \(P_1, P_2, \ldots, P_q\) such that

(i) \(P_i\) is an \(x-y_i\) path, for \(1 \leq i \leq q\), and
(ii) \(V(P_i) \cap Y = \{y_i\}\), \(1 \leq i \leq q\).

**Proof.** Consider a graph \(G'\) obtained by adding a new vertex \(y\) and joining it to all vertices \(x_1, x_2, \ldots, x_q\). A set of \(q - 1\) vertices cannot separate \(y\) and \(x\) because the same set would separate \(x\) and a vertex belonging to the set \(\{x_1, x_2, \ldots, x_q\}\), which contradicts the definition of \(\kappa(X)\). Thus, by Menger’s theorem, \(G'\) contains \(q\) internally disjoint \(x-y\) paths and the existence of the desired collection of paths is obvious.

**4. Proof of Theorem 5**

Let \(X = \{x_1, x_2, \ldots, x_q\}\) be a subset of \(Y\), where \(q \leq \kappa(Y)\) and \(|Y| \geq 2\). We may assume that \(Y\) is not a clique in \(G\).

The proof is by induction on \(q\). Assume \(q = 2\) and \(X = \{x_1, x_2\} \subset Y\). If \(x_1\) and \(x_2\) are not adjacent we apply Menger’s theorem and we are done. Otherwise, since \(Y\) is not a clique, there exists another vertex \(u\) in \(Y\), so we may use the Fan Lemma (Lemma 2) and find a cycle containing \(x_1, x_2\) (and \(u\)). Suppose the assertion is true for every set \(Z\) of \(p\) vertices, \(p \leq q - 1 < \kappa(Y)\) and let \(X = \{x_1, x_2, \ldots, x_q\}\). By the induction hypothesis there is a cycle that contains the vertices \(x_1, x_2, \ldots, x_{q-1}\). Denote by \(C\) such a cycle with a given orientation. By Lemma 2 there is a collection \(P = P_1, P_2, \ldots, P_{q-1}\) of \(q - 1\) internally disjoint paths and there are \(q - 1\) different vertices \(y_1, y_2, \ldots, y_{q-1}\) in \(C\) such that for each \(i, 1 \leq i \leq q - 1\), \(P_i\) is an \(x_{i-1}-y_i\) path with \(V(P_i) \cap V(C) = \{y_i\}\). We may assume without loss of generality that the vertices \(y_i\) and \(x_j\) appear on \(C\) in the order indicated by \(C\) and the paths \(P_i\) are oriented from \(x_{q-1}\) to \(y_i\). If there is a cycle in \(G\) containing \(X\) we are done. So suppose the contrary. If \(y_i\) and \(y_{i+1}\) belong to \(x_j \rightarrow C \rightarrow x_{j+1}\) for some \(i\) and \(j\) (indices are taken \(\mod q - 1\)), then the cycle \(\rightarrow C \rightarrow y_i \rightarrow P_i \rightarrow x_{q-1} \rightarrow P_{i+1} \rightarrow y_{i+1}\) contains all the vertices of \(X\), a contradiction.

Moreover, if \(x_j = y_i\) for some \(s, t\) or \(x_j \rightarrow C \rightarrow x_{j+1}\) does not contain any vertex of \(X\), then, by the pigeonhole principle, there are two indices \(i\) and \(j\) such that \(y_i\) and \(y_{i+1}\) belong to \(x_j \rightarrow C \rightarrow x_{j+1}\), which leads to a contradiction. Thus, in each
segment \( x_j^+ \stackrel{C}{\rightarrow} x_{j+1} \) there is exactly one \( y_j \) and we may assume that \( x_q x_i \notin E(G) \) for \( 1 \leq i \leq q - 1 \). Since \( x_1 \) and \( x_q \) are not adjacent, it follows from Menger’s theorem that there is a collection \( \mathcal{Q} = Q_1, Q_2, \ldots, Q_{\kappa(Y)} \) of \( \kappa(Y) \) internally disjoint \( x_q - x_1 \) paths. We may assume that these paths are oriented from \( x_q \) to \( x_1 \). Denote by \( f(Q_j) \) the first vertex of \( Q_j \) on \( C \) (\( j = 1, 2, \ldots, \kappa(Y) \)). We claim that

\[
f(Q_j) \neq x_1
\]

for each \( j, \ k = 1, 2, \ldots, \kappa(Y) \). Indeed, suppose that for some \( s \), \( f(Q_s) = x_1 \). If \( Q_s \) is internally disjoint with any path \( P_i \), then the cycle \( y_j \stackrel{C}{\rightarrow} x_1 \stackrel{Q_i}{\rightarrow} x_q \stackrel{P_j}{\rightarrow} y_j \), where \( y_j \) is the only vertex of \( x_q^+ \stackrel{C}{\rightarrow} x_2^+ \), contains all the vertices of \( X \), a contradiction. Thus, \( Q_s \) contains an internal vertex of a path of \( \mathcal{Q} \). Let \( l(Q_s) \) be the last vertex of \( Q_s \) belonging to \( \bigcup V(P_i) \setminus \{x_q \cup V(C)\} \). Assume \( l(Q_s) \in P_r \) and let \( P_r' = x_q P_r l(Q_s) x_1 \). Therefore, the collection \( P_1, P_2, \ldots, P_{r-1} P_r', P_{r+1}, \ldots, P_{q-1} \) of \( q - 1 \) paths satisfies the condition of Lemma 2. Moreover, the terminal vertex of the path \( P_r' \) belongs to \( X \) and we can easily find a cycle in \( G \) passing through all the vertices of \( X \), which is a contradiction. This proves our claim.

Set \( z_j = V(Q_j) \cap V(C) \), \( j = 1, 2, \ldots, \kappa(Y) \). Since \( q - 1 < \kappa(Y) \), it follows by the pigeonhole principle that there are two vertices \( z_i \) and \( z_r \) belonging to the same segment of the form \( x_j \stackrel{C}{\rightarrow} x_{j+1} \), whence there is a cycle containing all the vertices of \( X \), a contradiction.

### 5. Proof of Theorem 6

By Theorem 5 the assertion is true if \( |X| \leq \kappa(X) \). Suppose \( \alpha(X) \leq \kappa(X) \) and \( X \) is not cyclable in \( G \). Hence \( |X| > \kappa(X) \geq 2 \). Let \( C \) be a cycle containing as many vertices of \( X \) as possible and let \( \overrightarrow{C} \) denote this cycle with a given orientation. Again by Theorem 5, \( C \) contains at least \( \kappa(X) \) vertices of \( X \). Since \( X \) is not cyclable, there exists at least one vertex of \( X \), say \( x \), that does not belong to \( V(C) \). By Lemma 2 there is a collection \( \mathcal{Q} = P_1, P_2, \ldots, P_{\kappa(X)} \) of \( \kappa(X) \) internally disjoint paths and there are \( \kappa(X) \) different vertices \( y_1, y_2, \ldots, y_{\kappa(X)} \) in \( C \) such that for each \( i \), \( 1 \leq i \leq \kappa(X) \), \( P_i \) is an \( x - y_i \) path with \( V(P_i) \cap V(C) = \{y_i\} \). Using the same argument as in the proof of Theorem 5 we conclude that in each segment \( y_j^+ \stackrel{C}{\rightarrow} y_{j+1}^- \) (indices are taken mod \( \kappa(X) \)) there is at least one vertex of \( X \). Let \( u_i \) be the first vertex of \( X \) in \( y_i^+ \stackrel{C}{\rightarrow} y_{i+1}^- \) (\( i = 1, 2, \ldots, \kappa(X) \)). Suppose, for instance, that \( u_i u_j \in E \). Let us consider the cycle

\[
u_i u_j \stackrel{C}{\rightarrow} y_i \stackrel{P_i}{\rightarrow} y_j \stackrel{C}{\rightarrow} u_i
\]

(see Fig. 1).

Since, by definition of \( u_i \), the segments \( y_j^+ \stackrel{C}{\rightarrow} u_j^- \) and \( y_i^+ \stackrel{C}{\rightarrow} u_i^- \) contain no vertex of \( X \), the above cycle contains more vertices of \( X \) than \( C \), a contradiction.

Also in the case \( x u_i \in E \), it is easy to show a cycle with more vertices of \( X \) than \( C \), which is a contradiction. Therefore, \( \{x, u_1, u_2, \ldots, u_{\kappa(X)}\} \) is an independent set of \( \kappa(X) + 1 \) vertices, again a contradiction.
6. Proof of Theorem 7

Let \( G = (V, E) \), \( X_i, i = 1, 2, \ldots, q \) and \( X \) be as in Theorem 7. Consider now the Bondy–Chvátal closure \( Cl_n(G) \). By assumption, every set \( X_i, i = 1, \ldots, q \) induces a clique in \( Cl_n(G) \). Let \( A \) be an independent set in the subgraph induced by \( X \). It is obvious that \( A \) can have at most one vertex in each clique \( X_i \), so \( |A| \leq q \). Hence, \( \alpha(X) \leq q \leq \kappa(X) \). Now, by Theorem 6, \( X \) is cyclable in \( Cl_n(G) \) and, by Theorem 9, \( X \) is cyclable in \( G \). This finishes the proof.

7. Remarks

Note that if we substitute \( X = V \) in Theorem 7 we get the following:

**Theorem 11.** Let \( G = (V, E) \) be a \( k \)-connected graph, \( k \geq 1 \), of order \( n \) and let \( V = X_1 \cup X_2 \cup \cdots \cup X_k \). If for each \( i, i = 1, 2, \ldots, k \), and for any pair of nonadjacent vertices \( x, y \in X_i \), we have

\[
d(x) + d(y) \geq n,
\]

then \( G \) is hamiltonian.

So, we get the hamiltonicity of a graph for which the Ore’s condition holds in each of the parts separately (regionally) provided that the graph is \( k \)-connected. For \( k = 1 \) we get the classical Ore’s theorem. Notice that, in this case the connectivity (even 2-connectivity) is implied by the condition itself.

In order to compare the regional Ore condition with other ones, consider the graph \( G \) on \( n \) vertices (\( n \geq 20, n \equiv 0 \pmod{4} \)) with \( X = V(G) = X_1 \cup X_2 \cup X_3 \), where \( |X_1| = n/2 + 2, |X_i| = n/4 - 1 \) for \( i = 2, 3 \), and such that \( X_2 \) and \( X_3 \) induce a clique of \( G \) and \( X_1 \) induces a clique without one edge. Moreover, \( G \) has five independent edges: one of them between \( X_1 \) and \( X_2 \), two of them joining \( X_2 \) and \( X_3 \) and the remaining two edges between \( X_1 \) and \( X_3 \), so \( G \) is 3-connected. It is easy to see that this graph satisfies none of the well-known conditions implying hamiltonicity as for instance the condition of Ore, Chvátal, Fan, Chvátal–Erdős, etc. but is hamiltonian by Theorem 11.

Another example of application of Theorem 8 is the following classical result which can be viewed as a corollary of Menger’s theorem.

**Theorem 12.** Let \( G = (V, E) \) be a graph and let \( X \) be a set of vertices of \( G \) with \( \kappa(X) \geq 2 \). If \( e, f \) are two edges of the subgraph of \( G \) induced by \( X \), then \( G \) contains a cycle passing through \( e \) and \( f \).

Finally, we would like to show another application of Theorem 5. In [17] Ota proved the following result.

**Theorem 13.** Let \( G \) be a graph of order \( n \), and let \( X \) be a set of vertices of \( G \) with \( \kappa(X) \geq k \geq 2 \). If for any \( s \geq k \) and for any independent subset \( S \) of \( X \) of \( s + 1 \) vertices we have

\[
\sum_{x \in S} d_G(x) \geq n + s^2 - s,
\]

then \( X \) is cyclable in \( G \).

Note that in the original paper [17] the author used internally disjoint paths in order to define the connectivity of a set of vertices. However, from Theorem 5 the assertion is obvious if \( |X| \leq \kappa(X) \) and by Lemma 1, the two conditions used to define the connectivity are equivalent if \( |X| > \kappa(X) \). It is also clear that Theorem 13 implies Theorem 4.

References