APPLICATION OF A TWO-DIMENSIONAL HINDMARSH-ROSE TYPE MODEL FOR BIFURCATION ANALYSIS

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In this study, we examine the bifurcation scenarios of a two-dimensional Hindmarsh-Rose type model [Tsuji et al., 2007] with four parameters and simulate some resemblances of neurophysiological features for this model using spike-and-reset conditions. We present possible classifications based on the results of the following assessments: 1) the number and stability of the equilibrium points are analyzed in detail using a table to demonstrate the matter in which the stability of the equilibrium changes and to determine which two equilibria collapse through the saddle-node bifurcation; 2) the sufficient conditions for an Andronov-Hopf bifurcation and a saddle-node bifurcation are mathematically confirmed; and 3) we elaborately evaluate the sufficient conditions for the Bogdanov-Takens (BT) and Bautin bifurcations. Several numerical simulations for these conditions are also presented. In particular, two types of bistable behaviors are numerically demonstrated: the BT and Bautin bifurcations. Notably, all of the bifurcation curves in the domain of the remaining parameters are similar when the time scale is large. Additionally, to show the potential for a limit cycle, the existence of a trapping region is demonstrated. These results present a variety of diverse behaviors for this model. The results of this study will be helpful in assessing suitable parameters for fitting the resemblances of experimental observations.

Keywords: spike-and-reset conditions; neuro-computational features; bifurcation analysis

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1. Introduction

Neuroscientists have constructed a variety of mathematical models that mimic the activities of neurons. In the absence of appropriate models, researchers have used a variety of experimental designs and statistical methods to characterize mechanisms underlying the neurotransmission of electrical signals. Thus far, the potential behavior of neurons has been characterized using several mathematical models [Hodgkin & Huxley, 1952; Connor & Stevens, 1971; Connor et al., 1977; FitzHugh, 1961; Nagumo et al., 1962; Hindmarsh & Rose, 1982, 1984; Rose & Hindmarsh, 1989], with which some neuronal behaviors have been observed at bifurcation points. Numerical simulations have been examined to understand the potential dynamics of these models and to assess the behavior of neuronal systems. However, the number of theoretical studies on these models has been far lower than the number of numerical studies. Additionally, many parameters affect the accuracy of these models; therefore, it is important to determine which parameters are critical to analytically explaining the dynamics of the model and to characterizing the behavior of neurons at bifurcations.

To understand the behavior of neurons [Hodgkin & Huxley, 1952; Connor & Stevens, 1971; Connor et al., 1977], it is important to clarify the behavior of reduced neuronal models. Herein, we discuss a simple neuronal model called the Hindmarsh-Rose (HR) model, which was first implemented in the 1970s. Connor et al. [Connor & Stevens, 1971; Connor et al., 1977] were the first to establish a model for the alternative generation of action potentials. This model, which is similar to the classical four-dimensional Hodgkin-Huxley (HH) model [Hodgkin & Huxley, 1952], contains fast sodium, delayed rectifier potassium, leakage, and additional potassium conductance (i.e., the transient A-current). The properties of the fast sodium and the delayed rectifier potassium conductance are somewhat different from those of the HH model, especially the brief action potentials induced through faster kinetics. In 1989, Rose and Hindmarsh [Rose & Hindmarsh, 1989] simplified the six-dimensional Connor-Stevens model to the two-dimensional Hindmarsh-Rose (2DHR) model by a transformation of variables. Furthermore, they extended the two-dimensional HR model to a three-dimensional HR model with the addition of a slow variable to describe the subthreshold of the inward and outward current. With suitable parameters, the models also resemble repetitive firing [Hindmarsh & Rose, 1982], bursting [Hindmarsh & Rose, 1984] and thalamic neurons [Rose & Hindmarsh, 1989] with detailed ionic currents, where repetitive firing is primarily induced through quadratic recovery. On a physiological level, the HR model can simulate the bursting neurons of the pond snail Lymnaea [Chay, 1985b,a; Chay & Rinzel, 1985; Chay & Keizer, 1985; Sherman et al., 1988]. The HR model is considered as a generalization of the FitzHugh-Nagumo (FN) model [FitzHugh, 1961; Nagumo et al., 1962], which is a polynomial model that mimics the HH model [Hodgkin & Huxley, 1952]. The major difference between the FN and HR models is the relation between the rate of change of the recovery variable and the membrane potential: the relation of the former is linear, and the relation of the latter is quadratic. With respect to one or two bifurcation parameters, the HR model can reproduce dynamic behaviors, such as quiescence, spiking, irregular spiking, bursting and irregular bursting [Terman, 1991, 1992; González-Miranda, 2003, 2007; Innocentia et al., 2007]. There are two major advantages for using the HR type model: only two nonlinear terms are used to describe its vector field and there exist circuit syntheses [Denker et al., 2005; Lee et al., 2007]. In 2007, Tsuji et al. [2007] proposed a 2DHR type model that preserves both a time-scale parameter and the first component of vector fields for FN [FitzHugh, 1961; Nagumo et al., 1962]. Bifurcation analysis for this model was numerically studied, and mathematical descriptions and physiological simulations were lack. Therefore, the 2DHR-type model should be analytically discussed for modeling neuro-computational features.

Neuronal behaviors are diverse. Variations of these behaviors have been observed through bifurcations such as saddle-node (SN) and Andronov-Hopf (AH) bifurcations. Herein, we discuss correlations between neuron types and behaviors near bifurcations. In 1948, Hodgkin [1948] proposed two types of neurons, Class I and Class II, which are characterized by their response to a constant current applied to the cell body. Class I neurons switch from steady state to oscillatory behavior through a saddle-node bifurcation. Additionally, for Class I neurons, repetitive firing occurs with zero frequency (homoclinic bifurcation), latency may be arbitrarily long, and intermediate-sized responses (in amplitude) are not possible. For Class II neurons, spiking is initiated through a (subcritical) Hopf bifurcation, resulting in the onset of
oscillations with a well-defined, non-zero frequency and a small amplitude; moreover, the latency for firing is finite. With respect to classification based on bifurcation, these two types of neurons have also been called Type I and Type II, respectively. Bifurcation methodologies have enabled the reduction of neuron models to a two-dimensional system of ordinary differential equations. Notably, in a plane system, class I excitability is essentially characterized using quadratic nonlinearity. To date, many two- and three-dimensional HR models [Hindmarsh & Rose, 1982, 1984; Innocentia et al., 2007; Storace et al., 2008; Tsuji et al., 2007; Ma & Feng, 2011] have been studied. However, most of these studies use computer simulations to assess the bifurcations of the HR model. Therefore, it is important to mathematically characterize bifurcations for the HR type model.

In this paper, we present the results of mathematical computations, diagrams of bifurcations and resemblances of neurophysiological behaviors. Based on the bifurcation analysis, several behaviors for the 2DHR model with spike-and-reset conditions are simulated near a bifurcation point. In Sec. 2, the 2DHR type model of Tsuji et al. [2007] is introduced. In Sec. 3, we discuss the number of equilibria and their stabilities; several categories are shown, and each diagram is divided into several zones. We assign a letter to each zone and list the number of equilibria and their stabilities in a table. Using this table, we determine the manner in which the number of solutions and their stabilities change with the current parameter. The conditions of the SN bifurcation are given in Sec. 4. We also show an example of a saddle-node bifurcation on a limit cycle. In Sec. 5, the AH and Bautin bifurcations are analyzed. A numerical simulation with a bistable behavior is presented. In Sec. 6, the Bogdanov-Takens bifurcation is analyzed. A numerical simulation is used to show homoclinic behavior. In Sec. 7, several of the most prominent features of biological spiking neurons are simulated using the 2DHR type model with spike-and-reset conditions. The goal of this section is to illustrate the richness and complexity of the spiking behavior of individual neurons in response to simple current input and pulses of current. The conclusions drawn from the results of this study are then discussed.

2. A Two-Dimensional Hindmarsh-Rose Type Model

Let us consider a 2DHR type model [Tsuji et al., 2007] of the following form:

\[ \dot{x} = F(x, \mu), \]

(1)

where \( \mu = (a, b, c, d)^T \), the dot denotes a derivative with respect to the independent variable \( t \) and \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is a smooth function defined by \( F(x, y) = (f(x, y), g(x, y)) \) with functions \( f \) and \( g \) as follows:

\[ f(x, y) = c\left( x - \frac{x^3}{3} - y + I \right), \]

\[ g(x, y) = \left( x^2 + dx - by + a \right)/c. \]

The two variables \( x \) and \( y \) denote the cell membrane potential and a recovery variable, respectively. The parameters \( a, b, c \) and \( d \) are positive. The parameter \( c \) represents the time scale. The parameter \( I \) denotes the membrane current or external stimulus. When the value of \( I \) is increased and the other parameters are unchanged, the graph of the cubic function moves up. However, when the parameter \( a \) is decreased and the other parameters are unchanged, the graph of the quadratic function moves down. Hence, the effect of \( I \) is reflected through parameter \( a \). Herein, we can assume that \( I = 0 \).

This system is simple; however, many more complicated systems of two equations can be reduced to this form. Thus, this system may exhibit typical nonlinear phenomena, such as AH and SN bifurcations, separatix loops, and hard oscillations, depending on its nonlinearity and parameters.

3. Equilibria and their stability

In this section, we attempt to discuss the equilibrium and its stability for Eq. (1). Examining the equilibrium for Eq. (1) is equivalent to determining the zero solution of the following cubic function:

\[ h(x) = (b/3)x^3 + x^2 + (d - b)x + a, \]

(2)
with $b \neq 0$. For the function $h$, $a$ is replaced with $a - bI$ if $I \neq 0$. Notably, if $b = 0$, the function $g$ in Eq. (1) has no $y$ term, and the system is of no interest. Because the function $h$ has three unknown parameters, it is difficult to determine the existence and number of zero solutions using Cardano’s method. Thus, we will show the existence and number of zero solutions for $h$ in this section. Let us define

$$x^e_\ell = \left(-1 - \sqrt{D}\right)/b, \quad x^e_r = \left(-1 + \sqrt{D}\right)/b,$$

where $D = 1 + b^2 - bd$ and $b \neq 0$. When $D$ is positive, these variables represent the right and left critical numbers of $h(x)$, respectively. Obviously, $x^e_\ell < x^e_r$, and the number of zero solutions can be analyzed using $x^e_\ell$, $x^e_r$, and the sign of $D$. When $D \leq 0$, the cubic function $h$ is monotonic. Therefore, only one zero solution exists, i.e., there is only one equilibrium for Eq. (1). When $D > 0$, the two critical numbers $x^e_\ell$ and $x^e_r$ exist. When the zero is between $h(x^e_\ell)$ and $h(x^e_r)$, there are three equilibria for Eq. (1). If either $h(x^e_\ell) = 0$ or $h(x^e_r) = 0$, then there are two equilibria for Eq. (1). If neither of the two previous cases holds, then there is only one equilibrium for Eq. (1). These relationships are illustrated in Fig. 1. The notation $(SN)^\pm$ indicates a triple set of parameters $(a, b, d)$ that satisfies the following relationship:

$$(SN)^\pm: a = \left(1 - 3D \pm 2D\sqrt{D}\right)/(3b^2).$$

If the condition $(SN)^+$ [resp. $(SN)^-$] holds, then $h(x^e_\ell) = 0$ [resp. $h(x^e_r) = 0$]. Thus, the existence and number of equilibria have been analyzed.

![Fig. 1. Illustration of the number of equilibria. A zero solution of $h$ represents the first component of an equilibrium, and three types of intersections between the graphs of $y = h(x)$ and $y = 0$ are plotted. The top, middle, and bottom functions intersect the $x$-axis at three points, two points, and one point, respectively. If $D > 0$, then two values, $x^e_\ell$ and $x^e_r$, exist. When the zero is between $h(x^e_\ell)$ and $h(x^e_r)$, there are three equilibria for Eq. (1). If either $h(x^e_\ell) = 0$ or $h(x^e_r) = 0$, then there are two equilibria for Eq. (1). If neither of the two previous cases holds, then there is only one equilibrium for Eq. (1).]

Next, the eigenvalues of the equilibrium for a plane system are determined. In the two-dimensional case, the Jacobian matrix $L_0$ of Eq. (1) at an equilibrium $x_0 = (x_0, y_0)$ is given by

$$
\begin{bmatrix}
c - cx_0^2 & -c \\
(c + d)x_0/c & c - b/c
\end{bmatrix}.
$$

$\delta(x_0)$ represents the trace of $L_0$ and $\Delta(x_0)$ is the determinant of $L_0$, where the functions $\delta$ and $\Delta : \mathbb{R} \to \mathbb{R}$ are represented by the following equations:

$$\begin{align*}
\delta(x_0) &= -cx_0^2 + c - b/c, \\
\Delta(x_0) &= bx_0^2 + 2x_0 - b + d.
\end{align*}$$

The characteristic polynomial of $L_0$ is $\lambda^2 - \delta(x_0)\lambda + \Delta(x_0)$. Notably, a comparison between Eq. (2) and $\Delta$ shows that $\delta'(x) = \Delta(x)$ for all $x$. The zero solutions of $\Delta$ are $x^e_\ell$ and $x^e_r$. If the first component of the equilibrium $x_0$ is either $x^e_\ell$ or $x^e_r$, then at least one of the eigenvalues is zero. With respect to the $(SN)^\pm$ condition, when $c^2 > b$, the zero solutions of $\delta$ are $x^0_\ell = -m_0$ and $x^0_r = m_0$, where $m_0 = \sqrt{1 - b/c^2}$ and $x^0_\ell < x^0_r$. If the first component of the equilibrium $x_0$ is either $x^0_\ell$ or $x^0_r$, then the sum of the eigenvalues is
zero. Thus, the eigenvalues must be imaginary. The notation \((AH)^\pm\) indicates a quadruple set of parameters \((a, b, c, d)\) that satisfies the following relationship:

\[
(AH)^\pm : c^2 > b \text{ and } a = \pm \left( d - \frac{2b}{3} - \frac{b^2}{3c^2} \right) m_0 - m_0^2,
\]

where \((AH)^+\) and \((AH)^-\) are determined using the equations \(h(x_0^\ell) = 0\) and \(h(x_0^r) = 0\), respectively. Using a previously derived theorem [Robinson, 2004], the stability of the equilibrium in the plane can be described using the following relationships: if \(\Delta(x_0^\ell) > 0\) and \(\delta(x_0^\ell) > 0\), then \(x_0\) is an unstable node; if \(\Delta(x_0^\ell) < 0\), then \(x_0\) is a saddle point; if \(\Delta(x_0^\ell) = 0\) and \(\delta(x_0^\ell) < 0\), then there are zero and negative eigenvalues; if \(\Delta(x_0^\ell) = 0\) and \(\delta(x_0^\ell) = 0\), then there are double zero eigenvalues; and if \(\Delta(x_0^\ell) = 0\) and \(\delta(x_0^\ell) > 0\), then there are zero and positive eigenvalues. Although these criteria are given, the number of equilibria and their stabilities are unknown for a given parameter in the 2DHR type model. Thus, the relative position of \(x_0\) among \(x_0^\ell\), \(x_0^r\), \(x_c^\ell\) and \(x_c^r\) must be determined. Consequently, we separate the parameter domain into several parts.

**Lemma 1.** Let \(c^2 > b\) and \(D > 0\).

(a) If \(0 < b \leq 1\), then \(-\frac{1}{b} < x_0^\ell\).

(b) If \(1 < b\) and \(\sqrt{b} < c \leq \frac{b^{3/2}}{\sqrt{b^2 - 1}}\), then \(-\frac{1}{b} \leq x_0^\ell\).

(c) If \(1 < b\) and \(\frac{b^{3/2}}{\sqrt{b^2 - 1}} < c\), then \(x_0^\ell < -\frac{1}{b}\).

**Proof.** The proof is trivial. ■

The relationship between \(-1/b\) and \(x_0^\ell\) is confirmed using Lemma 1. The number \(-1/b\) or zero is the middle point between \(x_c^\ell\) and \(x_0^\ell\) or \(x_0^r\) and \(x_c^r\), respectively. Because \(b > 0\), the number \(-1/b\) is always less than zero. Notably, if

\[
c = \frac{b^{3/2}}{\sqrt{b^2 - 1}},
\]

then \(x_0^\ell = -1/b\). The interfaces of these conditions for Lemma 1 are shown in Fig. 2, where the regions \(\Omega_2\) and \(\Omega_1\) satisfy Condition (a) and (b), respectively, and Condition (c) is represented by regions \(\Omega_3\) and \(\Omega_4\). None of the three conditions satisfy the region \(\Omega_5\).

![Diagram](https://via.placeholder.com/150)

**Fig. 2.** Illustration of the conditions of Lemma 1 in the parameter domain \((b, c)\). Regions \(\Omega_2\) and \(\Omega_1\) correspond to Conditions (a) and (b), respectively, in Lemma 1. Regions \(\Omega_3\) and \(\Omega_4\) correspond to Conditions (c) and (b), respectively, in Lemma 1. The relationship between \(b\) and \(c\) in the region \(\Omega_5\) satisfies the inequality \(c^2 < b\).
Next, we will discuss the relationship among $x^0_\ell$, $x^0_\ell$, $x^c_\ell$ and $x^c_c$. The two critical numbers $\hat{L}_{b,c}$ and $\hat{L}_{b,c}$ are defined by the following equations:

$$\hat{L}_{b,c} = \frac{b}{2} + 2c^2 \sqrt{\frac{(b + c^2)/c^2)}{(c^2)}},$$
$$\hat{L}_{b,c} = \frac{b}{2} - 2c^2 \sqrt{\frac{(b + c^2)/c^2)}{(c^2)}}.$$

**Lemma 2.** Consider the following conditions where $c^2 > b$ and $D > 0$.

1. Let $x^0_\ell \leq -1/b$.
   
   (a) If $\hat{L}_{b,c} \leq d$, then $x^0_\ell < x^c_\ell < x^0_\ell$.
   
   (b) If $\hat{L}_{b,c} < d < \hat{L}_{b,c}$, then $x^c_\ell < x^0_\ell < x^c_\ell$.
   
   (c) If $d \leq \hat{L}_{b,c}$, then $x^c_\ell < x^0_\ell < x^0_\ell < x^c_\ell$.

2. Let $x^0_\ell > -1/b$.
   
   (a) If $\hat{L}_{b,c} \leq d$, then $x^0_\ell < x^c_\ell < x^0_\ell < x^0_\ell$.
   
   (b) If $\hat{L}_{b,c} < d < \hat{L}_{b,c}$, then $x^c_\ell < x^0_\ell < x^0_\ell < x^c_\ell$.
   
   (c) If $d \leq \hat{L}_{b,c}$, then $x^c_\ell < x^0_\ell < x^0_\ell < x^c_\ell$.

**Proof.** Only Case (1) is discussed. The proof of Case (2) is the same as that of Case (1).

First, we consider Condition (a) in Case (1). We have the equality $x^0_\ell = x^c_\ell$, provided that $\hat{L}_{b,c} = d$. If $\hat{L}_{b,c} < d$, then $x^0_\ell < x^c_\ell$. Therefore, $x^0_\ell < x^c_\ell < x^0_\ell$ when $-1/b$ is less than zero. Second, we consider Condition (c) in Case (1). It follows from $\hat{L}_{b,c} = d$ that $x^0_\ell = x^c_\ell$. If $d < \hat{L}_{b,c}$, then $x^0_\ell < x^c_\ell$. Therefore, $x^0_\ell < x^c_\ell < x^0_\ell < x^c_\ell$ when $-1/b$ is less than zero. Finally, we consider Condition (b) in Case (1). Based on the previous two cases, $x^c_\ell < x^0_\ell$ and $x^0_\ell < x^c_\ell$; therefore, $x^c_\ell < x^c_\ell < x^c_\ell < x^0_\ell$ when $x^0_\ell \leq -1/b$. Thus, the proof of Case (1) is complete.

The order among $x^0_\ell$, $x^0_\ell$, $x^c_\ell$ and $x^c_\ell$ is confirmed using Lemma 2. It follows from their relationship that the stability of an equilibrium can be determined using the signs of $\Delta$ and $\delta$. Between Cases (1) and (2) in Lemma 2 and among $x^0_\ell$, $x^0_\ell$, $x^c_\ell$ and $x^c_\ell$, the order only differs in Case (a), and the orders for the remaining cases are the same. For Case (1), if the two curves (AH)$_+$ and (SN)$_-$ intersect, then $x^c_\ell = x^0_\ell$. If the two curves (AH)$_-$ and (SN)$_+$ intersect, then $x^0_\ell = x^c_\ell$. For Case (2), if the two curves (AH)$_+$ and (SN)$_+$ intersect, then $x^0_\ell = x^c_\ell$. If the two curves (AH)$_-$ and (SN)$_+$ intersect, then $x^c_\ell = x^0_\ell$.

Next, for a pair of given parameters $(b, c)$, the parameter domain $(a, d)$ is divided into several regions. For each region, the number and stability of the equilibria vary. This basic category is based on the number and stability of the equilibria. We determine at which parameter the four conditions (AH)$_\pm$ or (SN)$_\pm$ can be equal. If the parameters $b$, $c$ and $d$ satisfy both (AH)$_+$ and (AH)$_-$, we obtain the following equation:

$$- \left( d - \frac{2b}{3} - \frac{b^2}{3c^2} \right) m_0 = m_0^2.$$ 

This equation implies

$$d = \frac{b(b + 2c^2)}{3c^2}.$$ 

(6)

If the parameters $b$, $c$ and $d$ satisfy both (AH)$_+$ and (SN)$_-$ with Eq. (6), then we obtain the following equation:

$$- \left( d - \frac{2b}{3} - \frac{b^2}{3c^2} \right) m_0 = m_0^2 = \frac{3 - 3D - 2D \sqrt{D}}{(3b^2)}.$$ 

A combination of this equation with Eq. (6) yields $c = \pm \frac{b^{3/2}}{\sqrt{b^2 - 1}}$, where the sign “–” is unreasonable because $b > 0$ and $c > 0$. For any positive parameter $c$, where $0 < b \leq 1$ (i.e., the region $\Omega_2$ in Fig. 2), the bifurcation diagram of $a$ and $d$ is as shown in Fig. 3(a). If a pair of parameters $(b, c)$ belongs to the region $\Omega_1$ in Fig. 2, where the pair satisfies the relations $\sqrt{b} < c < b^{3/2}/\sqrt{b^2 - 1}$ and $b > 1$, then the
bifurcation diagram of parameters $a$ and $d$ is similar to the diagram shown in Fig. 3(a), where the cusp is to the right of $AH^+$ and the curve $AH^+$ is tangent to the curve $SN^+$. If $c = \frac{b^{3/2}}{\sqrt{b^2 - 1}}$, then the three lines $(SN)\pm$ and $(AH)^+$ in Fig. 3(b) intersect. For the region $\Omega_3$, the parameters $b$ and $c$ satisfy either the inequality $\frac{b^{3/2}}{\sqrt{b^2 - 9}} > c > \frac{b^{3/2}}{\sqrt{b^2 - 1}}$ with $b > 3$ or the inequality $c > \frac{b^{3/2}}{\sqrt{b^2 - 1}}$ with $b \leq 3$. Their bifurcation diagram is shown in Fig. 3(c). The cusp is to the left side of $AH^+$, and the curve $AH^+$ is tangent to the curve $SN^-$. Consider the case where $c = b^{3/2}/\sqrt{b^2 - 9}$. The corresponding bifurcation diagram is presented in Fig. 3(d). The two lines $AH^-$ and $SN^-$ intersect at the end point of $AH^+$. The bifurcation diagram resulting from $c > \frac{b^{3/2}}{\sqrt{b^2 - 9}}$ is shown in Fig. 3(e). The intersection of $AH^+$ and $SN^-$ is to the left of $AH^-$. The bifurcation diagram for parameters $a$ and $d$ is shown in Fig. 4, where the parameters $b$ and $c$ satisfy the relationship $c < \sqrt{b}$ (i.e. the region $\Omega_5$ in Fig. 2). In this case, there is no $AH$ bifurcation.

![Fig. 3](image_url)

**Theorem 1.** For the two positive parameters $b$ and $c$, $c^2 > b$. For the parameter domain $(a, d)$ in Fig. 3, the number and stability of the equilibria are listed for different regions in Table 1.

**Proof.** We only consider regions A, B, C, D, E, F, G and H. The discussion for the remaining regions is similar and has therefore been omitted from the results in Table 1.

The three regions A, B, and C are shown in Fig. 3(b). The interface between A and B satisfies Condition $AH^-$, where $h(x_0^B) = 0$, and $\delta(x_0^B) = 0$. The interface between B and C satisfies Condition $AH^+$, where $h(x_0^C) = 0$, and $\delta(x_0^C) = 0$. In the region A, $D < 0$; therefore, Eq. (2) is an increasing function and $\Delta(x_0) > 0$ is at the equilibrium $(x_0, y_0)$. Recall that $h'(x) = \Delta(x)$. Therefore, only one equilibrium exists. The same
result is observed in regions B and C. The equilibrium is stable in the region A, unstable in the region B, and stable in the region C because \( \delta(x_0) < 0, \delta(x_0) > 0, \) and \( \delta(x_0) < 0, \) respectively (Fig. 5(a)). The terms \((s,N,N), (N,N,u)\) and \((N,N,s)\) represent the number of equilibria and their stabilities, where “N” denotes no equilibrium point, “s” denotes a stable node, and “u” denotes an unstable node.

Let us consider regions D, E, F, G and H, which lie between \( \hat{L}_{b,c} \) and \( \check{L}_{b,c} \) in Fig. 3(b). Because \( c = \frac{b^3}{2}\sqrt{b^2-1} \) and \( \hat{L}_{b,c} > d > \check{L}_{b,c}, \) it follows from Lemmas 1 and 2 that the order of the four numbers is \( x_c^l < x_0^l < x_c^r < x_0^r. \) For convenience of this proof, an auxiliary line is shown in Fig. 3(b). The region D (resp. F) is divided into D\(_1\) and D\(_2\) (resp. F\(_1\) and F\(_2\)). The black lines from the bottom to the top are, respectively, \( \text{SN}^- \), \( \text{AH}^- \), and \( \text{SN}^+ \), as shown in Fig. 5(b). The black lines from the bottom to the top are, respectively, \( \text{AH}^- \), \( \text{SN}^- \), and \( \text{SN}^+ \), as shown Fig. 5(c). The signs of \( \Delta(x_0) \) and \( \delta(x_0) \) are shown in Figs. 5(b) and 5(c). Accordingly, we confirm the stabilities of the equilibria in these regions and find that the stabilities of the equilibria are \((N,N,s), (N,N,u), (s,sd,u), (s,N,N)\) and \((s,sd,s)\) in D, E, F, G and H, respectively, where “sd” denotes a saddle point. The proof is complete.

**Remark.** For positive parameters \( b \) and \( c \) with \( c^2 < b \), we plot a bifurcation diagram with \( b = 5 \) and \( c = 1 \) in Fig. 4.

In this paragraph, we explain how to interpret Table 1. All forms, such as “\((s,N,N)\)”, “\((N,N,u)\)” or “\((s,sd,s)\)”, show the change in the stability of equilibria and which two equilibria collapse at the SN bifurcation. For example, if the parameter \( a \) increases from the region F to G, there exists a SN bifurcation and the left equilibrium still exists. If the parameter \( a \) increases from the region L to M in the parameter domain \((a,d)\), then the stability of the left equilibrium is changed and the remaining two points disappear. The stability of one equilibrium is changed both from C to B and from B to A. Therefore, Table 1 shows us which two points collapse and the manner in which the stability of an equilibrium point changes.
Fig. 5. Illustration for an auxiliary proof of Theorem 1. Different capital letters (e.g., A, B, etc.) correspond to different regions in Figs. 3 and 4. The graph of Eq. (2) for given values of $a$, $b$, $c$, $d$ is plotted by a black line when a bifurcation occurs in Eq. (1) at the given parameters; otherwise, the graph is given by a black dotted line. For example, the left and right black regions in Figs. 3 and 4. The graph of Eq. (2) for given values of $a$
Fig. 5. Illustration for an auxiliary proof of Theorem 1. Different capital letters (e.g., A, B, etc.) correspond to different marked below the double arrows. The stability of the equilibrium ($x$ near the parameter domain at the same time. In this section, the conditions of a saddle-node bifurcation if we want to find the SNLC in a parameter domain, a limit cycle and a SN bifurcation should be found period of this cycle tends to infinity as the parameter approaches its bifurcation value. In planar ODEs, The SNLC results in the occurrence of a limit cycle when the saddle-node disappears. For the SNLC, the a limit point or fold bifurcation. More interestingly, if this bifurcation occurs for an invariance set, it is linear part for the ODEs at an equilibrium must have one zero eigenvalue. This bifurcation is also called theoretically, a necessary condition for the emergence of such a bifurcation in autonomous ODEs is that the A saddle-node bifurcation occurs if a dynamic system has a collision and two equilibria disappear. Math-

4. Saddle-Node Bifurcation

A saddle-node bifurcation occurs if a dynamic system has a collision and two equilibria disappear. Mathematically, a necessary condition for the emergence of such a bifurcation in autonomous ODEs is that the linear part for the ODEs at an equilibrium must have one zero eigenvalue. This bifurcation is also called a limit point or fold bifurcation. More interestingly, if this bifurcation occurs for an invariance set, it is called a saddle-node homoclinic bifurcation, such as a saddle-node bifurcation on a limit cycle (SNLC). The SNLC results in the occurrence of a limit cycle when the saddle-node disappears. For the SNLC, the period of this cycle tends to infinity as the parameter approaches its bifurcation value. In planar ODEs, if we want to find the SNLC in a parameter domain, a limit cycle and a SN bifurcation should be found near the parameter domain at the same time. In this section, the conditions of a saddle-node bifurcation are computed while the detection of a SNLC considered in a later section.

The following theorem is provided to demonstrate the occurrence of a SN bifurcation. Recall that
Theorem 2. Assume $D > 0$. If the parameters satisfy Condition $(SN)^+$, then a SN bifurcation exists, and there is an equilibrium at

$$
\left(\frac{-1 \pm \sqrt{D}}{b}, \frac{4 - 3bd \pm (3b^2 - 4 - b(d - b))\sqrt{D}}{3b^3}\right).
$$

Proof. Let us consider Condition $(SN)^+$. The proof for Condition $(SN)^-$ is the same as that for Condition $(SN)^+$. Assume that in Eq. (1), a SN bifurcation with a fixed point $(x_0, y_0)$ occurs. Because there exists at least one zero eigenvalue, we obtain Eq. (4). In other words, $bx_0^3/3 + x_0^2 + (d - b)x_0 + a = 0$ and $D > 0$, we obtain the following condition

$$
a = \left(1 - 3D \pm 2D\sqrt{D}\right)/(3b^2).
$$

Next, we consider the case in which the system has nonvanishing quadratic terms along the vector $v_1 = \left(\frac{b}{c}, \frac{2 + 2\sqrt{D}b + bd}{bc}\right)^T$ which is the eigenvector of $L_0$ with respect to a single zero eigenvalue. The vector $\omega_1$ belonging to the dual space with respect to the right eigenspace of $L_0$ is defined as

$$
w_1 = (b, -c^2)/\varrho,
$$

where $\varrho = \left[b_3 - c^2\left(-2 + bd + 2\sqrt{D}\right)\right]/(bc)$. It follows from the first two conditions that, in Condition $(SN)^+$,

$$
w_1D^2\mathcal{F}(v_1, v_1) = \frac{-2b^2c^2b}{\varrho^2 + 2 - 2\sqrt{D} - bd} \neq 0.
$$

Finally, the transversality condition is trivial. Therefore, by the definition of a SN bifurcation, we confirm that there exists a SN bifurcation when Condition $(SN)^+$ holds. The proof is complete.

Corollary 4.1. If $d = b + \frac{1}{b}$ and $a = \frac{1}{3c^2}$, then a cusp bifurcation exists.

Proof. Following the definition of a cusp bifurcation, the two conditions of $(SN)^+$ and $(SN)^-$ simultaneously hold. In other words, $D = 0$. Therefore, we obtain the conditions $d = b + \frac{1}{b}$ and $a = \frac{1}{3c^2}$.

5. Andronov-Hopf and Bautin Bifurcation

In this section, we analyze an AH bifurcation, which describes the situation in which a stable or unstable limit cycle appears out of an unstable or stable equilibrium, respectively. Recall that Condition $(AH)^\pm$ only implies that the trace of $L_0$ is zero. The following theorem will show that the eigenvalues of $L_0$ are complex.

Theorem 3. Assume that $c^2 > b$. If the parameters satisfy Condition $(AH)^\pm$ and $d > \frac{b^2}{c^2} \pm 2m_0$, then an AH bifurcation exists, and there is an equilibrium at

$$
\left(\pm m_0, \mp \left(\frac{2}{3} + \frac{b}{3c^2}\right)m_0\right).
$$

Proof. Let us consider Condition $(AH)^+$. The proof for Condition $(AH)^-$ is the same as that for Condition $(AH)^+$.
Assume that Eq. (1) is at an AH bifurcation with a fixed point \((x_0, y_0)\). One of the conditions for the emergence of an AH bifurcation is that the real parts of the pair of complex conjugate eigenvalues are zero. By Eq. (3), we have \(x_0 = -m_0\). Because \(x_0\) also satisfies \(bx_0^3/3 + x_0^2 + (d - b)x_0 + a = 0\), we obtain the following condition:

\[
a = \left(d - \frac{2b}{3} - \frac{b^2}{3c^2}\right)m_0 - m_0^2.
\]

Because \(d > b^2/c^2 + 2m_0\), their imaginary part of the pair of complex conjugate eigenvalues is not zero. Finally, we compute the transversality condition at \((x_0, y_0)\):

\[
\left(\frac{\partial \text{Re}(\lambda)}{\partial a} , \frac{\partial \text{Re}(\lambda)}{\partial b} , \frac{\partial \text{Re}(\lambda)}{\partial c} , \frac{\partial \text{Re}(\lambda)}{\partial d}\right) = \left(\frac{cx_0}{\Delta(x_0)} , \frac{-cx_0 y_0}{\Delta(x_0)} , -\frac{1}{2c} - x_0^2 + 1 + \frac{b}{c^2} , \frac{cx_0^2}{\Delta(x_0)}\right).
\]

Recall that \(\Delta(x_0) = bx_0^2 + 2x_0 + d - b\). We are only interested in the parameter \(a\) because changing this parameter is equivalent to changing the external stimulus. Finally, \(\partial \text{Re}(\lambda)/\partial a \neq 0\) because \(c > 0\), \(c^2 > b\) and \(d > b^2/c^2 + 2m_0\). Therefore, the proof is complete.

Next, a Bautin bifurcation is considered. For a nearby bifurcation parameter, there exist two limit cycles that collide and disappear via a SN bifurcation of periodic orbits. We analyze the bifurcation using the process described in Ref. [Kuznetsov, 2004].

**Theorem 4.**

(a) Let the quadruple set of parameters \((a, b, c, d)\) satisfy Condition AH\(^+\). If \(m_0 \geq 1/b\), then a Bautin bifurcation occurs when \(d = 2b - b^2/c^2\) and \(c^2 \neq b^3/(b^2 - 1)\).

(b) Let the quadruple set of parameters \((a, b, c, d)\) satisfy Condition AH\(^-\). If \(m_0 \geq 0\), then a Bautin bifurcation occurs when \(d = 2b - b^2/c^2\) and \(c^2 \neq b^3/(b^2 - 1)\).

**Proof.** The proofs for these two cases are presented in the appendix.

For an AH bifurcation, there are two types: a subcritical AH bifurcation and a supercritical AH bifurcation. Let us consider Cases (a) and (b) in Theorem 4. If \(d > 2b - b^2/c^2\), then \(\ell_1(0) < 0\) and the AH bifurcation is supercritical. If \(d < 2b - b^2/c^2\), then \(\ell_1(0) > 0\) and the AH bifurcation is subcritical. For Case (a), it follows from Fig. 6 that bistable behavior exists near the subcritical AH bifurcation for Eq. (1).

6. **Bogdanov-Takens bifurcation**

In this section, a BT bifurcation is analyzed using the process described in Ref. [Kuznetsov, 2004] and [Carrillo et al., 2010]. The normal form of the BT bifurcation is as follows:

\[
\dot{u} = v, \quad \dot{v} = \beta_1 + \beta_2 u + u^2 - uv
\]

We attempt to transform Eq. (1) into Eqs. (7) and (8). For the normal form, the following three results are obtained: if \(4\beta_1 - \beta_2^2 = 0\), then the system undergoes a SN bifurcation; if \(\beta_1 = 0\) and \(\beta_2 < 0\), then the system undergoes a supercritical AH bifurcation; if \(\beta_1 + (6/25)\beta_2^2 = 0(\beta_2^2)\) and \(\beta_2 < 0\), then the system undergoes a homoclinic bifurcation curve. Therefore, the bifurcation curves are found if \(\beta_1\) and \(\beta_2\) are represented by parameters \(a, b, c\) and \(d\). The following theorem provides the parameters \(\beta_1\) and \(\beta_2\) in terms of the parameters \(a, b, c\) and \(d\).

**Theorem 5.** (a) Assume that the parameter \(a\) satisfies Condition AH\(^+\) and \(d = \hat{L}_{b,c}\). Then,

\[
\beta_1 = \frac{16c^4m_0^4(\lambda_1 - m_0\lambda_2)}{(1 - bm_0)^3} , \quad \beta_2 = -\frac{4c^2m_0^2\lambda_2}{(bm_0 - 1)^2},
\]
where \( \lambda_1 = a - (b - c^2)(2bm_0 - 3)/(3c^2) \) and \( \lambda_2 = d - (b^2/c^2 + 2m_0) \).

(b) Assume that the parameter \( a \) satisfies Condition AH and \( d = \hat{L}_{b,c} \). Thus,

\[
\beta_1 = \frac{16c^4m_0^4(\lambda_1 + m_0\lambda_2)}{(1 + bm_0)^3}, \quad \beta_2 = -\frac{4c^2m_0^2\lambda_2}{(bm_0 + 1)^2},
\]

where \( \lambda_1 = a - (c^2 - b)(2bm_0 + 3)/(3c^2) \) and \( \lambda_2 = d - (b^2/c^2 - 2m_0) \).

Proof. For a pair of given positive parameters \( b \) and \( c \), let \( d = \hat{L}_{b,c} \), and let \( a \) satisfy Condition (AH)*. Let \( \mu_0 = ((b - c^2)(2bm_0 - 3)/(3c^2), b, c, b^2/c^2 + 2m_0) \) denote the bifurcation parameter. That is, the parameter \( \mu_0 \) satisfies the above two conditions. By Theorem 3, there is an equilibrium at \((x_0, y_0)\), where \( x_0 = -m_0 \) and \( y_0 = -(2/3 + b/(3c^2))m_0 \). Under these two conditions, the Jacobian matrix of Eq. (1) at \((x_0, y_0)\) is given by

\[
\begin{pmatrix}
\frac{b}{c^2} & -c \\
\frac{b}{c} & \frac{b}{c^2} - \frac{b}{c}
\end{pmatrix}.
\]

This result shows a double zero eigenvalue. Let \( p_1 = (-c, -\frac{b}{c^2})^T \), \( p_2 = (c^2, c)^T/(c - b) \), \( q_1 = (-1, c)^T/(c - b) \) and \( q_2 = (-b/c^2, 1)^T \), where \( p_1, q_2 \) (resp. \( p_2, q_1 \)) are the right and left (resp. generalized) eigenvectors associated with the zero eigenvalue. Using the above information, we can transform Eq. (1) into the following normal form using a series of smooth transformations

\[
\dot{u} = v, \quad \dot{v} = \alpha_1 + \alpha_2u + \alpha_3u^2 + \alpha_4uv, \]

Fig. 6. Illustration of phase portraits near the Bautin bifurcation. The limit cycle in black (resp. gray) is stable (resp. unstable). Here, \( a = 0.518 \), \( b = 1.55 \), \( c = 3 \) and \( d = 2.6 \).
where \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) are formulated using the following formulas:

\[
\begin{align*}
\alpha_1 &= S_1^T(\mu - \mu_0), \quad \alpha_2 = S_2^T(\mu - \mu_0), \\
\alpha_3 &= \frac{1}{2} p_1^T(q_1 \circ D^2\mathbf{F}(x_0, \mu_0)) p_1, \\
\alpha_4 &= p_1^T(q_1 \circ D^2\mathbf{F}(x_0, \mu_0)) p_1 + p_2^T(q_2 \circ D^2\mathbf{F}(x_0, \mu_0)) p_2, \\
S_1 &= \mathbf{F}_\mu^T(x_0, \mu_0) q_2 = (s_{11}, s_{12}, s_{13}, s_{14})^T, \\
S_2 &= \left[ \frac{2\hat{a}}{\hat{b}} \left( p_1^T(q_1 \circ D^2\mathbf{F}(x_0, \mu_0)) p_2 + p_2^T(q_2 \circ D^2\mathbf{F}(x_0, \mu_0)) p_2 \right) \right] \times \\
& \quad \mathbf{F}_\mu^T(x_0, \mu_0) q_1 - \frac{2\hat{a}}{\hat{b}} \sum_{i=1}^{2} (q_i \circ \mathbf{F}_{\mu x}(x_0, \mu_0)) p_i + \\
& \quad \left( q_2 \circ \mathbf{F}_{\mu x}(x_0, \mu_0) \right) p_1 = (s_{21}, s_{22}, s_{23}, s_{24})^T,
\end{align*}
\]

where \( \hat{a} = (1/2)p_1^T(q_2 \circ D^2\mathbf{F}(x_0, \mu_0)) p_1, \hat{b} = p_1^T(q_1 \circ D^2\mathbf{F}(x_0, \mu_0)) p_1 + p_2^T(q_2 \circ D^2\mathbf{F}(x_0, \mu_0)) p_2. \) The notation \( \circ \) is defined by \( v \circ A = \sum_{i=1}^{2} v_i A_i, \) where \( v = (v_1, v_2)^T \in \mathbb{R}^2 \) and \( A = (A_1, A_2)^T \) with the same size for matrices \( A_1, A_2 \in \mathbb{R}^{r \times s}. \) Thus, we can obtain

\[
\begin{align*}
\alpha_1 &= \frac{m_0b^2 + (2c^2m_0 - 3)b + 3c^2(a - dm_0 + 1)}{3c^3}, \\
\alpha_2 &= \frac{b^2}{c^2} - d + 2m_0, \\
\alpha_3 &= c - bcm_0, \quad \alpha_4 = -2c^2m_0.
\end{align*}
\]

We also provide the following values: \( s_{11} = 1/c, \quad s_{12} = (2c^2 + b) m_0/(3c^3), \quad s_{13} = -(bc^2m_0^3 + 3c^2m_0^2 + \beta_3m_0 + 3ac^2)/(3c^4), \quad s_{14} = -m_0/c, \quad s_{21} = 0, \quad s_{22} = (2bm_0 - 1)/(c^2m_0), \quad s_{23} = -(3m_0^2c^2 + c^2 + b + (c^2d - 3b^2)m_0)/(c^3m_0) \) and \( s_{24} = -1. \)

Let \( t \to -\alpha_4 t/\alpha_3, \quad u \to \alpha_3 u/\alpha_3^2 \) and \( v \to -\alpha_3 v/\alpha_3^2. \) Thus, Eqs. (9) and (10) can be transformed into Eqs. (7) and (8) with \( \beta_1 = (\alpha_4^2/\alpha_3^2)\alpha_1 \) and \( \beta_2 = (\alpha_4^2/\alpha_3^2)\alpha_2 \) in Eqs. (7) and (8). For Case (b), we provide all of the values in the appendix. The proof is complete. \( \blacksquare \)

To show the practicability of this theorem, some numerical results concerning the existence of a homoclinic bifurcation are given. Let \( (b, c, d) = (2, 3, 2.25283 \cdots) \) be fixed. When \( a = -0.0965, \) one stable equilibrium and one saddle equilibrium exist. When \( a \approx -0.0969983648, \) one homoclinic orbit, one stable equilibrium and one saddle equilibrium exist. When \( a = -0.09715, \) one unstable limit cycle, one stable equilibrium and one saddle equilibrium exist. A numerical simulation is shown in Fig. 7. In particular, bistable behaviors are clearly presented in the subfigure (e) of Fig. 7, which is a part of subfigure (d). To clarify the location between the parameters and bifurcations, we plot subfigure (f), where the simulation parameter is represented by a pink point and the red [resp. blue and dotted blue] line indicates the SN [resp. AH and homoclinic] bifurcation. Therefore, we also numerically confirm that a homoclinic bifurcation exists.

7. Existence of a Limit Cycle and Resemblances of Neuro-Computational Features

In this section, we demonstrate some resemblances of neurophysiological signals. However, the existence of a stable limit cycle should be discussed before the resemblances are simulated. The existence of stable limit cycles can help us determine the tendency of a signal as time goes to infinity through the adjustment of parameters and the prediction of behaviors. If parameter \( b \) with \( c > 0 \) is not zero, then a trapping region exists, as shown in Fig. 8, with two nullclines and several vectors. The respective trapping regions are drawn using break regions. For regions B, E, N, O and X in Fig. 3, it follows from Theorem 1 that there
Fig. 7. Illustration of a homoclinic bifurcation. In subfigure (a) with $a = -0.0965$, one stable and one saddle equilibrium exist. In subfigure (b) with $a = -0.0969993468$, the homoclinic orbit shown in gray is unstable. In subfigure (c) with $a = -0.09715$, the limit cycle shown in black is unstable. In subfigure (d), a stable limit cycle is plotted. The parameter $c$ for subfigures (d), (e) and (f) is the same as that for subfigure (c). In subfigure (e), the rectangular part of subfigure (d) is magnified, and an unstable limit cycle and part of the stable limit cycle are plotted. In subfigure (f), the red [resp. blue and dotted blue] line indicates the SN [resp. AH and homoclinic] bifurcation and the pink point indicates the simulation parameter. Here, $b = 2$, $c = 3$ and $d = 2.2528$.

exists only one unstable fixed point. Therefore, according to the Poincaré Bendixson theorem, a stable limit cycle exists for these regions.

We aimed to mimic some neuro-computation features for the 2DHR type model using spike-and-reset conditions as neural activities in accordance with works by Izhikevich [2004, 2007] because the planar system is too simple to explain the diversity of biological neurons. The spike-and-reset condition is represented as
follows: If \( x(t^-) > \theta \), then
\[
\begin{align*}
x(t) &= x_r, \\
y(t) &= y(t^-) + y_r,
\end{align*}
\]
where \( \theta \) is a threshold, the subscript “\( r \)” represents “reset”, and \( x_r \) (resp. \( y_r \)) denotes the reset value of \( x \) (resp. \( y \)).

Twenty resemblances of neural activities are shown in Fig. 9, and the relative parameters are given in Table 2. In the previous paragraph, we demonstrated the existence of a stable limit cycle in regions B, E, N, O, and X. Hence, we constructed resemblances of neuronal phenomena for Cases (a), (c), (e), (f), (i) and (j) in Table 2, where the limit cycles are attractive. The parameters for Cases (a)-(d), (g), (h), (k), (l), (n), (s) and (t) in Table 2 can be located near the SN bifurcation. The parameters for Cases (o), (p) and (q) in Table 2 are close to the Hopf bifurcation. The parameters for Cases (m) [resp. (q)] in Table 2 are near the Bautin [resp. BT] bifurcation. For Cases (n), (o) and (q), the 2DHR type model without the spike-and-rest condition exhibits behaviors that differ from those of Izhikevich [Izhikevich, 2004]. Therefore, these results show we successfully mimicked features of neural activities.

![Fig. 9. Illustration of the diversity of the 2DHR type model with/without the spike-and-reset condition. For each case, a resemblance of the neurosignal is plotted in the upper panel, and information about the input current and the excitatory and inhibitory pulse is given in the lower panel. The corresponding parameters are listed in Table 2.](image)

### 8. Conclusion

For the 2DHR type model (1), the conditions for the AH, SN, BT and Bautin bifurcations are shown. The existence and number of equilibria for Eq. (1) are completely studied. Because the system is planar, an analysis of the trace and determinant of the linear part for the stability of the equilibria is discussed. In accordance with the conditions in Lemma 1, the parameter domain \((b, c)\) can be divided into three parts, \(\Omega_1, \Omega_2\) and \(\Omega_3 + \Omega_4\), which correspond to Conditions (a), (b) and (c), respectively, in Lemma 1. The order among \(x^0_\ell, x^0_r, x^f_\ell\) and \(x^f_r\) was confirmed by Lemma 2. Based on Lemma 2, we confirmed the signs of \(\delta(x^0_0)\).
and $\Delta(x_0)$, where $x_0$ is the first component of the equilibrium $(x_0, y_0)$ for Eq. (1). The two signs can help us determine the stability of the equilibrium. Using the categories for Lemmas 1 and 2, we can explore the AN and SN bifurcations. Furthermore, the normal forms for the BT and Bautin bifurcations were evaluated in Theorems 4 and 5 based on the results of the AH and SN bifurcations, respectively. We also found that bistable behaviors exist near the Bautin and BT bifurcations, which are shown in Figs. 6 and 7, respectively. The bistable behaviors can be used to construct (m) and (q), as shown in Fig. 9. Notably, $\lim_{c \to \infty} d = 2b$ in Theorem 4, $\lim_{c \to \infty} a = \pm(d - \frac{2b}{3}) - 1$ in Condition (AH)$^\pm$, and there was no parameter c in Condition (SN)$^\pm$. Therefore, as c is sufficiently large, all of the bifurcation curves in the parameter domains $(a, b, d)$ are almost the same. In this study, we have shown bifurcation scenarios for Eq. (1) and have simulated resemblances of neuro-activities for the model using spike-and-reset conditions, which may be useful in further biophysical studies, for instance, as a guide for choosing suitable parameters for model fitting to obtain qualitatively different resemblances of electro-physiological behaviors.

The advantages of the 2DHR type model are explained from two perspectives: to retain the original structure of the FH model and to include a quadratic recovery term. First, the model can be regarded as an intuitive extension of the FH model [FitzHugh, 1961]. We reserve the traditional type, which has the same structure of the first component for FH [FitzHugh, 1961] and which retains the time-scale parameter c. Because the 2DHR type model has a slow-fast form, we can adjust the time-scale parameter to generate a Canard-like waveform. It may be difficult to describe phenomena using other types of HR type models without the time-scale parameter. Secondly, the nullcline for the second component of Eq. (1) is quadratic. However, by properly adjusting the parameters $d$ and $b$, the model can be regarded as an FH model. If $b$ is large enough, the $x^2$ component can be omitted, i.e., the nullcline of the second term can be regarded as a linear function. For example, if we take $a = 0$, $b = 6$, $c = 3$ and $d = 10$, the nullcline for the second component is nearly linear near the significant region shown in Fig. (10). Therefore, we can also use the 2DHR type model to mimic neuropysiological signals, except in the FH model. Therefore, it is worthwhile to systematically analyze the 2DHR type model.

Our work provides results for both the characterization of bifurcation behaviors for Eq. (1) and the application of this model for resemblances using simulations under spike-and-reset conditions. The results

Table 2. Illustration of Parameter Setting for Fig. 9. Herein, SFA: spike frequency adaptation, IIB: inhibition-induced bursting, IIS: inhibition-induced spiking, and DAP: depolarizing after-potential. Note that the three types (n), (o) and (q) are simulated without spike-and-reset conditions; therefore, the values $x_r$, $y_r$ and $\theta$ for the cases are useless and given by the symbol “$. For Case (n) [resp. (o)], Class 1 [resp. 2] excitability can be shown if the parameter value of $I$ is changed from 0.16 to 0.2 [resp. from 0 to 0.5]. The value of $a - bI$ in the table represents the value of $a$ in Eq. (2).

<table>
<thead>
<tr>
<th>Name</th>
<th>$a$</th>
<th>$I$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$x_r$</th>
<th>$y_r$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) tonic spiking</td>
<td>0</td>
<td>0.23</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>-0.45</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>(b) phasic spiking</td>
<td>-0.37</td>
<td>0.01</td>
<td>2.7</td>
<td>15</td>
<td>2.4</td>
<td>-0.7</td>
<td>0.1</td>
<td>-0.6</td>
</tr>
<tr>
<td>(c) tonic bursting</td>
<td>0</td>
<td>0.235</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0.2612</td>
<td>-0.504</td>
<td>1</td>
</tr>
<tr>
<td>(d) phasic bursting</td>
<td>-0.37</td>
<td>0.01</td>
<td>2.7</td>
<td>10</td>
<td>2.4</td>
<td>-0.4</td>
<td>0.02</td>
<td>-0.2</td>
</tr>
<tr>
<td>(e) mixed mode</td>
<td>1.35</td>
<td>0.7</td>
<td>2</td>
<td>15</td>
<td>3</td>
<td>-0.8</td>
<td>0.15</td>
<td>0</td>
</tr>
<tr>
<td>(f) SFA</td>
<td>0.7</td>
<td>0.7</td>
<td>2</td>
<td>15</td>
<td>2.6</td>
<td>-1.4</td>
<td>0.05</td>
<td>-0.2</td>
</tr>
<tr>
<td>(g) rebound mode</td>
<td>0</td>
<td>0</td>
<td>2.13</td>
<td>10</td>
<td>2.42</td>
<td>-0.774</td>
<td>0.5</td>
<td>-0.7176</td>
</tr>
<tr>
<td>(h) rebound burst</td>
<td>0</td>
<td>0</td>
<td>2.13</td>
<td>10</td>
<td>2.42</td>
<td>-0.774</td>
<td>0.005</td>
<td>-0.7176</td>
</tr>
<tr>
<td>(i) IIB</td>
<td>-2.9</td>
<td>-0.65</td>
<td>2</td>
<td>3</td>
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<tr>
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<td>(r) subthreshold oscill.</td>
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<td>-1.04</td>
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and $\Delta(x_0)$, where $x_0$ is the first component of the equilibrium $(x_0, y_0)$ for Eq. (1). The two signs can help us determine the stability of the equilibrium. Using the categories for Lemmas 1 and 2, we can explore the AN and SN bifurcations. Furthermore, the normal forms for the BT and Bautin bifurcations were evaluated in Theorems 4 and 5 based on the results of the AH and SN bifurcations, respectively. We also found that bistable behaviors exist near the Bautin and BT bifurcations, which are shown in Figs. 6 and 7, respectively. The bistable behaviors can be used to construct (m) and (q), as shown in Fig. 9. Notably, $\lim_{c \to \infty} d = 2b$ in Theorem 4, $\lim_{c \to \infty} a = \pm(d - \frac{2b}{3}) - 1$ in Condition (AH)$^\pm$, and there was no parameter c in Condition (SN)$^\pm$. Therefore, as c is sufficiently large, all of the bifurcation curves in the parameter domains $(a, b, d)$ are almost the same. In this study, we have shown bifurcation scenarios for Eq. (1) and have simulated resemblances of neuro-activities for the model using spike-and-reset conditions, which may be useful in further biophysical studies, for instance, as a guide for choosing suitable parameters for model fitting to obtain qualitatively different resemblances of electro-physiological behaviors.

The advantages of the 2DHR type model are explained from two perspectives: to retain the original structure of the FH model and to include a quadratic recovery term. First, the model can be regarded as an intuitive extension of the FH model [FitzHugh, 1961]. We reserve the traditional type, which has the same structure of the first component for FH [FitzHugh, 1961] and which retains the time-scale parameter c. Because the 2DHR type model has a slow-fast form, we can adjust the time-scale parameter to generate a Canard-like waveform. It may be difficult to describe phenomena using other types of HR type models without the time-scale parameter. Secondly, the nullcline for the second component of Eq. (1) is quadratic. However, by properly adjusting the parameters $d$ and $b$, the model can be regarded as an FH model. If $b$ is large enough, the $x^2$ component can be omitted, i.e., the nullcline of the second term can be regarded as a linear function. For example, if we take $a = 0$, $b = 6$, $c = 3$ and $d = 10$, the nullcline for the second component is nearly linear near the significant region shown in Fig. (10). Therefore, we can also use the 2DHR type model to mimic neuropysiological signals, except in the FH model. Therefore, it is worthwhile to systematically analyze the 2DHR type model.

Our work provides results for both the characterization of bifurcation behaviors for Eq. (1) and the application of this model for resemblances using simulations under spike-and-reset conditions. The results
obtained from the characterization of bifurcation behaviors support theoretical results for the numerical simulations in Tsuji’s research; the application of the model produced neurophysiological signals using bifurcation mechanisms that differ from those presented in previously published results [Touboul, 2008; Izhikevich, 2004]. Under several specific parameters, Tsuji et al. [2007] numerically demonstrated the bifurcations and dynamics of neuronal behavior. Currently, few studies have attempted to use mathematical analysis for the 2DHR type model with the quadratic recovery term. Because some behaviors can be observed through bifurcation methods, we elaborately evaluated the condition of the bifurcations, especially that of the Bautin and BT bifurcations. Thus, our efforts can provide researchers with a model for assessing the potential behavior of neurons. In addition, we simulated interesting neural activities when the 2DHR type model was employed under spike-and-reset conditions. The physiological features were intensified, as demonstrated by the value obtained for Eq. (1). However, for Eq. (1), not all of the physiological analogs could be simulated under spike-and-reset conditions. The results shown in Table 2 reveal some behaviors of Eq. (1) are similar to those of the Class I and II neurons included in the supercritical Hopf bifurcation, which is not observed for the FH model. The phenomena observed for the model using the quadratic term were more significant than those for the model using the linear recovery term. For the development of models, the bifurcation phenomena and neurophysiological resemblances provided herein may be useful. However, the continuity of the model is disrupted under spike-and-reset conditions. In the future, we will examine the 2DHR type model with time-varying input or feedback control in advance. We would also like to extend the 2DHR type model to a model with a slow variable [González-Miranda, 2003, 2007], and we plan to characterize the behavior of the novel 3DHR model using bifurcation theory and assess its interior phenomena using chaos theory. In addition, we would also like to explore a related equation that adds a diffusion term for the conduction of action potentials along nerves in accordance with the studies of Connor and Stevens [Connor & Stevens, 1971; Connor et al., 1977] to provide more interesting information for modeling research.

Appendix

The second Lyapunov coefficient $\ell_2(0)$ is determined using the following equation:

$$\ell_2(0) = \frac{1}{12\omega_0^2}\text{Re}(\gamma_{32}) + \frac{1}{12\omega_0^2}T^2 + \frac{1}{12\omega_0^2}T^3 + \frac{1}{12\omega_0^2}T^4,$$
where

\[
T_2 = \Im \left( \frac{g_{31}g_{20} - g_{12}g_{30} - g_{11} (3g_{22} + 4g_{11}) - \frac{1}{3} g_{02} (g_{13} + g_{40})}{\omega_0 (b + ic \omega_0)^2} \right),
\]

\[
T_3 = 3 \Im (g_{11}g_{20}) \Im (g_{21}) + \Re \left[ g_{20} \left( \frac{1}{3} g_{02} g_{03} + \frac{g_{11} (3g_{12} - g_{30}) + g_{02} \left( \frac{g_{12} - g_{30}}{3} \right) + g_{11} \times \left( \frac{1}{3} g_{03} g_{02} + g_{02} \left( \frac{5g_{30}}{3} + 3g_{12} \right) - 4g_{11} g_{30} \right) \right) \right],
\]

\[
T_4 = \Im \left[ g_{11} \Im \left( \frac{g_{20}^2 - 3g_{12}g_{11} - 4g_{11}^2}{g_{12}} \right) + \Im (g_{20}g_{11}) \left[ 3 \Re (g_{20}g_{11}) - 2 |g_{02}|^2 \right] \right].
\]

We provide values for \( g_{mn} \) with \( 2 \leq m + n \leq 3 \) using the following calculations:

\[
g_{02} = -\frac{c^2 (ib + c \omega_0) (b m_0 + ic \omega_0 m_0 - 1)}{\omega_0 (b + ic \omega_0)^2},
\]

\[
g_{03} = \frac{c^4 (ib + c \omega_0)}{\omega_0 (b + ic \omega_0)^2}, \quad g_{11} = \frac{c^2 (-b m_0 - ic \omega_0 m_0 + 1)}{\omega_0 (c \omega_0 - ib)},
\]

\[
g_{12} = \frac{c^4}{\omega_0 (c \omega_0 - ib)}, \quad g_{20} = \frac{c^2 (b m_0 + i c \omega_0 m_0 - 1)}{\omega_0 (ib + c \omega_0)},
\]

\[
g_{21} = -\frac{c^4}{c \omega_0^2 + ib \omega_0}, \quad g_{30} = \frac{c^4 (c \omega_0 - ib)}{\omega_0 (ib + c \omega_0)^2},
\]

and use \( g_{mn} = 0 \) for the remain items.

**Proof of Theorem 4.** First, the computation of Case (a) is discussed. Using Theorem 3, the equilibrium is determined to be \((x_0, y_0)\), where \( x_0 = -m_0 \) and \( y_0 = -(2/3 + b/(3c^2)) m_0 \). Let \( \xi = x - x_0 \) and \( \eta = y - y_0 \). Eq. (1) then becomes

\[
f(\xi, \eta) = \frac{b}{c} \xi - c \eta + cm_0 \xi^2 - \frac{c}{3} \xi^3,
\]

\[
g(\xi, \eta) = (-b \eta + (d - 2m_0) \xi + \xi^2) / c.
\]

Its Jacobian matrix is

\[
L_0 = \begin{bmatrix}
\frac{b}{c} & -c \\
\omega_0^2 + b^2/c^2 & -b/c
\end{bmatrix},
\]

where \( \omega_0 \) is defined using the following formula:

\[
\omega_0 = \sqrt{d - b^2/c^2 - 2m_0}.
\]

The eigenvalues of \( L_0 \) are \( \lambda_{1,2} = \pm i \omega_0 \). Let

\[
q_0 = \left( \frac{-ic^2}{ib + c \omega_0}, 1 \right) \in \mathbb{C}^2
\]

be an eigenvector of \( L_0 \) corresponding to the eigenvalue \( i \omega_0 \) and let

\[
p_0 = \left( \frac{1}{2} - \frac{ib}{2c \omega_0} \right) \left( \frac{-b - ic \omega_0}{c^2}, 1 \right) \in \mathbb{C}^2
\]

be an eigenvector of the transposed matrix \( L_0^T \) corresponding to its eigenvalue \(-i \omega_0\). Notably, \( \langle p_0, q_0 \rangle = \langle \overline{p_0}, \overline{q_0} \rangle = 1 \) and \( \langle p_0, q_0 \rangle = \langle \overline{p_0}, q_0 \rangle = 0 \), where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{C}^2 \).

\[
\dot{z} = i \omega_0 z + h(z, \xi),
\]
where \( h(z, \bar{z}) = \sum_{2 \leq m + n \leq 3} \frac{1}{m!n!} g_{mn} z^m \bar{z}^n \). By direct computation, we can obtain

\[
h(z, \bar{z}) = \frac{1}{6\omega_0 (b^2 + c^2\omega_0^2)^2} \left[ c^2(z - b + ic\omega_0) + z(b + ic\omega_0) \right]^2 \times \left[ -3im_0b^2 + i(zc^2 + 3) b + c^2(zb + c\omega_0) - c\omega_0 (zc^2 + 3im_0\omega_0c - 3) \right].
\]

Therefore, using the formula in Ref. [Kuznetsov, 2004], the first Lyapunov coefficient \( \ell_1(0) \) is determined according to the following formula

\[
\text{Re} \left( \frac{\omega_0 g_{21} + ig_{11} g_{20}}{2\omega_0^2} \right) = - \left( c^3 \left( -2bm_0^2 + 2m_0 + \omega_0^2 \right) / (2\omega_0^3 \left( b^2 + c^2\omega_0^2 \right)) \right).
\]

If \( d = 2b - b^2/c^2 \), then \( \ell_1(0) = 0 \). By \( d > b^2/c^2 + 2m_0 \) in Theorem (3), \( m_0 \geq 1/b \).

Using the formula in Ref. [Kuznetsov, 2004], the second Lyapunov coefficient is determined according to the following calculation:

\[
\ell_2(0) = \frac{1}{18\omega_0^5 (b^2 + c^2\omega_0^2)^4} M(b, c),
\]

where the formula of \( \ell_2(0) \) in terms of \( g_{mn} \) with \( 2 \leq m + n \leq 3 \) is shown in the appendix, and \( M(b, c) \) is defined using the following formula:

\[
M(b, c) = c^9 \left[ 3 (b - 2c^2m_0^2) \omega_0^4 + (bm_0 - 1) (12c^2m_0^3 - 41bm_0 + 15) \omega_0^2 + 30m_0 (bm_0 - 1)^3 \right] = -40bc^5 (b - c^2) \left( b^3 - c^2b^2 + 2m_0c^2b - c^3 \right).
\]

In this formula, if \( c^2 \neq b^3/(b^2 - 1) \), then \( \ell_2(0) \neq 0 \).

Next, the computation for Case (b) is considered. Using Theorem 3, there is the equilibrium at \((x_0, y_0)\), where \( x_0 = m_0 \) and \( y_0 = (2/3 + b/(3c^2))m_0 \). Let \( \xi = x - x_0 \) and \( \eta = y - y_0 \). Eq. (1) then becomes

\[
f(\xi, \eta) = \frac{b}{c} \xi - c\eta - cm_0 \frac{\xi^2 - 2\xi^3}{3},
\]

\[
g(\xi, \eta) = (-b\eta + (d + 2m_0)\xi + \xi^2)/c.
\]

Its Jacobian matrix is

\[
L_0 = \begin{bmatrix} \frac{b}{c} & -c \\ (\omega_0^2 + b^2/c^2)/c - b/c \end{bmatrix},
\]

where \( \omega_0 = \sqrt{d - b^2/c^2 + 2m_0} \). The eigenvalues of \( L_0 \) are \( \lambda_{1,2} = \pm i\omega_0 \). Let

\[
q_0 = \left( \frac{ic^2}{ib + c\omega_0}, 1 \right)^T \in \mathbb{C}^2
\]

be an eigenvector of \( L_0 \) corresponding to the eigenvalue \( i\omega_0 \) and let

\[
p_0 = \left( \frac{1}{2} - \frac{ib}{2c\omega_0} \right) \left( - \frac{b - ic\omega_0}{\omega_0 c^2}, 1 \right)^T \in \mathbb{C}^2
\]

be an eigenvector of the transposed matrix \( L_0^T \) corresponding to its eigenvalue \( -i\omega_0 \). Notably, \( \langle p_0, q_0 \rangle = \langle p_0, \bar{q}_0 \rangle = 1 \) and \( \langle p_0, q_0 \rangle = \langle \bar{p}_0, q_0 \rangle = 0 \), where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{C}^2 \).

\[
\dot{z} = i\omega_0 z + h(z, \bar{z}),
\]
where \( h(z, \tau) = \sum_{2 \leq m+n \leq 3} \frac{1}{m!n!} g_{mn} z^m \tau^n \). By direct computation, we can obtain
\[
h(z, \tau) = \frac{1}{6\omega_0 (b^2 + c^2\omega_0^2)^2} \left[ ic^2 (z (b - ic\omega_0) + z (b + ic\omega_0))^2 + (3m_0b^2 + (ze^2 + 3) b + c^2 z (b - ic\omega_0) + ic\omega_0 (izc^2 + 3m_0c\omega_0 - 3i)) \right]
\]
Therefore, using the formula in Ref. [Kuznetsov, 2004], the first Lyapunov coefficient \( \ell_1(0) \) is determined as
\[
\Re (\omega_0 g_{21} + i g_{11} g_{20}) / (2\omega_0^3) = \left( c^5 (2bm^0 + 2m_0 - \omega_0^2) / (2\omega_0^3 (b^2 + c^2\omega_0^2)) \right).
\]
If \( d = 2b - b^2/c^2 \), then \( \ell_1(0) = 0 \). By \( d > b^2/c^2 - 2m_0 \) in Theorem (3), \( m_0 \geq 0 \).

Using the formula in Ref. [Kuznetsov, 2004], the second Lyapunov coefficient is determined according to the following formula:
\[
\ell_2(0) = \frac{1}{18\omega_0^4 (b^2 + c^2\omega_0^2)^4} M(b, c),
\]
where the formula of \( \ell_2(0) \) in terms of \( g_{mn} \) with \( 2 \leq m + n \leq 3 \) is shown in the appendix and \( M(b, c) \) is defined as follows:
\[
M(b, c) = c^3 (3 (b - 2c^2m_0^3) \omega_0^3 + (bm_0 + 1) \left( 12c^2m_0^3 - 41bm_0 - 15 \right) \omega_0^2 + 30m_0 (bm_0 + 1)^3) = -40b^5 (b - c^2) (b^3 - c^2b^2 - 2m_0c^2b - c^2)
\]
In this formula, if \( c^2 \neq b^3/(b^2 - 1) \), then \( \ell_2(0) \neq 0 \). The proof is complete.

We provide the values for \( g_{mn} \) with \( 2 \leq m + n \leq 3 \) using the following equations:
\[
\begin{align*}
g_{02} &= \frac{c^2 (ib + cw_0)(bm_0 + ic\omega_0m_0 + 1)}{\omega_0 (b + ic\omega_0)^2}, \\
g_{03} &= \frac{c^4 (ib + cw_0)}{\omega_0 (b + ic\omega_0)^2}, \quad g_{11} = \frac{c^2 (bm_0 + ic\omega_0m_0 + 1)}{\omega_0 (c\omega_0 - ib)}, \\
g_{12} &= \frac{c^4 (c\omega_0 - ib)}{\omega_0 (c\omega_0 - ib)}, \quad g_{20} = \frac{c^2 (-bm_0 - ic\omega_0m_0 - 1)}{\omega_0 (ib + cw_0)}, \\
g_{21} &= -\frac{c^4 (c\omega_0 - ib)}{\omega_0 (ib + cw_0)^2}, \quad g_{30} = \frac{c^4 (c\omega_0 - ib)}{\omega_0 (ib + cw_0)^2},
\end{align*}
\]
and set \( g_{mn} = 0 \) for the remaining items. \[ \blacksquare \]

**Proof for Case (b) in Theorem 5.** Using computation, we can obtain the following formulas:
\[
\begin{align*}
\alpha_1 &= -\frac{m_0b^2 + (2m_0c^2 + 3) b - 3c^2 (a + dm_0 + 1)}{3c^3}, \\
\alpha_2 &= \frac{b^2 - c^2 (d + 2m_0)}{c^2}, \\
\alpha_3 &= c + bcm_0, \quad \alpha_4 = 2c^2m_0.
\end{align*}
\]
Certain values can also be provided according to the following equations: \( s_{11} = 1/c, \ s_{12} = -(2c^2 + b) m_0/(3c^3), \ s_{13} = -(bc^2m_0^3 - 3c^2m_0^3 + (2b^2 + c^2b - 3c^2d) m_0 - 3ac^2)/(3c^4), \ s_{14} = m_0/c, \ s_{21} = 0, \ s_{22} = (2bm_0 + 1)/(c^2m_0), \ s_{23} = -(c^2 - 3m_0^2c^2 + b + (3b^2 - c^2d) m_0)/(c^3m_0) \) and \( s_{24} = -1. \)
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References


