Abstract—This paper presents an algorithm for transforming closed planar curves into a canonical form, independent of the viewpoint from which the original image of the contour was taken. The transformation that takes the contour to its canonical form is a member of the projective group PGL(2), chosen because PGL(2) contains all possible transformations of a plane curve under central projection onto another plane.

The scheme relies on solving computationally an “isoperimetric” problem in which a transformation is sought which maximises the area of a curve given unit perimeter. In the case that the transformation is restricted to the affine subgroup there is a unique extremising transformation for any piecewise smooth closed curve. Uniqueness holds, almost always, even for curves that are not closed. In the full projective case, isoperimetric normalization is well-defined only for closed curves. The question of uniqueness is more complex: we have found computational counterexamples for which there is more than one extremal transformation.

Numerical algorithms are described and demonstrated both for the affine and the projective cases. Once a canonical curve is obtained, its isoperimetric area can be regarded as an invariant descriptor of shape. Methods for discrimination of nonconvex shapes are already known. Our invariant descriptor is the first example, to our knowledge, of practical discrimination, up to projectivity, of convex, closed curves.

Index Terms—Object recognition, perspective projection, isoperimetric problem, normalization.

I. INTRODUCTION

Shapes measured in images depend not only on object shape, but on position, orientation and the intrinsic parameters of the camera. This paper addresses the problem of shape matching and discrimination for objects containing planar regions. The change in shape of a planar bounding contour under a change of view is tightly constrained: the contour and its image may differ by at most a perspective [12], [13].

Perspectivities are members of the projective group PGL(2) but do not themselves form a group. However it will be desirable to work within a group because the closure property makes iterative algorithms feasible—a sequence of transformations from the group must be equivalent to a certain single transformation. Hence the full projective group is used, being the smallest available group that contains the perspectivities. If plane curves are to be recognised successfully then the method used should be able to cope with the projective transformation induced by the imaging process.

There exist in the literature a wide variety of methods for comparing curves with candidate models. If an object plane has sufficiently small slant, given its angular dimensions in the image, then the “affine approximation” holds [7]. The assumption that the candidate contour differs from its target by at most an affine transformation reduces the search space and simplifies the matching procedure [9]. The search is then over the parameters of transformations in the affine group. It is often convenient to subdivide the affine group into the Euclidean group of rotations and translations (the rigid transformations) and the group of anisotropic scalings which affect contour shape and arise in the imaging process as the effect of varying the orientation of a contour and its distance from the camera [12]. Given two image contours, they may be compared by placing both in canonical frames. The shape of a contour in its canonical frame is independent of the initial viewing conditions (assuming that the affine approximation holds in this case). Depending on the method used to define the canonical frame the contour may still differ from its target by a Euclidean transformation [3], [4]. Canonical frames defined through multiple point correspondences will not in general have a rotational degree of freedom associated with them, unlike those defined through extremising some measure of the curve over the affine group, as done by Blake and Marinos [2]. The matching process between such curves and their targets may be performed via comparison of moments invariant to the Euclidean group or compared in a more direct way [8], [6].

In practice the affine approximation may not be adequate and the full projective group is then needed. Much work has been done in attempting to develop “invariant descriptors” for plane curves. Invariant descriptors are numbers which measure a shape property of a curve and are invariant to change of viewpoint. They are solely a function of the shape of the original curve and may, therefore, be used for direct access to a precompiled model-base as a means of object recognition. For instance, shape approximations in terms of algebraic curves [5], [11] can be used with algebraic invariants. Probably the most successful approach is one which is free of the need to approximate the original curve (approximation is undesirable because it introduces its own projective variability). It operates by obtaining four invariant points on a plane contour which can be used directly to define a transformation to a canonical frame [10]. The points must be invariant in the sense of remaining fixed with respect to the contour under arbitrary projectivities. They may be bitangent points, points of inflection or tangent points on the contour from either of the first two. Convex curves have no points of inflection and hence no self-bitangent points.
points. It is therefore hard to find four invariant points. An ellipse fitting technique which does this has been demonstrated [4] but appears to be combinatorially complex and does not in general obtain a unique fitted ellipse.

A. The Proposed Algorithm

This paper describes a transformation that maps a closed convex contour to a canonical image under projectivity. It is complementary to the 4-point canonical frame algorithm [10] in that it is most effective for convex curves, precisely the condition under which the 4-point algorithm fails. It operates by solving computationally an “isoperimetric” problem. The isoperimetric problem is the problem of finding a closed curve of maximal area, from some family of closed curves, given unit perimeter. For the family of smooth, closed, simple, planar curves, the unique solution is a circle [14]. In the context of normalization of a particular curve x(s) within a group G of transformations, the isoperimetric problem is restricted to the family

\[ Gx \equiv \{ gx(s), g \in G \} \]

of transformed curves. We are interested in the case that G is the projective group, and in the special case that it is the affine group.

Each curve, therefore, generates its own isoperimetric problem and it is desired to solve the problem computationally. Our algorithm for doing this minimizes the isoperimetric measure

\[ M = P^2/A - 4\pi. \]  

The isoperimetric measure\(^1\) is invariant to rotation, translation and isotropic scaling of the contour, transformations which do not change the “shape” of the contour. This is entirely to be expected as it is shape, rather than size or disposition, which is to be normalized.

Once normalization is achieved, the minimum value of the isoperimetric measure M itself constitutes an invariant shape descriptor, and we give a few examples now. For instance, in accordance with the classical isoperimetric theory, the circle is the unique figure for which M achieves its lower bound M = 0. Hence, it is already normalized with respect to projectivity (and indeed with respect to any family of diffeomorphisms).

A regular polygon with \( n \) sides is also already projectively normalized and has isoperimetric measure

\[ M = 2n\cot(\pi/n) - 4\pi. \]  

so that the isoperimetric measures for the first four regular polygons with \( n = 3, 4, 5, 6 \) are 8.21824, 3.43363, 1.96448, and 1.29004, approaching 0 as the polygons approximate a circle more and more closely.

II. MINIMIZATION UNDER AFFINE TRANSFORMATION

In this section, the affine group is considered, before going on, in the next section, to treat the full problem with projectivities. An affine transformation has six degrees of freedom, namely:

- two translational degrees of freedom,
- one isotropic scaling degree of freedom,
- one rotational degree of freedom,
- two anisotropic scaling degrees of freedom.

The first four of these do not change the “shape” of the contour. The first three can be fixed in a standard way by isotropically scaling the area of the figure to unity and translating the center of area over the origin. The last two degrees of freedom do affect the shape of a contour and can be constrained by minimizing M. The minimization, in the affine case, thus occurs in a two-parameter space. After setting up notation an extremality condition will be derived for the minimization of M. From this, a minimization algorithm is derived which is guaranteed to converge to a unique solution and which is then demonstrated on several closed contours.

A. Notation

Notation is introduced here for parameterised plane curves and for affine transformations. The planar curve to be normalized is \( x \), with general point \( x(s) \), and parameterised by arclength \( s \); \( x'(s) \) is the same curve after some transformation has been applied to it. The linear part of an affine transformation is denoted by the \( 2 \times 2 \) matrix \( U \), giving \( x'(s) = Ux(s) \). The remaining part of the affine transformation, the translation, has no effect on shape and on M in particular, and so does not call for a notation here.

Unit tangents vectors to the plane curve \( x(s) \) are denoted \( x_\alpha \equiv dx/ds \). In the case of the transformed curve \( x'(s) \), \( s \) is, in general, no longer an arclength for the curve \( x'(s) \), so a distinct notation

\[ t'(s) \equiv x'(s)/|x'(s)| \]  

is introduced for its unit tangent. The transformed arclength parameter \( s' \) is defined differentially by

\[ ds' = |x'|ds. \]  

The perimeter of the transformed curve \( x' \) is denoted

\[ P = \oint |x'_\alpha(s)|ds, \]  

and its area is

\[ A = \oint dA, \quad \text{where} \quad dA = \frac{1}{2}[x'(s) \times x'_\alpha(s)]ds. \]  

B. The Extremality Condition for Affine Minimization

The condition

\[ \frac{\partial M}{\partial U} = 0. \]  

for the extremality of the isoperimetric measure M with respect to \( U \) is expressed now in terms of moments of the
curve \( x'(s) \). This proves to be a suitable form from which an algorithm can be derived for the optimisation of \( M \). A similar analysis was done by Blake and Marinos [2] for polygons, and is adapted for smooth curves here.

Given that \( M = P^2/A - 4\pi \), extremising \( M \) is equivalent to the extremisation of \( A \) while holding the perimeter \( P \) constant. This constraint is imposed by means of a Lagrange multiplier \( \mu \) as follows:

\[
\frac{\partial}{\partial U} (\mu P - A) = 0,
\]

where \( \frac{\partial}{\partial U} \) denotes (tensor) differentiation by the elements of \( U \). Substituting (5) and (6) into (8) gives

\[
\frac{\partial}{\partial U} \left[ \mu \int |Ux| ds - \int |Ux \times Ux| ds \right] = 0.
\]

Now

\[
A = \det U \int |x \times x| ds,
\]

and

\[
\frac{\partial}{\partial U} \det U = (\det U) U^{-T},
\]

so that evaluating the derivative in (9) gives

\[
\mu \int \frac{1}{|Ux|} Ux_x^T ds = (\det U) U^{-T} \int |x \times x| ds, \tag{12}
\]

and, post-multiplying both sides by \( U^T \),

\[
\mu \int \frac{1}{|Ux|} Ux_x(Ux_x)^T ds = (\det U) I \int |x \times x| ds, \tag{13}
\]

where \( I \) is the \( 2 \times 2 \) identity matrix. Now, writing \( x'_x = Ux_s \), gives

\[
\mu \int \frac{1}{|x'|} x'_s x'_s^T ds = \left[ (\det U) \int |x \times x| ds \right] I. \tag{14}
\]

Note that the left-hand side of this equation is simply \( \mu J' \) where \( J' \) is the nonlinear second moment defined by

\[
J' = J[x'] \equiv \frac{1}{P} \int t' t'^T ds', \tag{15}
\]

therefore and equivalent extremality condition is that \( J' \) be an isotropic tensor—a multiple of \( I \). Since, from (15) and using the fact that \( |t'| = 1 \),

\[
\text{tr}(J') = \frac{1}{P} \int ds' = 1, \tag{16}
\]

the condition for extremality is simply that

\[
J' = \frac{1}{2} J. \tag{17}
\]

Specifying a minimum constrains two degrees of freedom of the affine transformation, corresponding to a uniform stretching of the contour in some fixed direction. Proofs of existence and uniqueness of the minimum, in this affine case, are straightforward adaptations for smooth curves of the proofs [2, Theorems 4.1 and 4.3] for polygons. Existence is guaranteed and so is uniqueness, provided the curve \( x \) is not simply a straight line segment.

C. Algorithm Description

The algorithm for minimizing \( M \) is based on the extremum condition that \( J \) should be an isotropic tensor. The intuitive explanation of its operation is that successive linear transformations are applied to the curve to make its second moment \( J \) closer and closer to isotropic. The degree to which \( J \) is anisotropic is measured in terms of its eigenvalues \( \lambda_1, \mu_2 \) which are equal at isotropy. A "correcting" transformation \( U \) is computed in terms of those eigenvalues and of the eigenvectors \( e, f \). Owing to the non-linear nature of the tensor functional \( J[x'] \) it turns out not to be possible to compute an exact correction in a single step. However a rapidly converging iterative algorithm is possible.

The iterative algorithm follows.

1) Given the data \( x(s) \), compute

\[
J_0 = \int x_x(s)(x_x(s))^T ds
\]

and set \( U_0 = I \), \( n = 1 \).

2) Compute the eigenvalues \( \lambda_{n-1}, \mu_{n-1} \) and the eigenvectors \( e_{n-1}, f_{n-1} \) of \( J'_{n-1} \).

3) Compute an incremental linear transformation

\[
\delta U_n = \lambda_{n-1}(f_{n-1})^T + \mu_{n-1}(e_{n-1})^T \]

4) Compute

\[
U_n = \delta U_n U_{n-1},
\]

the current estimate of the normalizing transformation.

5) Compute

\[
J'_n = J[U_n x]
\]

the second moment of the current estimate of the normalized curve.

Repeat steps 2 through 5 until \( U_n \) converges according to some suitable criterion.

Details of convergence properties are given elsewhere [2], where asymptotic convergence is proved (Theorem 5.5) for the polygonal case and easily adapted for the case of smooth curves.

D. Results

The affine normalization scheme is demonstrated here, first on an irregular computer-generated quadrilateral, then on the extremal boundary of a fifty-pence coin, extracted from an image. Figure 1(a) shows an irregular four sided figure. Figure 1(b) shows the same figure after affine normalization has been enforced. Note that mirror symmetry emerges in the normalized figure, in keeping with the result of Brady and Yuille [3] that minimizing \( M \) tends to recover mirror symmetry.

Figure 2 shows a CCD camera image of three coins and a gasket. The bounding contours (Fig. 3) of these objects were extracted using a standard edge detector and edge following routine. Figure 4 shows one coin and its normalized outline. Affine normalization appears to have restored the rotational symmetry of the coin. In fact, the restoration of symmetry is
rather than six so that the projective minimization of the isoperimetric measure $M$ takes place not in a two-parameter space as in the affine case, but a four-parameter space. The extra two degrees of freedom can be written as a two-component co-vector $v$, in a (non-affine) transformation of the form \[ (1.2) \]

It is straightforward to show that any projectivity can be expressed as a transformation of the form in (18) either followed by or preceded by an affine transformation. For the present purposes, the affine part is restricted to a linear transformation $U$, as in the previous section. Switching the order of the affine and projective parts is allowed by the simple relation

\[ V[U^T v] U = U[V[v]]. \] (19)

This means that the normalization problem itself can be decomposed into projective and affine parts. The values of $v$ can be adjusted to minimize the isoperimetric measure $M$. As $v$ varies, $U$ is tracked continuously, maintaining the transformation that affinely normalizes the curve $V[v]x$.

The goal of this section is to derive a scheme to determine the projective parameter $v$ and thus define the transformation to the canonical frame. An expression for the gradient of $M$ is derived and interpreted geometrically in terms of curve centroids. The gradient is then used to define an iterative algorithm, an extension of the affine minimization technique of the previous section.

**Derivatives of the isoperimetric measure**

As a preparation for obtaining the gradient of $M$ with respect to $v$, we derive some useful derivatives of the transformed curve $x'(s)$ and its perimeter and area. First, the projectivity is

\[ x' = U[V[v]]x = \frac{Ux}{1 + v \cdot x}. \] (20)

Its derivative with respect to the projective parameter $v$ is a tensor, easily shown to be:

\[ \frac{\partial}{\partial v} x' = -x'x'^T U^{-T}. \] (21)

Also, differentiating (21) with respect to $s$,

\[ \frac{\partial x'}{\partial v} = -\left(x'x'^T s + x's x'^T\right) U^{-T}. \] (22)

Now, from (5), the derivative of the perimeter is

\[ \frac{\partial P}{\partial v} = \left(\int \frac{1}{|x'|} x' s \cdot \frac{\partial x'}{\partial v} ds\right). \] (23)

which, using (3), (4), (21) and (22) gives

\[ \frac{\partial P}{\partial v} = -\left(\int (x'^T + t'(x'^T \cdot t')) ds\right) U^{-T}. \] (24)
This derivative of perimeter with respect to the projective parameter \( v \) has a concise interpretation in terms of moments and centroids. It is simply

\[
\frac{\partial P}{\partial v} = -P \left[ J' \frac{\partial}{\partial v} + J' \frac{\partial}{\partial v} \right] U^{-T}
\]

where \( J' \) is the nonlinear second moment defined in the previous section, \( \mathcal{X}_p \) is the centroid of the curve \( x'(s) \) and \( \mathcal{X}_c \) is a nonlinear centroid defined by

\[
\mathcal{X}_c = \frac{1}{P} \int \left( t' - x' \right) t' ds'.
\]

It can be shown that this definition gives a well-defined centroid in the sense of being covariant to translation and invariant to rotation of the curve itself.

Next, an expression is derived for the derivative of the area \( A \) bounded by the curve with respect to \( v \):

\[
\frac{\partial A}{\partial v} = \frac{1}{2} \frac{\partial}{\partial v} \int |x'(s) \times x'_s(s)| ds,
\]

which, using (21) and (22), gives

\[
\frac{\partial A}{\partial v} = -\left[ \int x'^T x'(s) x'_s(s) ds \right] U^{-T} = -3A \mathcal{X}'^T U^{-T},
\]

where \( \mathcal{X}' \) is yet another centroid, this time the centroid of the area bounded by the curve.

Now the derivative \( \partial M/\partial v \) can be computed:

\[
\frac{\partial M}{\partial v} = 2P \frac{\partial P}{\partial v} - P^2 \frac{\partial A}{A^2} \frac{\partial v}{\partial v}
\]

and, assuming that \( U \) is adjusted continuously to maintain affine extremity as the projective parameter \( v \) varies so that \( J' = \frac{1}{2}I \) as in (17),

\[
\frac{\partial M}{\partial v} = 2M \left[ 3 \mathcal{X}' - \mathcal{X}'_p - \frac{1}{2} \mathcal{X}_p \right] U^{-T}.
\]

The term in square brackets is a strikingly simple sum of centroids. Note that the coefficients of the centroids in (30) sum to zero; this is to be expected since a circle, for example, centred at \( e \), has equal centroids \( \mathcal{X}_p = \mathcal{X}_c = \mathcal{X}_c' = e \) and hence \( \partial M/\partial v = 0 \) — extremity as expected. Such a coincidence of centroids also occurs for any curve with rotational symmetry of order 4 or higher (regular polygons for instance, starting with the square) and so such figures are already projectively normalized. In general, it is not necessary for extremity that all three centroids coincide, just that their weighted vector sum, with weights as in (30), be zero.

The gradient (30) can now be used in an iterative gradient descent scheme to minimize \( M \). Such a scheme is described in Section III-D.

**B. Singular Projectivities**

In order to guarantee existence of a normalizing projectivity, the projective parameter \( v \) can be restricted to a compact set that is constructed naturally from the curve \( x(s) \). This set, denoted \( V(x) \), is a convex set in the \( v \)-plane consisting of those \( v \) for which the projectivity \( V[v] \), restricted to the curve \( x(s) \), is non-singular. Now singularity in the projectivity occurs when \( 1 + v \cdot x(s) = 0 \) so the non-singular set of projectivities with respect to a particular curve is given by

\[
\{ v : \forall s, 1 + v \cdot x(s) \neq 0 \}.
\]

Taking the closure of the positive subset of this nonsingular set defines

\[
V(x) = \{ v : \forall s, 1 + v \cdot x(s) \geq 0 \}
\]

which is convex because it is an intersection of half-planes. If the curve \( x \) is bounded, \( V(x) \) must clearly contain the identity transformation \( v = 0 \), the origin in \( v \)-space. It is straightforward to show that if the curve also encircles the origin then the set \( V(x) \) is also bounded and therefore compact. Then the function \( M(U(v), v) \), where \( U(v) \) tracks to maintain affine extremity, achieves a minimum on the set.

We have shown that a minimum of \( M \) must exist in \( V(x) \), provided that the curve \( x \) encircles the origin. Since \( M \) is invariant to translation, any given curve \( x \) can be translated so that it encircles the origin, without affecting its normalized shape. This completes the proof that at least one projective normalized form exists for any smooth closed curve. If the curve is also bounded, then the origin \( v = 0 \) is inside the set \( V(x) \) and can therefore be used as a starting point for an iterative minimization algorithm.

**C. Uniqueness for the Projective Minimization**

Existence of normalizing projectivities has been established, but the question of uniqueness is yet to be considered. We have been unable, so far, to establish mathematically conditions under which the projective minimization is unique. However, counterexamples have been generated for which more than one minimum exists within the set \( V(x) \). For a convex curve, when the function \( M(U(v), v) \) is plotted within the set \( V(x) \) as in Fig. 5, it appears to be convex except near the set boundaries. (Note that the set itself is convex, in accordance with theory.) Indeed, Astrom [1] has recently claimed that, for a large class of curves, there are minima on the boundary of the set and that, at these minima, \( M = 0 \). Clearly such minima would not be of practical use because they imply projective equivalence to a circle. There appears however to be a unique minimum in the interior of the set for examples of convex curves that we have seen, and they have proved to be practically useful for normalization. In practice, by starting with an affinely normalized shape which is normally well away from the set boundary, our gradient descent algorithm (see below) finds the minimum in the interior of the set.

In the case of nonconvex contours the minimization problem is more clearly nonconvex. The set \( V(x) \) continues, of course, to be convex, but the function \( M(U(v), v) \) can have several minima, all of which appear to be on, or very close to, the boundary of the set. Figure 6 shows four contours with 1, 2, 3, 4 concavities respectively. Contour plots of the function \( M(U(v), v) \) for the four cases are shown in Fig. 7. The

\[\text{it ought also to be the case that set of projectivities generated by } V(x) \text{ is invariant to translations of the curve. It is straightforward though laborious to prove this.}\]
Fig. 5. The convex curve in a) has isoperimetric measure $M(U(v), v)$ as a function of $v$, shown as a contour plot in (b). Note that the contours are displayed within the convex non-singular set $V(x)$ for the particular curve. It appears to be a convex function (the contours are convex) except near the set boundary. We conjecture that convexity of the function is associated with convexity of the curve.

number of minima of the function is 1, 2, 3, 4, respectively, equal, at least in these examples, to the number of concavities in the corresponding curve. This is intuitively reasonable. Each minimum corresponds to a projectivity which shrinks all but one convex protrusion of the curve to a short segment. Concavities tend to increase $M$ because they increase perimeter while decreasing area. Shrinking concavities, by applying a suitable projectivity, therefore tends to minimize $M$.

D. An Iterative Algorithm for Projective Normalization

The algorithm is an iterated series of Newton-Raphson steps in the variable $v$, alternating with the affine maximisation algorithm earlier, as follows.

1) Given a closed curve, define the function $M(U, v)$ for that curve as in (1), (5), and (6), and its derivative as in (30).
2) Set $U_0 = I$, $v_0 = 0$, $n=1$.
3) Apply the affine maximisation scheme in Section II-C to find $U_n$ such that $M_U(U_n, v_{n-1}) = 0$.
4) Evaluate $w$, the unit gradient of $M$: $w_n = M_v/M_v$ at $U = U_n$, $v = v_{n-1}$.
5) Set

$$
\delta v = \frac{-\epsilon M_v(U_n, v_{n-1})}{M_v(U_n, v_{n-1}) - M_v(U_n, v_{n-1} + \epsilon w_n)}.
$$

where $\epsilon$ is a suitably small number used for the numerical evaluation of derivatives.
6) Set $v_n = v_{n-1} + \delta v_n$.
7) Set $M_n = M(U_n, v_n)$.
8) Repeat steps 4 through 7 until $|M_{n+1} - M_n| < \delta$.

The algorithm functions by finding a local numerical approximation to the second derivative of $M$ and then using this to estimate the required step size $v_n$. The parameter $\epsilon$ is the size of a small local step in $v$ used to find the numerical approximation to the second derivative of $M$; a value of $\epsilon = 10^{-5}$ was found satisfactory in our experiments. From Fig. 8, a log-linear plot of $M_n$, it can be seen that the algorithm converged within five iterations. From the graph it is apparent that convergence is geometric and sufficiently fast that, given the relative error bound $\delta$, the absolute error in $M_n$ is also bounded approximately by $\delta$.

Details of the iterative backprojection algorithm used to enforce affine minimization are given in [2]. It is worth pointing out that it is possible to compute symbolically an expression for the second derivative of $M$ but that involves a large number of terms and is therefore expensive to evaluate. Therefore it is preferred to compute the second derivative numerically, as in the denominator of step 5.

E. Results of Projective Normalization

In this section, the full projective minimization scheme is demonstrated as a means of transforming convex contours into their canonical frames. Then the measure $M$ for the various contours is used to discriminate between contours.

As an illustration, the full projective minimization scheme is applied to the irregular quadrilateral in Fig. 9. The square is the quadrilateral with minimal $M$, and since any given convex quadrilateral can be transformed by some (nonsingular) projectivity into a square, the normalized form of any convex quadrilateral should be a square. This is what the figure shows in practice.

The full minimization scheme is demonstrated on the boundary of a fifty pence piece in Fig. 10. Then three views of the objects in Fig. 2 were taken from different vantage points. The extremal boundaries were extracted and the resulting contours were transformed into their canonical frames. Table I shows the starting and finishing values of the isoperimetric measure.
Fig. 7. Isoperimetric measure $M(U(v), v)$ as a function of $v$, for each of the series of nonconvex test curves in figure 6. In each case $M$ is nonconvex. It appears that there is one local minimum of $M$ for each convex lobe of the curve.

\[ \log_{10}(M_n - M_{\infty}) \]

Fig. 8. This graph shows the convergence of the isoperimetric measure $M_n$, in successive iterations, on a log-linear plot, for data consisting of the contour of Fig. 14. The algorithm converges within five iterations, after which fluctuations in $M_n$ are dominated by rounding error.

Fig. 9. (a) An irregular four sided convex figure. (b) Affine normalization performed on (a). (c) Projective normalization performed on (a). All convex four-sided figures become squares in their canonical frames.

$M$ for each of the three views of the four objects in Fig. 2. The final values are displayed in graphical form in Fig. 11. The isoperimetric measure $M$ can be used as a discriminant between qualitatively similar curves.

Projective normalization has been demonstrated now for some convex curves. It could also be used to transform nonconvex curves into their canonical frames. One way to do this is to fit a convex hull to the curve and then normalize the convex hull. This is illustrated on the outline of a leaf, shown Fig. 12. The bounding curve of the leaf was extracted as before by using an edge detector and edge following routine, and a convex hull was then fitted as shown in Fig. 13. Three views of this leaf were taken and convex hulls fitted to their bounding

<table>
<thead>
<tr>
<th>Object</th>
<th>View</th>
<th>Initial $M$</th>
<th>Final $M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50 pence</td>
<td>1</td>
<td>1.855</td>
<td>0.0832</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.018</td>
<td>0.0816</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.111</td>
<td>0.0993</td>
</tr>
<tr>
<td>20 pence</td>
<td>1</td>
<td>2.094</td>
<td>0.0278</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.852</td>
<td>0.0302</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.043</td>
<td>0.0380</td>
</tr>
<tr>
<td>2 pence</td>
<td>1</td>
<td>1.887</td>
<td>0.0082</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.186</td>
<td>0.0102</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.043</td>
<td>0.0031</td>
</tr>
<tr>
<td>Widget</td>
<td>1</td>
<td>8.299</td>
<td>0.3535</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.323</td>
<td>0.3385</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3.048</td>
<td>0.3316</td>
</tr>
</tbody>
</table>

Fig. 10. The extremal boundary of the fifty pence in Fig. 2(b) together with its affinely normalized image and the full projectively normalized image. The isoperimetric measure $M$ for each of the three outlines is 0.7558, 0.0835, and 0.0832.

Fig. 11. Graphical display of the final values of isoperimetric measure $M$ for three views of the objects in Fig. 2.
IV. CONCLUSION

This paper sets out a framework for normalization of planar shapes in images by solving a constrained isoperimetric problem. In the affine case, the problem is well understood, uniqueness is established and a provably correct algorithm is available. Convex, nonconvex and even nonclosed or disconnected shapes can satisfactorily be normalized.

The more general projective case is mathematically less straightforward. The most interesting situation involves convex curves that cannot be readily normalized by other means. We have demonstrated robust isoperimetric normalization of convex curves. Furthermore, the isoperimetric measure itself, after normalization, is a powerful shape discriminant. Of course, other shape descriptors—higher moments for instance—when applied to the normalized shape, constitute an inexhaustible supply of invariant shape measures.

ACKNOWLEDGMENT

We gratefully acknowledge discussions with A. Zisserman, C. Rothwell, and M. Brady.

REFERENCES

D. Sinclair is a native of Scotland. He received the B.Sc. degree in mathematical physics with honors from Edinburgh University in 1988. He worked for the British Ministry of Defence from 1988 to 1989. He studied for his D.Phil degree in the Department of Engineering at the University of Oxford from 1989 to 1993. He was then awarded a European Community Human Capital Mobility Grant to perform research into active viewpoint selection at the Laboratoire d'Informatique et d'Intelligence Artificielle in the Institut National Polytechnique de Grenoble.

He is currently working as a research assistant on the Vision as Process project in Aalborg University. He has published papers on vision. His research interests include active structure recovery, image representation and semantic dynamic scene description.

A. Blake was born in England in 1956. He received the B.A. degree in mathematics and electrical sciences from Trinity College, Cambridge, in 1977.

After a year as a Kennedy Scholar at MIT and two years in the electronics industry, he studied for a doctorate at the University of Edinburgh which was awarded in 1983. Until 1987, he was a lecturer at the University of Edinburgh and a Royal Society Research Fellow. He is currently on the faculty of the Robotics Research Group in the University of Oxford. His research interests are in Robot Vision and also in human visual psychophysics. He has published a number of papers in vision, a book with A. Zisserman, Visual Reconstruction (MIT Press), and recently edited Active Vision with A. Yuille (MIT Press). He currently serves on the program committees of the International and European Conferences on Computer Vision and on the editorial boards of Vision Research, Image and Vision Computing, and International Journal of Computer Vision.