

# Existence of Equilibria in Games with Arbitrary Strategy Spaces and Payoffs: A Full Characterization

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## Abstract

This paper provides a complete solution to the question of the existence of equilibria in games with general strategy spaces that may be discrete, continuum, non-compact or non-convex and payoff functions that may be discontinuous or do not have any form of quasi-concavity. We establish a single condition, called *recursive diagonal transfer continuity*, which is both necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs. As such, our result strictly generalizes all the existing results on the existence of pure strategy Nash equilibrium. Moreover, recursive diagonal transfer continuity also permits full characterization of symmetric, mixed strategy, and Bayesian Nash equilibria in games with general strategy spaces and payoffs. The approach and main result developed in the paper can also allow us to ascertain the existence of equilibria in important classes of economic games. As an illustration, we show how they can be employed to fully characterize the existence of competitive equilibrium for economies with excess demand functions. The method of proof adopted to obtain our main result is also new and elementary — a non-fixed-point-theorem approach.

*Keywords:* existence of pure, symmetric, mixed strategy, and Bayesian Nash equilibrium; recursive diagonal transfer continuity; discontinuity; non-quasiconcavity; nonconvexity.

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# 1 Introduction

The notion of Nash equilibrium introduced by Nash (1950, 1951) is probably one of the most important solution concepts in game theory that has wide applications in almost all areas in economics. The early theorems of Nash (1950, 1951), Debreu (1952), and Fan (1953) reveal that games possess a pure strategy Nash equilibrium if (1) the strategy spaces are nonempty, convex and compact, and (2) players have continuous and quasiconcave payoff functions. The theorems say nothing about equilibrium in games with discontinuous and/or non-quasiconcave payoffs. Glicksberg (1952) shows that games with compact Hausdorff strategy spaces and continuous payoffs possess mixed strategy Nash equilibria.

Many economic models such as the classic price competition models of Bertrand (1883), Hotelling (1929), and auction models such as the one in Milgrom and Weber (1982), patent race models of Fudenberg et al. (1983), etc., however, frequently exhibit discontinuities or non-quasiconcavity in payoffs. Consequently, the standard theorems cannot be applied to establishing the existence of a pure or mixed strategy Nash equilibrium. While in many of these games equilibria can be constructed, there are other models in which this is not the case. Indeed, even for multi-dimensional auction models, the existence of an equilibrium in this type of games has been at issue since payoffs are not continuous, see Jackson (2005). Also, many economic models do not have convex strategy spaces, and payoff functions under consideration may not be quasiconcave or even do not have any form of quasi-concavity.

Accordingly, economists have been struggling to seek weaker conditions that can still guarantee the existence of an equilibrium. So far, two approaches have been adopted to weaken the continuity and/or quasiconcavity. The first approach is to relax the quasi-concavity of payoffs and convexity of strategy spaces. In a homogeneous product setting McManus (1964) and Roberts and Sonnenschein (1976) show the existence of a symmetric Cournot equilibrium allowing for a general downward sloping demand when there are  $n$  identical firms with convex costs. Nishimura and Friedman (1981) assume, in addition to some conditions on best reply correspondences, that payoff functions are continuous. Topkis (1979), Vives (1990), and Milgrom and Roberts (1990) establish the existence in games where payoffs are upper semi-continuous and satisfy certain monotonicity properties.

The central idea of this approach is based upon lattice-theoretical concepts, and at its heart lies Tarski's (1955) fixed point theorem. An advantage of Vives' approach is that it does not require the convexity of strategy sets. Payoffs need not be quasiconcave in all of the papers mentioned above, and additionally payoffs need not be continuous in some of them. The key property is that best-replies are increasing in the opponents' strategies, which guarantees the existence of pure

strategy Nash equilibrium. However, the lattice-theoretic approach requires payoff functions must be upper semi-continuous in one's own strategy in order to guarantee the existence of best replies. This assumption fails to hold in virtually all auctions, as well as in the classic games of Bertrand and Hotelling.

The second approach is the topological approach proposed to weaken the continuity of payoff functions. Dasgupta and Maskin (1986) are the first to establish an existence theorem valid for games with discontinuous payoff functions. Their results reveal that such games possess a pure strategy equilibrium, provided (1) the strategy spaces are nonempty, convex and compact, and (2) players have payoff functions that are quasiconcave, upper semi-continuous, and graph continuous. They also investigate the existence of mixed strategy Nash equilibrium in games with discontinuous payoffs.

Simon (1987) obtains the existence of mixed strategy Nash equilibria in discontinuous games through introducing the notion of reciprocal upper semi-continuity (under the name of "complementary discontinuities") and thus strictly generalizes the result of Dasgupta and Maskin (1986). Reciprocal upper semi-continuity requires that some player's payoff jump up whenever some other player's payoff jumps down, which generalizes the condition that the sum of the players' payoffs is upper semicontinuous.

Simon and Zame (1990) establish the existence of a Nash equilibrium in mixed strategies with an endogenous sharing rule. While in some settings involving discontinuities this approach is remarkably helpful, in others it is less so. In an auction design environment where discontinuities are sometimes deliberately introduced, the participants must be presented with a game that fully describes the strategies and payoffs, since one cannot leave some of the payoffs unspecified or somehow endogenously determined. In addition, this method is only useful in establishing the existence of a mixed, as opposed to pure, strategy equilibrium.

Baye, Tian, and Zhou (1993) investigate the existence of pure strategy Nash equilibrium and dominant-strategy equilibrium by weakening both continuity and quasi-concavity of payoffs. It is shown that diagonal transfer quasi-concavity is necessary, and further, under diagonal transfer continuity and compactness, sufficient for the existence of pure strategy Nash equilibrium. Both diagonal transfer quasi-concavity and diagonal transfer continuity are very weak notions of quasi-concavity and continuity, which adopt a basic idea of transferring a set of strategy profile(s) to another set of strategy profile(s).

Reny (1999) establishes the existence of Nash equilibrium in compact and quasiconcave games that are better-reply secure, which is also a weak notion of continuity. Reny (1999) shows that better-reply security can be imposed separately as reciprocal upper semi-continuity introduced by

Simon (1987) and payoff security. Bagh and Jofre (2006) further weaken reciprocal upper semi-continuity to weak reciprocal upper semi-continuity and show that it, together with payoff security, implies better-reply security. Both better-reply security and payoff security use a similar idea of transferring a (non-equilibrium) strategy to a securing strategy, and therefore they are also in the form of transfer continuity.

Nessah and Tian (2008) introduce an even weaker form of continuity, called *weak transfer quasi-continuity*, which is weaker than diagonal transfer continuity in Baye, Tian, and Zhou (1993) and better-reply security in Reny (1999), and holds in a large class of discontinuous games. They show that weak transfer quasi-continuity, together with the compactness of strategy spaces and quasiconcavity or (strong/weak) diagonal transfer quasiconcavity of payoffs, permits the existence of pure strategy Nash equilibrium so that it strictly generalizes the results of Baye, Tian, and Zhou (1993) and Reny (1999). They provide some sufficient conditions for weak transfer quasi-continuity by introducing notions of weak transfer continuity, weak transfer upper continuity and weak transfer lower continuity. These conditions are satisfied in many economic games and are often quite simple to check. They also study the existence of mixed strategy Nash equilibria in discontinuous and nonconvex games.

Recently, Barelli and Soza (2009) further significantly weaken the continuity and quasiconcavity conditions. They generalize most existing results and establish the existence of pure strategy Nash equilibria in discontinuous quasiconcave games and qualitative convex games.

However, all the existing results only give sufficient conditions for the existence of equilibrium, and no full characterization has been given yet in the literature.<sup>1</sup> The existing results use two separated conditions: continuity and quasi-concavity/monotonicity. Neither single unified condition nor full characterization approach has been given. A question is then whether or not there exists a single unified condition that can be used to prove the existence of (pure/mixed) Nash equilibrium in games with arbitrary strategy spaces and payoff functions, and if so, what the weakest condition is. This paper provides a complete answer to these questions by giving a necessary and sufficient condition for the existence of equilibrium in games with arbitrary strategy spaces and payoffs.

This paper fully characterizes the existence of pure strategy Nash equilibrium in games with general topological strategy spaces that may be discrete, continuum, non-compact or non-convex and payoff functions that may be discontinuous or do not have any form of quasi-concavity. It is shown that the condition, *recursive (or called sequential) diagonal transfer continuity* introduced

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<sup>1</sup>In the mechanism design literature, a lot of studies on full characterizations of Nash implementation of a social choice correspondence have been given such as those in Maskin (1999), Moore and Repullo (1990), Dutta and Sen (1991), etc. Also, Rahman (2008) recently provides a full characterization of correlated equilibrium.

in the paper, is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoff functions. As such, it strictly generalizes all the existing theorems on the existence of pure strategy Nash equilibrium. Recursive diagonal transfer continuity defined on respective spaces also permits full characterization of symmetric pure strategy, mixed strategy Nash, and Bayesian Nash equilibria in games with general strategy spaces and payoffs. Our full characterization results provide not only a way of understanding equilibrium, but also a way of checking the existence/nonexistence of pure strategy Nash equilibrium in games with discontinuous or nonconcave payoffs. We use many known examples to illustrate the usefulness of our main theorems, especially its usefulness in checking the nonexistence of equilibrium. The approach and main result developed in the paper can also allow us to ascertain the existence of equilibria in important classes of economic games. As an application, we show how they can be employed to fully characterize the existence of competitive equilibrium with excess demand functions.

The logic of recursive diagonal transfer continuity, which generalizes the notion of “diagonal transfer continuity” to allow for recursive dominance (sequential security) , can be roughly described as follows. Whenever a profile of strategies is not an equilibrium, there is a strategy profile that will be transferred to any finite set of “sequential securing strategies”, each of which upsets deviation strategy profiles uniformly and locally. This means whenever a strategy profile  $x$  is not an equilibrium, there is a deviation strategy profile  $y$  and an open set of candidate strategy profiles containing  $x$ , all of which are dominated by any recursive deviation (sequential securing) strategy profiles that directly or indirectly dominates  $y$ .

The relation of the recursive diagonal transfer continuity and diagonal transfer continuity is somewhat similar to that of the weak axiom of revealed preference (WARP) and strong axiom of revealed preference (SARP) in the revealed preference theory on the rational behavior of individual decision making. Directly revealing a preference by WARP is not enough to fully reveal individuals’ preferences, and then one may use SARP – recursive sequences of indirect revealed preferences (transitive closure) to fully reveal the rational behavior. Similarly, diagonal transfer continuity or better-reply security alone is not enough to guarantee the existence of Nash equilibrium -a description of rational behavior of individuals’ strategic decision making, one then may need to use recursive diagonal transfer continuity to fully characterize the existence of equilibrium.

The method of proof employed to obtain our main result is new. While there are different ways of establishing the existence of Nash equilibria, all the existing proofs use the fixed-point-theorem related approaches. As such, previous techniques fail for two reasons. First, all the approaches in the literature either use two types of conditions: continuity and quasi-convexity, or only provide

sufficient conditions to guarantee the existence of a pure strategy Nash equilibrium. The presence of discontinuity and non-quasi-concavity in payoffs may preclude the existence of best replies so that best reply correspondences need not be nonempty-valued or convex-valued. Consequently, both lattice-theoretical techniques and topological techniques (standard application of Kakutani's fixed point theorem) to best reply correspondences fail. Second, to weaken the continuity, and thus, to obtain the existence of an equilibrium, the existing approaches have only considered a direct deviation from a non-equilibrium strategy. Moreover, a remarkable advantage of our proof is that it is simple and elementary without using advanced math.

The remainder of the paper is organized as follows. Section 2 describes the notation, and provides a number of preliminary definitions used in our study of noncooperative games, including the definition of the aggregate function that underlies our analysis of noncooperative games. Section 3 introduces the new condition, recursive diagonal transfer continuity, which is used in our full characterization of pure strategy Nash equilibrium. We prove our main result that recursive diagonal transfer continuity is a necessary and sufficient condition for the existence of pure strategy Nash equilibrium for arbitrary strategy spaces and payoffs. We also provide sufficient conditions for recursive diagonal transfer continuity to be true. Section 4 extends the full characterization result to symmetric pure strategy Nash equilibrium. Section 5 fully characterizes the existence of mixed strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs. Section 6 shows recursive diagonal transfer continuity is also a necessary and sufficient condition for the existence of Bayesian Nash equilibrium in games. Section 7 shows how our main result can be employed to fully characterize the existence of competitive equilibrium for economies with excess demand functions. Finally, concluding remarks are offered in Section 8.

## 2 Preliminaries

Consider the following noncooperative game in the normal form:

$$G = (X_i, u_i)_{i \in I} \tag{1}$$

where  $I = \{1, \dots, n\}$  is the finite set of players,<sup>2</sup>  $X_i$  is player  $i$ 's strategy space which is a nonempty subset of a topological space  $E_i$ , and  $u_i : X \rightarrow \mathbb{R}$  is the payoff function of player  $i$ . Denote by  $X = \prod_{i \in I} X_i$  the Cartesian product of the sets of strategy profiles of the game. For each player  $i \in I$ , denote by  $-i$  all other players rather than player  $i$ . Also denote by  $X_{-i} = \prod_{j \neq i} X_j$  the Cartesian product of the sets of strategies of players  $-i$ .

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<sup>2</sup>All the results in the paper hold for a countable infinity of players.

A strategy profile  $x^* \in X$  is a *pure strategy Nash equilibrium* of a game  $G$  if,

$$u_i(y_i, x_{-i}^*) \leq u_i(x^*) \quad \forall i \in I, \forall y_i \in X_i.$$

A game  $G = (X_i, u_i)_{i \in I}$  is compact, convex, bounded, and upper (lower) semi-continuous if, for all  $i \in I$ ,  $X_i$  is compact, convex, and  $u_i$  is bounded and upper (lower) semi-continuous on  $X$ , respectively. A game  $G = (X_i, u_i)_{i \in I}$  is quasiconcave if, for every  $i \in I$ ,  $X_i$  is convex and the function  $u_i$  is quasiconcave in  $x_i$ .

To fully characterize the existence of equilibria, our strategy is to consider a mapping of individual payoffs into an aggregator function, and then provide a condition on the aggregator function that guarantees the existence of pure strategy Nash equilibrium. This kind of approach is pioneered by Nikaido and Isoda (1955), and is also used by Baye, Tian, and Zhou (1993). Dasgupta and Maskin (1986) also use a similar approach to prove the existence of mixed strategy Nash equilibrium in games with discontinuous payoff functions.

Our full characterization of pure strategy Nash equilibrium is based on the aggregator function,  $U : X \times X \rightarrow \Re$  defined by<sup>3</sup>

$$U(y, x) = \sum_{i=1}^n u_i(y_i, x_{-i}), \quad \forall (y, x) \in X \times X, \quad (2)$$

which refers to the aggregate payoff across individuals where for every player  $i$  assuming she or he deviates to  $y_i$  given that all other players follow the strategy profile  $x$ . We may call  $U(y, x)$  the virtual deviation aggregate payoff, which may not be realizable. Immediately, we have the following observation.

**LEMMA 2.1**  $x^* \in X$  is a *pure strategy Nash equilibrium* of a game  $G$  if and only if  $U(y, x^*) \leq U(x^*, x^*)$  for all  $y \in X$ .

**Proof:** Suppose  $U(y, x^*) \leq U(x^*, x^*)$  for all  $y \in X$ . Let  $y = (y_i, x_{-i}^*)$ . We then have

$$u_i(y_i, x_{-i}^*) \leq u_i(x^*) \quad \forall y_i \in X, \quad (3)$$

which means that  $x^*$  is a Nash equilibrium. The converse is obvious by summing up (3) for all players.

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<sup>3</sup>When  $I$  is a countably infinite set, one may define  $U$  according to  $U(y, x) = \sum_{i \in I} \frac{1}{2^i} u_i(y_i, x_{-i})$ . This is a more general formulation.

### 3 Full Characterization of Pure Strategy Nash Equilibria

In this section we provide a complete solution to the question of the existence of pure strategy Nash equilibrium in games with arbitrary compact strategy spaces and payoffs by providing a necessary and sufficient condition for the existence of pure strategy Nash equilibrium.

#### 3.1 Main Result

We begin with stating diagonal transfer continuity introduced by Baye, Tian, and Zhou (1993) since the notion of recursive diagonal transfer continuity is somewhat an extension of diagonal transfer continuity.

**DEFINITION 3.1** A game  $G = (X_i, u_i)_{i \in I}$  is diagonally transfer continuous if, whenever  $U(y, x) > U(x, x)$  for  $x, y \in X$ , there exists a deviation strategy profile  $z \in X$  and a neighborhood  $\mathcal{V}_x \subset X$  of  $x$  such that  $U(z, x') > U(x', x')$  for all  $x' \in \mathcal{V}_x$ .

Note that “ $U(y, x) > U(x, x)$  for  $x, y \in X$ ” means “ $x \in X$  is not an equilibrium”. We will use these terms interchangeably. We say that a strategy profile  $y$  *upsets* strategy profile  $x$  if  $U(y, x) > U(x, x)$ . Also, since the deviation strategy profile  $z$  results in a strictly higher payoff in a neighborhood of  $x$ , we call such a deviation strategy profile  $z$  a *securing* strategy profile. The diagonal transfer continuity then simply means that, whenever  $x$  is not an equilibrium, there exists a securing strategy profile  $z$  that upsets all points in some neighborhood of  $x$ . For convenience of exposition, let  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$  denote  $U(z, x') > U(x', x')$  for all  $x' \in \mathcal{V}_x$ , where  $\mathcal{V}_x$  is a neighborhood of  $x$ .

**DEFINITION 3.2** (Recursive Upsetting) A strategy profile  $y^0 \in X$  is said to be *recursively* (or called *sequentially*) *upset* by  $z \in X$  if there exists a finite set of deviation strategy profiles  $\{y^1, y^2, \dots, y^{m-1}, z\}$  such that  $U(y^1, y^0) > U(y^0, y^0)$ ,  $U(y^2, y^1) > U(y^1, y^1)$ ,  $\dots$ ,  $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$ .

We say that a strategy profile  $y^0 \in X$  is *m-recursively upset* by  $z \in X$  if the number of such deviation strategy profiles is  $m$ . For convenience, we say  $y^0$  is *directly upset* by  $z$  when  $m = 1$ , and *indirectly upset* by  $z$  when  $m > 1$ . Recursive upsetting says that a strategy profile  $y^0$  can be directly or indirectly upset by a strategy profile  $z$  through sequential deviation strategy profiles  $\{y^1, y^2, \dots, y^{m-1}\}$  in a recursive way that  $y^0$  is upset by  $y^1$ ,  $y^1$  is upset by  $y^2$ ,  $\dots$ , and  $y^{m-1}$  is upset by  $z$ . The assertion that  $y^0$  is directly upset by  $z$  is a dominance relation that the virtual aggregate payoff with individuals' deviation exceeds the realizable total payoff for a given



strategy profile  $y^0$ . Then the recursive upsetting says this dominance holds for any finite sequence of strategy profiles originated from a given strategy point.

We are now ready to introduce the notion of recursive diagonal transfer continuity.

**DEFINITION 3.3** (Recursive Diagonal Transfer Continuity) A game  $G = (X_i, u_i)_{i \in I}$  is said to be *recursively (or called sequentially) diagonal transfer continuous* if, whenever  $U(y, x) > U(x, x)$  for  $x, y \in X$ , there exists a strategy profile  $y^0 \in X$  (possibly  $y^0 = x$ ) and a neighborhood  $\mathcal{V}_x$  of  $x$  such that  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$  for any  $z$  that recursively upsets  $y^0$ .

Similarly, we can define  $m$ -recursive diagonal transfer continuity. A game  $G = (X_i, u_i)_{i \in I}$  is  *$m$ -recursively diagonal transfer continuous* if the phrase “for any  $z$  that recursively upsets  $y^0$ ” in the above definition is replaced by “for any  $z$  that  $m$ -recursively upsets  $y^0$ ”. Thus, a game  $G = (X_i, u_i)_{i \in I}$  is recursively diagonal transfer continuous if it is  $m$ -recursively diagonal transfer continuous on  $X$  for all  $m = 1, 2, \dots$

**REMARK 3.1** Under recursive diagonal transfer continuity, when  $U(z, y^{m-1}) > U(y^{m-1}, y^{m-1})$ ,  $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$ ,  $\dots$ ,  $U(y^1, y^0) > U(y^0, y^0)$ , we have not only  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ , but also  $U(y^{m-1}, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ ,  $\dots$ ,  $U(y^1, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$  since it is also  $k$ -recursively diagonal transfer continuous for  $k = 1, 2, \dots, m-1$ . That means all of the points in  $\mathcal{V}_x$  are upset by the sequence of securing strategy profiles  $\{y^1, \dots, y^{m-1}, y^m\}$  that directly or indirectly upset  $y^0$ .

Recursive diagonal transfer continuity means that whenever  $x$  is not an equilibrium, there is a strategy profile  $y^0$  and an open set of candidate strategy profiles containing  $x$ , all of which are upset by all securing strategy profiles that directly or indirectly upset  $y^0$ . This implies that, if equilibrium fails to exist, then there is a nonequilibrium strategy profile  $x$  such that for every  $y^0 \in X$  and every neighborhood  $\mathcal{V}_x$  of  $x$ , some deviation strategy profiles in the neighborhood cannot be upset by a securing strategy profile  $z$  that directly or indirectly upsets  $y^0$ .

Recursive diagonal transfer continuity refers to the fact that, when  $U(y, x) > U(x, x)$ ,  $y$  may be transferred to a sequence of securing strategy profiles  $\{y^1, y^2, \dots, y^m\}$  in order for all points in a neighborhood of  $x$  to be upset by these securing strategy profiles. The usual notion of continuity would require that this dominance hold at  $y$  for all points in a neighborhood of  $x$ .

**REMARK 3.2** When  $m = 0$ , there is no recursive upsetting. 0-recursive diagonal transfer continuity becomes the diagonal transfer continuity whenever there is a securing strategy profile  $y^0$ . Also, recursive diagonal transfer continuity neither implies nor is implied by continuity for games

with two or more players.<sup>4</sup> This point becomes clear when one sees recursive diagonal transfer continuity is a necessary and sufficient condition for the existence of pure strategy Nash equilibrium for arbitrary strategy spaces and payoff functions while continuity of the aggregate payoff function is not a necessary nor sufficient condition for the existence of pure strategy Nash equilibrium.

Before proceeding to our main result, we describe the basic idea why recursive diagonal transfer continuity ensures the existence of pure strategy Nash equilibrium for a compact game. When a compact game fails to have pure strategy Nash equilibrium, every strategy profile  $x$  will be upset by another strategy profile  $y^0$ . Then, by recursive diagonal transfer continuity, there is some open set of candidate profiles containing  $x$ , all of which will be upset by some securing strategy profile  $z$  that directly or indirectly upsets  $y^0$ . Then there are finite strategy profiles  $\{x^1, x^2, \dots, x^n\}$  whose neighborhoods cover  $X$ . Thus, all of the points in a neighborhood, say  $\mathcal{V}_{x^1}$ , will be upset by a corresponding deviation profile  $z^1$ , which means  $z^1$  cannot be an element in  $\mathcal{V}_{x^1}$ . If it is in some other neighborhood, say,  $\mathcal{V}_{x^2}$ , then it can be shown that  $z^2$  will upset all strategy profiles in the union of  $\mathcal{V}_{x^1}$  and  $\mathcal{V}_{x^2}$  so that  $z^2$  is not in the union of  $\mathcal{V}_{x^1}$  and  $\mathcal{V}_{x^2}$ . Suppose  $z^2 \in \mathcal{V}_{x^3}$ . Then we can similarly show that  $z^3$  is not in the union of  $\mathcal{V}_{x^1}$ ,  $\mathcal{V}_{x^2}$  and  $\mathcal{V}_{x^3}$ . Repeating such arguments, we can show that  $z^k \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \dots \cup \mathcal{V}_{x^k}$ , i.e.,  $z^k$  is not in the union of  $\mathcal{V}_{x^1}$ ,  $\mathcal{V}_{x^2}$ ,  $\dots$ ,  $\mathcal{V}_{x^k}$  for  $k = 1, 2, \dots, n$ . In other words,  $z^k \notin \mathcal{V}_{x^j}$  for all  $j \leq k$  and  $k = 1, 2, \dots, n$  so that the securing strategy profile  $z^n$  will not be in the strategy space  $X$ , which is impossible. Thus recursive diagonal transfer continuity guarantees the existence of a pure strategy Nash equilibrium.

Now we are ready to state our main result that strictly generalizes all the existing results on the existence of pure strategy Nash equilibrium in compact games.

**THEOREM 3.1** *Suppose  $G = (X_i, u_i)_{i \in I}$  is compact. Then, the game  $G$  possesses a pure strategy Nash equilibrium if and only if it is recursively diagonal transfer continuous on  $X$ .*

**PROOF.** *Sufficiency ( $\Leftarrow$ ).* Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium. Then, by recursive diagonal transfer continuity, for each  $x \in X$ , there exists  $y^0$  and a neighborhood  $\mathcal{V}_x$  such that  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$  whenever  $y^0 \in X$  is recursively upset by  $z$ , i.e., for any sequence of recursive securing strategy profiles  $\{y^1, \dots, y^{m-1}, y^m\}$  with  $U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1})$ ,  $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$ ,  $\dots$ ,  $U(y^1, y^0) > U(y^0, y^0)$  for  $m \geq 1$ , we have  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ . Since there is no equilibrium by the contrapositive hypothesis and

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<sup>4</sup>In one-player games recursive diagonal transfer continuity is equivalent to the player's utility function possessing a maximum on a compact set, and consequently it implies transfer weak upper continuity introduced in Tian and Zhou (1995), which is weaker than continuity.

the game is recursively diagonal transfer continuous on  $X$ , such a sequence of recursive securing strategy profiles  $\{y^1, \dots, y^{m-1}, y^m\}$  exists for some  $m \geq 1$ .

Since  $X$  is compact and  $X \subseteq \bigcup_{x \in X} \mathcal{V}_x$ , there is a finite set  $\{x^1, \dots, x^L\}$  such that  $X \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$ . For each of such  $x^i$ , the corresponding initial deviation profile is denoted by  $y^{0i}$  so that  $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$  whenever  $y^{0i}$  is recursively upset by  $z^i$ .

Since there is no equilibrium, for each of such  $y^{0i}$ , there exists  $z^i$  such that  $U(z^i, y^{0i}) > U(y^{0i}, y^{0i})$ , and then, by 1-recursive diagonal transfer continuity, we have  $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ . Now consider the set of securing strategy profiles  $\{z^1, \dots, z^n\}$ . Then,  $z^i \notin \mathcal{V}_{x^i}$ , otherwise, by  $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$ , we will have  $U(z^i, z^i) > U(z^i, z^i)$ , a contradiction. So we must have  $z^1 \notin \mathcal{V}(x^1)$ .

Without loss of generality, we suppose  $z^1 \in \mathcal{V}_{x^2}$ . Since  $U(z^2, z^1) > U(z^1, z^1)$  by noting that  $z^1 \in \mathcal{V}_{x^2}$  and  $U(z^1, y^{01}) > U(y^{01}, y^{01})$ , then, by 2-recursive diagonal transfer continuity, we have  $U(z^2, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1})$ . Also,  $U(z^2, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$ . Thus  $U(z^2, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2})$ , and consequently  $z^2 \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2}$ .

Again, without loss of generality, we suppose  $z^2 \in \mathcal{V}_{x^3}$ . Since  $U(z^3, z^2) > U(z^2, z^2)$  by noting that  $z^2 \in \mathcal{V}_{x^3}$ ,  $U(z^2, z^1) > U(z^1, z^1)$ , and  $U(z^1, y^{01}) > U(y^{01}, y^{01})$ , by 3-recursive diagonal transfer continuity, we have  $U(z^3, \mathcal{V}_{x^1}) > U(\mathcal{V}_{x^1}, \mathcal{V}_{x^1})$ . Also, since  $U(z^3, z^2) > U(z^2, z^2)$  and  $U(z^2, y^{02}) > U(y^{02}, y^{02})$ , by 2-recursive diagonal transfer continuity, we have  $U(z^3, \mathcal{V}_{x^2}) > U(\mathcal{V}_{x^2}, \mathcal{V}_{x^2})$ . Thus, we have  $U(z^3, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}) > U(\mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}, \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3})$ , and consequently  $z^3 \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \mathcal{V}_{x^3}$ .

Going on this process, we can show that  $z^k \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \cup \dots \cup \mathcal{V}_{x^k}$ , i.e.,  $z^k$  is not in the union of  $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \dots, \mathcal{V}_{x^k}$  for  $k = 1, 2, \dots, L$ . In particular, for  $k = L$ , we have  $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \dots \cup \mathcal{V}_{x^L}$  and so  $z^L \notin X \subseteq \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \dots \cup \mathcal{V}_{x^L}$ , a contradiction.

*Necessity* ( $\Rightarrow$ ). Suppose  $x^*$  is a pure strategy Nash equilibrium and  $U(y, x) > U(x, x)$  for  $x, y \in X$ . Let  $y^0 = x^*$  and  $\mathcal{V}_x$  a neighborhood of  $x$ . Since  $U(y, x^*) \leq U(x^*, x^*)$  for all  $y \in X$ , it is impossible to find any securing strategy profile  $y^1$  such that  $U(y^1, y^0) > U(y^0, y^0)$  (then it of course is impossible to find any sequence of strategy profiles  $\{y^1, y^2, \dots, y^m\}$  such that  $U(y^1, y^0) > U(y^0, y^0)$ ,  $U(y^2, y^1) > U(y^1, y^1)$ ,  $\dots$ ,  $U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1})$ ). Hence, the recursive diagonal transfer continuity holds trivially. ■

Theorem 3.1 needs to assume that the strategy space of a game is compact. This may be still a restrictive assumption. For some important economic games, they may not have compact strategy spaces. For instance, it is well known that Walrasian mechanism can be regarded as a generalized game. As we will discuss in Section 7, when preferences are strictly monotone, excess demand functions are not well defined for zero prices. In this case, Walras' law are only defined for interior

points of unit simplex, and thus we cannot use Theorem 3.1 to fully characterize the existence of competitive equilibrium.

In the following we show that compactness of strategy space in Theorem 3.1 can also be removed.<sup>5</sup> To do so, we first introduced the following weak notion of recursive diagonal transfer continuity.

**DEFINITION 3.4** Let  $B$  be a subset of  $X$ . A game  $G = (X_i, u_i)_{i \in I}$  is said to be *recursively diagonal transfer continuous on  $X$  with respect to  $B$*  if, whenever  $U(y, x) > U(x, x)$  for  $x, y \in X$ , there exists a strategy profile  $y^0 \in X$  (possibly  $y^0 = x$ ) and a neighborhood  $\mathcal{V}_x$  of  $x$  such that for any finite subset of securing strategy profiles  $\{y^1, \dots, y^m\} \subset B$  with  $y^m = z$ , we have  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ .

Note that, while  $\{y^1, \dots, y^m\}$  are required to be in  $B$ ,  $y^0$  is not necessarily in  $B$  but can be any point in  $X$ . We then have the following theorem that generalizes Theorem 3.1 by relaxing compactness of games.

**THEOREM 3.2** A game  $G = (X_i, u_i)_{i \in I}$  possesses a pure strategy Nash equilibrium if and only if there exists a compact set  $B \subseteq X$  such that it is recursively diagonal transfer continuous on  $X$  with respect to  $B$ .

**PROOF.** *Sufficiency* ( $\Leftarrow$ ). The proof of sufficiency is essentially the same as the one of Theorem 3.1 and we just outline the proof here.

To show the existence of a pure strategy Nash equilibrium on  $X$ , it suffices to show that there exists a pure strategy Nash equilibrium  $x^* \in B$  if the game  $G$  is recursively diagonal transfer continuous on  $X$  with respect to  $B$ . Suppose, by way of contradiction, that there is no pure strategy Nash equilibrium in  $B$ . Then, since the game  $G$  is recursively diagonal transfer continuous on  $X$  with respect to  $B$ , for each  $x \in B$ , there exists  $y^0$  and a neighborhood  $\mathcal{V}_x$  such that for any finite subset of securing strategy profiles  $\{y^1, \dots, y^m\} \subset B$  with  $y^m = z$ , we have  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$

Since  $B$  is compact and  $B \subseteq \bigcup_{x \in X} \mathcal{V}_x$ , there is a finite set  $\{x^1, \dots, x^L\} \subseteq B$  such that  $B \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$ . For each of such  $x^i$ , the corresponding initial deviation profile is denoted by  $y^{0i}$  so that  $U(z^i, \mathcal{V}_{x^i}) > U(\mathcal{V}_{x^i}, \mathcal{V}_{x^i})$  whenever  $y^{0i}$  is recursively upset by  $z^i$  through any finite subset of securing strategy profiles  $\{y^{1i}, \dots, y^{mi}\} \subset B$  with  $y^{mi} = z^i$ . Then, by the same argument as in the proof of Theorem 3.1, we will obtain that  $z^k$  is not in the union of  $\mathcal{V}_{x^1}, \mathcal{V}_{x^2}, \dots, \mathcal{V}_{x^k}$  for  $k = 1, 2, \dots, L$ . For  $k = L$ , we have  $z^L \notin \mathcal{V}_{x^1} \cup \mathcal{V}_{x^2} \dots \cup \mathcal{V}_{x^L}$  and so  $z^L \notin B \subseteq \bigcup_{i=1}^L \mathcal{V}_{x^i}$ , which contradicts that  $z^L$  is a securing strategy profile in  $B$ .

<sup>5</sup>I thank David Rahman raises this issue to me.

*Necessity* ( $\Rightarrow$ ). Suppose  $x^*$  is a pure strategy Nash equilibrium. Let  $B = \{x^*\}$ . Then, the set  $B$  is clearly compact. Now, for any  $x, y \in X$  such that  $U(y, x) > U(x, x)$  for  $x, y \in X$ , let  $y^0 = x^*$  and  $\mathcal{V}_x$  be a neighborhood of  $x$ . Since  $U(y, x^*) \leq U(x^*, x^*)$  for all  $y \in X$  and  $y^0 = x^*$  is a unique element in  $B$ , it is no other securing strategy profiles  $y^1$  such that  $U(y^1, y^0) > U(y^0, y^0)$ . Hence, the game is recursively diagonal transfer continuous on  $X$  with respect to  $B$ . ■

Thus, Theorem 3.2 fully characterizes the existence of equilibrium in a games with an arbitrary strategy space that may be discrete, continuum, non-convex or non-compact and an arbitrary payoff function that may be discontinuous or nonquasiconcave, and consequently it strictly generalizes all the existing results on the existence of pure strategy Nash equilibrium.

Theorems 3.1 and 3.2 provide not only a way of understanding equilibrium, but also a way of checking the existence/nonexistence of pure strategy Nash equilibrium in games with discontinuous or nonconcave payoffs. Five examples that illustrate the usefulness of the above results will be given in Subsection 3.3 below. In particular, in Example 3.2, although the strategy space is an unit open interval, highly discontinuous and nonquasiconcave, we can use Theorem 3.2 to argue the existence of equilibrium.

In general, the weaker the conditions in an existence theorem are, the harder it is to verify whether the conditions are satisfied in a particular game. Although these examples show the usefulness of Theorems 3.1 and 3.2, especially in proving the nonexistence of pure strategy Nash equilibrium, given the generality of the condition, it is not surprising that the condition is, in general, not easy to check from a practical point of view. Since the main purpose of the paper is to characterize the existence of equilibria in games with general strategy spaces and payoffs, the condition does suggest a way to interpret equilibrium existence. It mainly shows what is possible for a game to have a pure strategy Nash equilibrium. Nevertheless, adding specificity to the model, we may get sufficient conditions for recursive diagonal transfer continuity, and consequently provide new sufficient conditions for the existence of pure strategy Nash equilibrium.

**DEFINITION 3.5** (Deviation Transitivity)  $G = (X_i, u_i)_{i \in I}$  is said to be *deviational transitive* if  $U(y^2, y^1) > U(y^1, y^1)$  and  $U(y^1, y^0) > U(y^0, y^0)$  imply that  $U(y^2, y^0) > U(y^0, y^0)$ . That is, the upsetting dominance relation is transitive.

**COROLLARY 3.1** Suppose  $G = (X_i, u_i)_{i \in I}$  is compact and deviational transitive. Then, there exists a pure strategy Nash equilibrium point if and only if  $G$  is 1-recursively diagonal transfer continuous.

**PROOF.** We only need to show that, when  $G$  is deviational transitive, 1-recursive diagonal transfer continuity implies  $m$ -recursive diagonal transfer continuity for  $m \geq 1$ . Suppose  $x$  is not an

equilibrium. Then, by 1-recursive diagonal transfer continuity, there exists a strategy profile  $y^0 \in X$  and a neighborhood  $\mathcal{V}_x$  of  $x$  such that  $U(z, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$  whenever  $U(z, y^0) > U(y^0, y^0)$  for any  $z \in X$ .

Now, for any sequence of deviation profiles  $\{y^1, \dots, y^{m-1}, y^m\}$ , if  $U(y^m, y^{m-1}) > U(y^{m-1}, y^{m-1})$ ,  $U(y^{m-1}, y^{m-2}) > U(y^{m-2}, y^{m-2})$ ,  $\dots$ ,  $U(y^1, y^0) > U(y^0, y^0)$ , we then have  $U(y^m, y^0) > U(y^0, y^0)$  by deviation transitivity of  $U$ , and thus by 1-recursive diagonal transfer continuity,  $U(y^m, \mathcal{V}_x) > U(\mathcal{V}_x, \mathcal{V}_x)$ . Since  $m$  is arbitrary,  $G$  is recursively diagonal transfer continuous. ■

### 3.2 Discussion and Related Work

The recursive diagonal transfer continuity we propose is of transfer type. The basic transfer method is developed in Tian (1992, 1993), Tian and Zhou (1992, 1995), Zhou (1992), and Baye, Tian, and Zhou (1993) for studying the maximization of binary relations that may be nontotal or nontransitive and the existence of equilibrium in games that may have discontinuous or nonquasiconcave payoffs. They develop three types of transfers: transfer continuities, transfer convexities, and transfer transivities. This kind of properties have provided milestones in the literature on the maximization of binary relations and the existence of equilibrium in games with discontinuous and/or nonquasiconcave payoffs. Various notions of transfer continuities, transfer convexities and transfer transivities provide complete solutions to the question of the existence of maximal elements for complete preorders and interval orders — cf. Tian (1993) and Tian and Zhou (1995). The recursive diagonal transfer continuity proposed in this paper extends the static transfer approach to a dynamic (sequential) one so that it enables us to provide a complete solution to the question of the existence of equilibrium in games with arbitrary compact strategy spaces and payoffs.

We now discuss how Theorem 3.1 yields the results of Baye, Tian, and Zhou (1993), Reny (1999), Nessah and Tian (2008), and Barelli and Soza (2009).

Baye, Tian, and Zhou (1993) study the existence of pure strategy Nash equilibria in games with discontinuous and nonquasiconcave payoffs by introducing the concepts of diagonal transfer continuity and diagonal transfer quasiconcavity.

**DEFINITION 3.6** A game  $G = (X_i, u_i)_{i \in I}$  is diagonally transfer quasiconcave in  $y$  if, for any finite subset  $Y^m = \{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $X^m = \{x^1, \dots, x^m\} \subset X$  such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , and any  $x \in \text{co}\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\}$ , we have  $\min_{1 \leq l \leq s} U(y^{k^l}, x) \leq U(x, x)$ .

Theorem 1 in Baye, Tian, and Zhou (1993) shows that diagonal transfer quasi-concavity is necessary, and further, under diagonal transfer continuity and compactness, sufficient for the existence of pure strategy Nash equilibrium.

Reny (1999) studies the existence of pure strategy Nash equilibria in discontinuous games by introducing the concepts of payoff security and better-reply security.

Let  $\Gamma = \{(x, u) \in X \times \mathbb{R}^n : u_i(x) = u_i, \forall i \in I\}$  be the graph of the game. The closure of  $\Gamma$  in  $X \times \mathbb{R}^n$  is denoted by  $\bar{\Gamma}$ . The frontier of  $\Gamma$ , which is the set of points in  $\bar{\Gamma}$  but not in  $\Gamma$ , is denoted by  $\text{Fr } \Gamma$ .

**DEFINITION 3.7** A game  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if, whenever  $(x^*, u^*) \in \bar{\Gamma}$ ,  $x^*$  is not an equilibrium implies that there is some player  $i$ ,  $\bar{x}_i \in X_i$ , and an open neighborhood  $\mathcal{V}_{x_{-i}}$  of  $x_{-i}$  such that  $u_i(\bar{x}_i, y_{-i}) > u_i(x^*)$  for all  $y_{-i} \in \mathcal{V}_{x_{-i}}$ .

The notion of better-reply security also uses the same idea of transferring a non-equilibrium strategy to a securing strategy, and thus it actually falls in the form of transfer continuity.

Theorem 3.1 in Reny (1999) shows that a game  $G = (X_i, u_i)_{i \in I}$  possesses a Nash equilibrium if it is compact, bounded, quasiconcave and better-reply secure. Reny (1999) and Bagh and Jofre (2006) provide sufficient conditions for a game to be better-reply secure.

**DEFINITION 3.8** A game  $G = (X_i, u_i)_{i \in I}$  is payoff secure if for every player  $i$ ,  $x \in X$ , and  $\epsilon > 0$ , there exists  $\bar{x}_i \in X_i$  and an open neighborhood  $\mathcal{V}_{x_{-i}}$  of  $x_{-i}$  such that  $u_i(\bar{x}_i, y_{-i}) \geq u_i(x) - \epsilon$  for all  $y_{-i} \in \mathcal{V}_{x_{-i}}$ .

**DEFINITION 3.9** A game  $G = (X_i, u_i)_{i \in I}$  is reciprocally upper semicontinuous if, whenever  $(x, u) \in \bar{\Gamma}$  and  $u_i(x) \leq u_i$  for every player  $i$ ,  $u_i(x) = u_i$  for every player  $i$ .

**DEFINITION 3.10** A game  $G = (X_i, u_i)_{i \in I}$  is weakly reciprocal upper semi-continuous if, for any  $(x, u) \in \text{Fr } \Gamma$ , there is a player  $i$  and  $\hat{x}_i \in X_i$  such that  $u_i(\hat{x}_i, x_{-i}) > u_i$ .

Reny (1999) shows that a game  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if it is payoff secure and reciprocally upper semi-continuous. Bagh and Jofre (2006) further show that  $G = (X_i, u_i)_{i \in I}$  is better-reply secure if it is payoff secure and weakly reciprocal upper semi-continuous.

To provide more sufficient conditions for recursive diagonal transfer continuity, as a complement of this paper, Nessah and Tian (2008) recently introduce a new notion of weak continuity, called *weak transfer quasi-continuity*, which is weaker than the most known weak notions of continuity, including diagonal transfer continuity in Baye, Tian, and Zhou (1993) and better-reply security in Reny (1999), and holds in a large class of discontinuous games.

**DEFINITION 3.11** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer quasi-continuous* if, whenever  $x \in X$  is not an equilibrium, there exists a strategy profile  $y \in X$  and a neighborhood  $\mathcal{V}(x)$  of  $x$  so that for every  $x' \in \mathcal{V}(x)$ , there exists a player  $i$  such that  $u_i(y_i, x'_{-i}) > u_i(x')$ .

They also provide some sufficient conditions, each of which implies weak transfer quasi-continuity: (1) transfer continuity, (2) weak transfer continuity, (3) weak transfer upper continuity and payoff security, and (4) upper semicontinuity and weak transfer lower continuity.

**DEFINITION 3.12** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *transfer continuous* if for all player  $i$ , whenever  $u_i(z_i, x_{-i}) > u_i(x)$  for  $z_i \in X_i$  and  $x \in X$ , there is some neighborhood  $\mathcal{V}(x)$  of  $x$  and  $y_i \in X_i$  such that  $u_i(y_i, x'_{-i}) > u_i(x')$  for all  $x' \in \mathcal{V}(x)$ .

**DEFINITION 3.13** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer continuous* if, whenever  $x \in X$  is not an equilibrium, there exists a player  $i$ ,  $y_i \in X_i$  and a neighborhood  $\mathcal{V}_x$  of  $x$  such that  $u_i(y_i, x'_{-i}) > u_i(x')$  for all  $x' \in \mathcal{V}_x$ .

**DEFINITION 3.14** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer upper continuous* if, whenever  $x \in X$  is not an equilibrium, there exists a player  $i$ ,  $\hat{x}_i \in X_i$  and a neighborhood  $\mathcal{V}_x$  of  $x$  such that  $u_i(\hat{x}_i, x_{-i}) > u_i(x')$  for all  $x' \in \mathcal{V}_x$ .

**DEFINITION 3.15** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly transfer lower continuous* if, whenever  $x$  is not a Nash equilibrium, there exists a player  $i$ ,  $y_i \in X_i$ , and a neighborhood  $\mathcal{V}_{x_{-i}}$  of  $x_{-i}$  such that  $u_i(y_i, x'_{-i}) > u_i(x)$  for all  $x'_{-i} \in \mathcal{V}_{x_{-i}}$ .

They also introduce some new notions of transfer type of quasiconcavity, called *strong diagonal transfer quasiconcavity* and *weak diagonal transfer quasiconcavity*, each of which is a necessary condition for the existence of pure strategy Nash equilibrium, and consequently a necessary condition for recursive diagonal transfer continuity.

**DEFINITION 3.16** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *strongly diagonal transfer quasi-concave* if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset X$  such that for any subset  $\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\} \subset X^m$ ,  $1 \leq s \leq m$ , and any  $x \in \text{co}\{x^{k^1}, x^{k^2}, \dots, x^{k^s}\}$ , there exists  $y^h \in \{y^{k^1}, \dots, y^{k^s}\}$  satisfying  $u_i(y_i^{k^h}, x_{-i}) \leq u_i(x)$  for all  $i \in I$ .

**DEFINITION 3.17** A game  $G = (X_i, u_i)_{i \in I}$  is said to be *weakly diagonal transfer quasiconcave* if for any finite subset  $\{y^1, \dots, y^m\} \subset X$ , there exists a corresponding finite subset  $\{x^1, \dots, x^m\} \subset$



$X$  such that for each  $\tilde{x} = \sum_{i,j} \lambda_{i,j} x^j \in \text{co}\{x^h, h = 1, \dots, m\}$ , we have

$$\min_{(i,j) \in J} [u_i(y_i^j, \tilde{x}_{-i}) - u_i(\tilde{x})] \leq 0,$$

where  $\lambda_{i,j} \geq 0$  with  $\sum_{i,j} \lambda_{i,j} = 1$  and  $J = \{(i, j) : \lambda_{i,j} > 0\}$ .

It may be remarked that a game  $G$  is: (1) weakly transfer quasi-continuous if it is weakly transfer continuous, diagonally transfer continuous or better reply secure; (2) weakly transfer upper continuous if it is upper semicontinuous; and (3) weakly diagonal transfer quasiconcave and diagonally transfer quasiconcave if it is strongly diagonal transfer quasiconcave.

Nessah and Tian (2008) show that a game  $G = (X_i, u_i)_{i \in I}$  possesses a Nash equilibrium if it is convex, compact, weakly transfer quasi-continuous, and quasiconcave or strongly/weakly diagonal transfer quasiconcave.

As a source of reference, we summarize these results as corollaries of Theorem 3.1 that provide sufficient conditions for recursive diagonal transfer continuity.

**COROLLARY 3.2** *Suppose  $G = (X_i, u_i)_{i \in I}$  is compact and convex. Then any one of the following conditions is sufficient for a game to be recursively diagonal transfer continuous, and consequently it possesses a pure strategy Nash equilibrium:*

1. *the game is quasiconcave, upper semicontinuous in  $x_i$ , and group continuous in  $x$ <sup>6</sup> [Dasgupta and Maskin (1986)];*
2. *the game is diagonally transfer quasiconcave and diagonally transfer continuous [Baye, Tian and Zhou (1993)];*
3. *the game is bounded, quasiconcave, and better-reply secure [Reny (1999)];*
4. *the game is bounded, quasiconcave, payoff secure, and reciprocally upper semicontinuous [Reny (1999)];*
5. *the game is bounded, quasiconcave, payoff secure, and weakly reciprocal upper semicontinuous [Bagh (2006)];*
6. *the game is bounded, quasiconcave, and weakly transfer quasi-continuous [Nessah and Tian (2008)];*

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<sup>6</sup>A payoff function  $u_i : X \rightarrow \mathbb{R}$  is graph-continuous if for all  $x \in X$ , there exists a function  $F_i : A_{-i} \rightarrow A_i$  with  $F_i(\bar{x}_{-i}) = \bar{x}_i$  such that  $u_i(F_i(x_{-i}), x_{-i})$  is continuous at  $x_{-i} = \bar{x}_{-i}$ .

7. *the game is strongly diagonal transfer quasiconcave and weakly transfer quasi-continuous [Nessah and Tian (2008)];*
8. *the game is bounded, weakly diagonal transfer quasiconcave, and weakly transfer quasi-continuous [Nessah and Tian (2008)];*
9. *the game is bounded, weakly diagonal transfer quasiconcave, and diagonally transfer continuous [Nessah and Tian (2008)];*
10. *the game is bounded, weakly diagonal transfer quasiconcave, and better-reply secure [Nessah and Tian (2008)];*
11. *the game is bounded, weakly diagonal transfer quasiconcave, weakly transfer upper continuous, and payoff secure [Nessah and Tian (2008)];*
12. *the game is bounded, weakly diagonal transfer quasiconcave, weakly transfer lower continuous, and upper semicontinuous [Nessah and Tian (2008)].*

In addition, the conditions imposed in Nash (1951), Debreu (1952), Nikaido and Isoda (1955), Nishimura and Friedman (1981), Dasgupta and Maskin (1986), Vives (1990), Carmona (2005), Morgan and Scalzo (2007), etc., which guarantee the existence of pure strategy Nash equilibria, can be regarded as sufficient conditions of recursive diagonal transfer continuity.

**REMARK 3.3** From Baye, Tian, and Zhou (1993), one knows that a game is diagonally transfer quasiconcave if any one of the following conditions is satisfied: (1) each  $u_i(x_i, x_{-i})$  is concave in  $x_i$ ; (2)  $U(x, y)$  is concave in  $x$ ; (3)  $U(x, y)$  is quasiconcave in  $x$ ; and (4)  $U(x, y)$  is diagonally quasiconcave in  $x$ . A game is diagonally transfer continuous if any one of the following conditions is satisfied: (1) each  $u_i(x_i, x_{-i})$  is continuous; (2) each  $u_i(x_i, x_{-i})$  is upper semicontinuous in  $x_i$  and continuous in  $x_{-i}$ ; (3)  $U(x, y)$  is continuous; and (4)  $\phi(x, y) \equiv U(x, y) - U(y, y)$  is lower semicontinuous in  $y$ . As a result, any pair of conditions, one each from the sufficient conditions for diagonal transfer quasi-concavity and diagonal transfer continuity, are sufficient for the existence of a pure strategy Nash equilibrium for games with compact strategy spaces.

### 3.3 Examples

In this subsection we provide five examples to illustrate how our main theorem fully characterizes the existence or nonexistence of pure strategy Nash equilibrium in games that do not satisfy the conditions of existing theorems. Examples 1 and 2 are games with pure strategy equilibria that are accounted for by our Theorem 3.1, but which violate the conditions of existing theorems.

Examples 3-5 are games that do not have an equilibrium because they fail to satisfy our recursive diagonal transfer continuity condition.

**EXAMPLE 3.1** Consider the two-player game with the following payoff functions defined on  $[0, 1] \times [0, 1]$  studied by Barelli and Soza (2009).

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i \in (0, 1) \\ 1 & \text{if } x_i = 0 \text{ and } x_{-i} \in \mathbb{Q} \\ 1 & \text{if } x_i = 1 \text{ and } x_{-i} \notin \mathbb{Q} \\ 0 & \text{otherwise} \end{cases},$$

where  $\mathbb{Q} = \{x \in [0, 1] : x \text{ is a rational number}\}$ .

This game is convex, compact, bounded and quasiconcave, but it is not weakly transfer quasi-continuous, and consequently, it is not diagonally transfer continuous, better-reply secure, or weakly transfer continuous, either. Thus, there is no existing theorem that can be applied.

To see the game is not weakly transfer quasi-continuous, consider the nonequilibrium  $x = (1, 1)$ . We then cannot find any  $y \in X$  and any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$  such that for every  $x' \in \mathcal{V}_x$ , there is a player  $i$  with  $u_i(y_i, x'_{-i}) > u_i(x')$ . We show this by considering two cases.

Case 1.  $y_2 \neq 0$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choosing  $x' \in \mathcal{V}_x$  with  $x'_1 = 1$  and  $x'_2 \notin \mathbb{Q}$ , we have  $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$  and  $u_2(x'_1, y_2) = u_2(x'_1, x'_2) = 0$ .

Case 2.  $y_2 = 0$ . When  $y_1 \neq 0$ , choosing  $x' \in \mathcal{V}_x$  with  $x'_2 = 1$  and  $x'_1 \notin \mathbb{Q}$ , we have  $u_1(y_1, x'_2) = u_1(x'_1, x'_2) = 0$  and  $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$ . When  $y_1 = 0$ , choosing  $x' \in \mathcal{V}_x$  with  $x'_1 \notin \mathbb{Q}$  and  $x'_2 \notin \mathbb{Q}$ , we have  $u_1(y_1, x'_2) = u_1(x'_1, x'_2) = 0$  and  $u_2(x'_1, y_2) = u_2(x'_1, x'_2) = 0$ .

Thus, the game is not weakly transfer quasi-continuous, and Theorems 3.1-3.3 of Nessah and Tian (2008) can not be applied.

However, it is recursively diagonal transfer quasi-continuous. Indeed, suppose  $U(y, x) > U(x, x)$  for  $x = (x_1, x_2) \in X$  and  $y = (y_1, y_2) \in X$ . Let  $y^0 = (0, 0)$  and  $\mathcal{V}_x$  be a neighborhood of  $x$ . Since  $U(y, y^0) \leq U(y^0, y^0)$  for all  $y \in Y$ , it is impossible to find any securing strategy profile  $y^1$  such that  $U(y^1, y^0) > U(y^0, y^0)$ . Hence, the recursive diagonal transfer continuity holds. Therefore, by Theorem 3.1, this game has a pure strategy Nash equilibrium.

**EXAMPLE 3.2** Consider a game with  $n = 2$ ,  $X_1 = X_2 = (0, 1)$  that is an open unit interval set, and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \mathbb{Q} \times \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2,$$

where  $\mathbb{Q} = \{x \in (0, 1) : x \text{ is a rational number}\}$ .

Then the game is not compact, nor quasiconcave. It is not weakly transfer quasi-continuous either (so it is not diagonally transfer continuous, better-reply secure, or weakly transfer continuous either). To see this, consider any nonequilibrium  $x$  that consists of irrational numbers. Then, for any neighborhood  $\mathcal{V}_x$  of  $x$ , choosing  $x' \in \mathcal{V}_x$  with  $x'_1 \in \mathbb{Q}$  and  $x'_2 \in \mathbb{Q}$ , we have  $u_1(y_1, x'_2) \leq u_1(x'_1, x'_2) = 1$  and  $u_2(x'_1, y_2) \leq u_2(x'_1, x'_2) = 1$  for any  $y \in X$ . So the game is not weakly transfer quasi-continuous. Thus, there is no existing theorem that can be applied.

However, it is recursively diagonal transfer continuous on  $X$ . Indeed, suppose  $U(y, x) > U(x, x)$  for  $x = (x_1, x_2) \in X$  and  $y = (y_1, y_2) \in X$ . Let  $y^0$  be any vector with rational numbers,  $B = \{y^0\}$ , and  $\mathcal{V}_x$  be a neighborhood of  $x$ . Since  $U(y, y^0) \leq U(y^0, y^0)$  for all  $y \in Y$ , it is impossible to find any securing strategy profile  $y^1$  such that  $U(y^1, y^0) > U(y^0, y^0)$ . Hence, the game is not recursively diagonal transfer continuous on  $X$  with respect to  $B$ . Therefore, by Theorem 3.2, this game has a pure strategy Nash equilibrium. In fact, the set of pure strategy Nash equilibria consists of all rational numbers on  $(0, 1)$ .

Our full characterization result is especially useful to check the nonexistence of equilibrium of economic games. No such result is available in the literature.

**EXAMPLE 3.3 (DASGUPTA AND MASKIN)** Consider the following game studied by Dasgupta and Maskin (1986):  $n = 2$ ,  $X_1 = X_2 = [0, 1]$ , and the payoff functions are defined by

$$u_i(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 = 1 \\ x_i & \text{otherwise} \end{cases} \quad i = 1, 2.$$

The game is not recursively diagonal transfer continuous on  $X$ . To see this, for  $x = (1, 1)$  and  $y \in (0, 1) \times (0, 1)$ , we have  $U(y, x) > U(x, x)$ . We then cannot find any  $y^0 \in X$  and neighborhood  $\mathcal{V}_x$  of  $x$  such that  $U(z, x') > U(x', x')$  for every deviation profile  $z$  that is upset directly or indirectly by  $y^0$  for all  $x' \in \mathcal{V}_x$ . We show this by considering two cases.

Case 1.  $y^0 \neq (1, 1)$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choosing strategy profiles  $z \in X$  and  $x' \in \mathcal{V}_{(1,1)}$  such that  $y_1^0 + y_2^0 < z_1 + z_2 < x'_1 + x'_2$ , we then have  $U(z, y^0) > U(y^0, y^0)$  but  $U(z, x') < U(x', x')$ .

Case 2.  $y^0 = (1, 1)$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choosing strategy profiles  $z \in X$  and  $x' \in \mathcal{V}_{(1,1)}$  such that  $0 < z_1 + z_2 < x'_1 + x'_2 < y_1^0 + y_2^0$ , we then have  $U(z, y^0) > U(y^0, y^0)$  but  $U(z, x') < U(x', x')$ .

Thus, we cannot find any  $y^0 \in X$  and any neighborhood  $\mathcal{V}_0$  of  $(1, 1)$  such that  $U(z, x') > U(x', x')$  for every deviation profile  $z$  that is upset by  $y^0$  for all  $x' \in \mathcal{V}_x$ . Hence, the game is not recursively diagonal transfer continuous on  $X$ , and therefore, by Theorem 3.1, there is no pure strategy Nash equilibrium on  $X$ .

**EXAMPLE 3.4 (KARLIN)** Consider games of “timing” or “silent duel”, which have been studied by Karlin (1959), Owen (1968), Jones (1980), and Dasgupta and Maskin (1986). These are symmetric two-person zero-sum games on the unit square so that  $n = 2$ ,  $X_1 = X_2 = [0, 1]$ , and  $U(x, x) = 0$  for all  $x \in X$ . The version called the “silent duel” has player 1’s payoff function of the form:

$$u_1(x_1, x_2) = \begin{cases} x_1 - x_2 + x_1x_2, & \text{if } x_1 < x_2 \\ 0, & \text{if } x_1 = x_2 \\ x_1 - x_2 - x_1x_2, & \text{if } x_1 > x_2 \end{cases} .$$

Consider  $x = (x_1, x_2) = (1, 1)$ . It can be verified that  $U(y, x) > U(x, x)$  implies that  $y$  must satisfy one of the following three sets of conditions: (1)  $y_1 + y_2 > 1$ ,  $y_1 < 1$ , and  $y_2 < 1$ ; (2)  $y_1 = 1$  and  $y_2 > 1/2$ , and (3)  $y_2 = 1$  and  $y_1 > 1/2$ . We then cannot find any  $y^0 \in X$  and neighborhood  $\mathcal{V}_x$  of  $x$  such that  $U(z, x') > U(x', x')$  for every deviation profile  $z$  that is upset directly or indirectly by  $y^0$  for all  $x' \in \mathcal{V}_x$ . To show this, four cases need to be considered.

Case 1.  $y_1^0 < 1$  and  $y_2^0 < 1$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choose strategy profiles  $z \in X$  and  $x' \in \mathcal{V}_{(1,1)}$  such that  $y_1^0 < z_1 < x'_1$  and  $y_2^0 < z_2 < x'_2$ . Since  $u_1(y_1, y_2)$  and  $u_2(y_1, y_2) = -u_1(y_1, y_2)$  are both increasing in  $y_1$  and  $y_2$ , respectively, we have  $u_1(z_1, y_2^0) - u_1(y_1^0, y_2^0) > 0$  and  $u_2(y_1^0, z_2) - u_2(y_1^0, y_2^0) = u_1(y_1^0, y_2^0) - u_1(y_1^0, z_2) > 0$ ,  $u_1(z_1, x'_2) - u_1(x'_1, x'_2) < 0$  and  $u_2(x'_1, z_2) - u_2(x'_1, x'_2) = u_1(x'_1, x'_2) - u_1(x'_1, z_2) < 0$ . Thus, we have  $U(z, y^0) > U(y^0, y^0)$  but  $U(z, x') < U(x', x')$ .

Case 2.  $y_1^0 = 1$  and  $y_2^0 < 1$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choose strategy profiles  $z \in X$  and  $x' \in \mathcal{V}_{(1,1)}$  such that  $y_1^0 = z_1 = x'_1$  and  $y_2^0 < z_2 < x'_2$ . Then, by the monotonicity of  $u_1(y_1, y_2)$  and  $u_2(y_1, y_2) = -u_1(y_1, y_2)$ , we have  $U(z, y^0) > U(y^0, y^0)$  but  $U(z, x') < U(x', x')$ .

Case 3.  $y_1^0 < 1$  and  $y_2^0 = 1$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choose strategy profiles  $z \in X$  and  $x' \in \mathcal{V}_{(1,1)}$  such that  $y_1^0 < z_1 < x'_1$  and  $y_2^0 = z_2 = x'_2$ . Then, by similar reasoning, we have  $U(z, y^0) > U(y^0, y^0)$  but  $U(z, x') < U(x', x')$ .

Case 4.  $y_1^0 = 1$  and  $y_2^0 = 1$ . Then, for any neighborhood  $\mathcal{V}_{(1,1)}$  of  $(1, 1)$ , choose strategy profiles  $z \in X$  and  $x' \in \mathcal{V}_{(1,1)}$  such that  $1/2 < z_1 < x'_1$  and  $1/2 = z_2 = x'_2$ . We then have  $u_1(z_1, y_2^0) - u_1(y_1^0, y_2^0) = 2z_1 - 1 > 0$  and  $u_2(y_1^0, z_2) - u_2(y_1^0, y_2^0) = u_1(y_1^0, y_2^0) - u_1(y_1^0, z_2) = 2z_2 - 1 > 0$ ,  $u_1(z_1, x'_2) - u_1(x'_1, x'_2) < 0$  and  $u_2(x'_1, z_2) - u_2(x'_1, x'_2) = u_1(x'_1, x'_2) - u_1(x'_1, z_2) < 0$ , and consequently,  $U(z, y^0) > U(y^0, y^0)$  but  $U(z, x') < U(x', x')$ .

Thus, we cannot find any  $y^0 \in X$  and any neighborhood  $\mathcal{V}_0$  of  $(1, 1)$  such that  $U(z, x') > U(x', x')$  for every deviation profile  $z$  that is upset by  $y^0$  for all  $x' \in \mathcal{V}_x$ . Hence, the game is not recursively diagonal transfer continuous on  $X$ , and therefore, by Theorem 3.1, there is no pure

strategy Nash equilibrium on  $X$ .

The version called the “noisy duel” has player 1’s payoff function of the form:

$$u_1(x_1, x_2) = \begin{cases} 2x_1 - 1, & \text{if } x_1 < x_2 \\ 0, & \text{if } x_1 = x_2 \\ 1 - 2x_2, & \text{if } x_1 > x_2 \end{cases} .$$

In this game, the payoff function  $u_i(x_1, x_2)$  is neither diagonally transfer continuous nor quasi-concave in  $y_i$  for  $i = 1$ . Therefore, theorems in Baye, Tian, and Zhou (1993) and Reny (1999) are not applicable.

Suppose  $U(y, x) > U(x, x)$  for  $x = (x_1, x_2) \in X$  and  $y = (y_1, y_2) \in X$ . Let  $y^0 = (1/2, 1/2)$  and  $\mathcal{V}_x$  be a neighborhood of  $x$ . Since  $U(y, y^0) \leq U(y^0, y^0)$  for all  $y \in Y$ , it is impossible to find any securing strategy profile  $y^1$  such that  $U(y^1, y^0) > U(y^0, y^0)$ . Hence, the recursive diagonal transfer continuity holds. Therefore, by Theorem 3.1, this game has a pure strategy Nash equilibrium.

**EXAMPLE 3.5 (VARIAN)** Consider a two-person game with nonnegative price strategies  $p_1$  and  $p_2$ . Thus  $Z_1 = Z_2 = [0, r]$  with  $r > 0$ . The payoffs  $u_i(p_1, p_2)$  are given by the functions

$$u_i(p_1, p_2) = \begin{cases} p_i(I + \mu)k & \text{if } p_i < p_j \\ p_i(\frac{I}{2} + \mu)k & \text{if } p_1 = p_2 \\ p_i\mu k & \text{if } p_i > p_j \end{cases} \quad i = 1, 2.$$

This game has a number of interpretations. Varian (1980) interprets  $I$  to be the number of informed consumers, who will shop at the firm charging the lowest price, while  $2\mu$  is the number of uninformed consumers, who allocate themselves equally across the two firms. Thus each firm sells to  $\mu$  uninformed consumers automatically, but gets the  $I$  informed consumers only if it succeeds in setting the lowest price.

It is well known that this game has no pure strategy Nash equilibrium (cf. Varian (1980); Baye, Kovenock, and de Vries (1992)), and in fact we can similarly verify that  $U$  is not recursively diagonal transfer continuous. Thus, by Theorem 3.1, the blame unambiguously lies squarely on the fact that the game is not recursively diagonal transfer continuous.

## 4 Full Characterization of Symmetric Pure Strategy Nash Equilibria

The techniques developed in the previous section can be used to fully characterize the existence of symmetric pure strategy Nash equilibrium. Throughout this section, we assume that the strategy spaces for all players are the same. As such, let  $X_0 = X_1 = \dots = X_n$ . If in addi-

tion,  $u_1(y, x, \dots, x) = u_2(x, y, x, \dots, x) = \dots, u_n(x, \dots, x, y)$  for all  $x, y \in X$ , we say that  $G = (X_i, u_i)_{i \in I}$  is a quasi-symmetric game.

**DEFINITION 4.1** A Nash equilibrium  $(x_1^*, \dots, x_n^*)$  of a game  $G$  is said to be *symmetric* if  $x_1^* = \dots = x_n^*$ .

For convenience, we denote, for each player  $i$ , and for all  $x, y \in X_0$ ,  $u_i(x, \dots, y, \dots, x)$  the function  $u_i$  evaluated at the strategy in which player  $i$  chooses  $y$  and all others choose  $x$ .

Define a quasi-symmetric function  $\psi : X_0 \times X_0 \rightarrow \mathbb{R}$  by

$$\psi(y, x) = u_i(x, \dots, y, \dots, x). \quad (4)$$

Since  $G$  is quasi-symmetric,  $x^*$  is a symmetric pure strategy Nash equilibrium if and only if  $\psi(y, x^*) \leq \psi(x^*, x^*)$  for all  $y \in X_i$ .

We then have the following theorem.

**THEOREM 4.1** *Suppose a game  $G = (X_i, u_i)_{i \in I}$  is quasi-symmetric and compact. Then it possesses a symmetric pure strategy Nash equilibrium if and only if  $\psi(y, x)$  defined by (4) is recursively diagonal transfer continuous on  $X$ .*

**PROOF.** The proof is the same as that of Theorem 3.1 provided  $U$  is replaced by  $\psi$ , thus it is omitted here. ■

Theorem 4.1 strictly generalizes all the existing results on the existence of symmetric pure strategy Nash equilibrium such as those in Reny (1999).

**EXAMPLE 4.1 (BAGH AND JOFRE)** The following two-person concession quasi-symmetric game on the unit square considered by Bagh and Jofre (2006) is a special case of a class of timing games on the unit square considered by Reny (1999). The payoffs are:

$$u_i(x_1, x_2) = \begin{cases} 10, & \text{if } x_i < x_{-i} \\ 1, & \text{if } x_i = x_{-i} < 0.5 \\ 0, & \text{if } x_i = x_{-i} \geq 0.5 \\ -10, & \text{if } x_i > x_{-i} \end{cases}.$$

Note that the payoffs are not quasiconcave (nor are they quasiconcave along the diagonal of the unit square). We now show that  $\psi$  is recursively diagonal transfer continuous, and thus the game possesses a symmetric pure strategy equilibrium. Indeed, let  $\psi(y, x) = u_i(y, x)$ . Suppose  $\psi(y, x) > \psi(x, x)$  for  $x = (x_1, x_2) \in X$  and  $y = (y_1, y_2) \in X$ . Let  $y^0 = (0, 0)$  and  $\mathcal{V}_x$  be a neighborhood of  $x$ . It is clear that  $\psi(y, y^0) \leq \psi(y^0, y^0)$  for all  $y \in Y$ , and thus it is impossible

to find any securing strategy profile  $y^1$  such that  $\psi(y^1, y^0) > \psi(y^0, y^0)$ . Hence, the recursive diagonal transfer continuity holds, and thus by Theorem 4.1, this game has a pure strategy Nash equilibrium.

**EXAMPLE 4.2 (HENDRICKS AND WILSON)** Consider the concession quasi-symmetric game between two players studied by Hendricks and Wilson (1983), Simon (1987), and Reny (1999). The players must choose a time  $x_1, x_2 \in [0, 1]$  to quit the game. The player who quits last wins, although conditional on winning, quitting earlier is preferred. If both players quit at the same time, the unit prize is divided evenly between them. Then payoffs are:

$$u_i(x_1, x_2) = \begin{cases} -x_i, & \text{if } x_i < x_{-i} \\ 1/2 - x_i, & \text{if } x_i = x_{-i} \\ 1 - x_i, & \text{if } x_i > x_{-i} \end{cases} .$$

Note that the payoffs are not quasiconcave (nor are they quasiconcave along the diagonal of the unit square) although  $U$  is diagonally transfer continuous by Proposition 2.(e) in Baye, Tian, and Zhou (1993). We now show that  $\psi$  is not recursively diagonal transfer continuous, and thus the game does not possess a symmetric pure strategy equilibrium. To see this, consider  $x = 0$ . It is clear that  $\psi(y, x) = u_i(y, 0) > u_i(0, 0)$  implies that  $0 < y < 1/2$ . We then cannot find any  $y^0 \in X_0$  and neighborhood  $\mathcal{V}_x$  of  $x$  such that  $\psi(z, x') > \psi(x', x')$  for every deviation profile  $z$  that is upset by  $y^0$  for all  $x' \in \mathcal{V}_x$ . We show this by considering two cases.

Case 1.  $y^0 \neq 0$ . Then, for any neighborhood  $\mathcal{V}_0$  of 0, choose a strategy profile  $z \in [0, 1]$  and a strategy profile  $x' \in \mathcal{V}_0$  such that  $\max\{1/2 + \epsilon, y^0\} < z < 1/2 + y^0$  and  $x' < \epsilon$ , where  $0 < \epsilon < \min\{1/2, y^0\}$ . Then, by  $z > y^0$  and  $1 - z > 1/2 - y^0$ , we have  $\psi(z, y^0) > \psi(y^0, y^0)$ . However, since  $z > x'$  and  $1/2 + x' < 1/2 + \epsilon < z$ , we have  $1 - z < 1/2 - x'$ , and consequently  $\psi(z, x') < \psi(x', x')$ .

Case 2.  $y^0 = 0$ . Note that  $\psi(z, y^0) > \psi(y^0, y^0)$  if and only if  $0 < z < 1/2$ . Then, for any neighborhood  $\mathcal{V}_0$  of 0, choosing a positive number  $\epsilon$  such that  $(\epsilon/2, \epsilon) \subset \mathcal{V}_0$ ,  $z = \epsilon/2$  and a strategy profile  $x' \in \mathcal{V}_0$  such that  $x' \in (\epsilon/2, \epsilon)$ , we have  $\psi(z, y^0) > \psi(y^0, y^0)$  but  $\psi(z, x') = -z < 1/2 - x' = \psi(x', x')$ .

Thus, we cannot find any  $y^0 \in [0, 1]$  and any neighborhood  $\mathcal{V}_0$  of 0 such that  $\psi(z, x') > \psi(x', x')$  for every deviation profile  $z$  that is upset by  $y^0$  for all  $x' \in \mathcal{V}_x$ . Therefore,  $\psi$  is not recursively diagonal transfer continuous on  $X_0$ , and thus, by Theorem 4.1, there is no symmetric pure strategy Nash equilibrium on  $X$ .

Besides, since all the games in Examples 3.1-3.5 are quasisymmetric, it is even easier to show the existence/nonexistence of pure strategy (symmetric) Nash equilibrium by working on a single



payoff function  $\psi$ , instead of the aggregate payoff function  $U$  that is the sum of individual payoff functions.

Similar to Corollary 3.1, the following corollary is reached.

**COROLLARY 4.1** *Suppose a game  $G = (X_i, u_i)_{i \in I}$  is quasi-symmetric, compact, and deviational transitive. Then it possesses a pure strategy symmetric Nash equilibrium if and only if  $\psi(x, y)$  defined by (4) is 1-recursively diagonal transfer continuous on  $X$ .*

We now provide some sufficient conditions for a game to be deviational transitive and 1-recursively diagonal transfer continuous.

**DEFINITION 4.2** A game  $G = (X_i, u_i)_{i \in I}$  is diagonally monotonic if for each  $\bar{x} \in X_0$ ,  $u_i(\bar{x}, \dots, x, \dots, \bar{x})$  is either: (i) decreasing in  $x$  on  $X_0 \setminus \bar{x}$  or (ii) increasing in  $x$  on  $X_0 \setminus \bar{x}$ .

The following propositions provide sufficient conditions for a game  $G = (X_i, u_i)_{i \in I}$  to be deviational transitive and 1-recursively diagonal transfer continuous.

**PROPOSITION 4.1** *Suppose  $X_i$  is a subset of  $\mathbb{R}$ . If a game  $G = (X_i, u_i)_{i \in I}$  is diagonally monotonic,  $\psi$  is deviational transitive.*

**PROOF.** We only need to show the case where  $\psi$  is nondecreasing in  $x$ . The proof of the case where  $\psi$  is non-increasing in  $x$  is similar.

We need to show that  $\psi(z, y) > \psi(y, y)$  and  $\psi(y, x) > \psi(x, x)$  imply that  $\psi(z, x) > \psi(x, x)$ . Indeed, when  $\psi(y, x) > \psi(x, x)$ , i.e.,  $u_i(x, \dots, y, \dots, x) > u_i(x, \dots, x)$ , we have  $y > x$  by monotonicity of  $u_i(x, \dots, y, \dots, x)$ . When  $\psi(z, y) > \psi(y, y)$ , we have  $z > y$  by monotonicity of  $u_i(x, \dots, y, \dots, x)$ . Thus we have  $z > y > x$ . Then, by monotonicity of  $u_i(x, \dots, y, \dots, x)$ , we have

$$u_i(x, \dots, z, \dots, x) > u_i(x, \dots, y, \dots, x) > u_i(x, \dots, x)$$

and therefore  $\psi(y, x) = u_i(x, \dots, z, \dots, x) > \psi(x, x)$ , which means  $\psi$  is deviational transitive.

■

**PROPOSITION 4.2** *Suppose  $X_i$  is a subset of  $\mathbb{R}$  and a game  $G = (X_i, u_i)_{i \in I}$  is diagonally monotonic. Any of the following conditions implies it is 1-recursively diagonal transfer continuous in  $x$ .*

- (i)  $u_i(\bar{x}, \dots, x, \dots, \bar{x})$  is continuous in  $x$ ;
- (ii)  $u_i(\bar{x}, \dots, x, \dots, \bar{x})$  is upper semi-continuous in  $x$ ;

**PROOF.** We only need to prove the case of upper semi-continuity and the case where  $\psi$  is increasing in  $x$ . The proof of the case where  $\psi$  is decreasing in  $x$  is similar.

Suppose  $\psi(y, x) > \psi(x, x)$  for  $x, y \in X_0$ . We need to show that there exists a point  $y^0 \in X_0$  and a neighborhood  $\mathcal{V}_x$  of  $x$  such that  $\psi(z, \mathcal{V}_x) > \psi(\mathcal{V}_x, \mathcal{V}_x)$  whenever  $\psi(z, y^0) > \psi(y^0, y^0)$ . Indeed, since  $\psi(y, x) > \psi(x, x)$ , we have  $y > x$  by diagonal monotonicity of  $\psi$ . Let  $y^0 = x + \delta < y$  for some positive  $\delta > 0$ . We have  $\psi(y^0, x) > \psi(x, x)$  by diagonal monotonicity of  $\psi$ . Then, by upper semi-continuity, there is a neighborhood  $\mathcal{V}_x = \{x' \in X_0 : |x' - x| < \epsilon\}$  such that  $\psi(y^0, x) > \psi(x', x)$  for all  $x' \in \mathcal{V}_x$ . Since  $u_i(\bar{x}, \dots, x, \dots, \bar{x})$  is nondecreasing in  $x$  on  $X_0 \setminus \bar{x}$  for all  $\bar{x} \in X_0$ , we particularly have  $\psi(y^0, x') > \psi(x', x')$  for all  $\bar{x} = x' \in \mathcal{V}_x$ . Thus, whenever  $\psi(z, y^0) > \psi(y^0, y^0)$ , we have  $z > y^0$  by diagonal monotonicity of  $\psi$ , and therefore, we have  $\psi(z, x') > \psi(y^0, x') > \psi(x', x')$  for all  $x' \in \mathcal{V}_x$ , which means  $\psi$  is 1-recursively diagonal transfer continuous in  $x$ . ■

**EXAMPLE 4.3 (BAYE AND KOVENOCK; BAYE, TIAN, AND ZHOU)** Consider the two-player quasi-symmetric game studied by Baye and Kovenock (1993), and Baye, Tian, and Zhou (1993). Two duopolists have zero costs and set prices  $(p_1, p_2)$  on  $Z = [0, T] \times [0, T]$ . The payoff functions are (for  $0 < c < T$ ):

$$u_i(p_1, p_2) = \begin{cases} p_i & \text{if } p_i \leq p_{-i} \\ p_i - c & \text{otherwise} \end{cases}.$$

One can interpret the game as a duopoly in which each firm has committed to pay brand loyal consumers a penalty of  $c$  if the other firm beats its price.<sup>7</sup> These payoffs are neither quasiconcave nor continuous. However, the game is diagonally monotonic and upper semicontinuous, and thus it is 1-recursively diagonal transfer continuous. Thus, by Corollary 4.1, this game possesses a symmetric pure strategy equilibrium.

We end this section by discussing how Theorem 4.1 covers Reny's result as a corollary.

The game  $G = (X_i, u_i)_{i \in I}$  is diagonally quasiconcave if  $X_i$  is convex, and for every player  $i$ , all  $x^1, \dots, x^m \in X$  and all  $x \in \text{co}\{x^1, \dots, x^m\}$ ,  $u_i(x, \dots, x) \geq \min_{1 \leq k \leq m} u_i(x, \dots, x^k, \dots, x)$ .

**DEFINITION 4.3** A game  $G = (X_i, u_i)_{i \in I}$  is diagonally better-reply secure if, whenever  $(x^*, u^*) \in X \times \mathbb{R}$  is in the closure of the graph of its payoff function and  $(x^*, \dots, x^*)$  is not an equilibrium, there is some player  $i$ ,  $y \in X_0$ , and an open neighborhood  $\mathcal{V}_{x^*}$  of  $x^*$  such that  $\psi(y, x') > \psi(x^*)$  for all  $x' \in \mathcal{V}_{x^*}$ .

<sup>7</sup>See Baye and Kovenock (1993) for an alternative formulation with both brand loyal and price conscious consumers, whereby a firm commits to pay a penalty if it does not provide the best price in the market.

**COROLLARY 4.2** (RENY (1999)) *If  $G = (X_i, u_i)_{i \in I}$  is quasi-symmetric, compact, diagonally quasiconcave, and better-reply secure, then it possesses a symmetric pure strategy Nash equilibrium.*

## 5 Full Characterization of Mixed Strategy Nash Equilibria

In this section, we fully characterize the existence of mixed strategy Nash equilibrium as corollaries to the pure strategy existence results derived in the previous sections. We assume throughout this section that each  $u_i$  is both bounded and measurable, and  $X_i$  is a compact Hausdorff space so we call  $G = (X_i, u_i)_{i \in I}$  a compact, Hausdorff game. Consequently, if  $M_i$  denotes the set of (regular, countably additive) probability measures on the Borel subsets of  $X_i$ ,  $M_i$  is compact in the weak\* topology. Extend each  $u_i$  to  $\prod_{i \in I} M_i$  by defining  $u_i(\mu) = \int_X u_i(x) d\mu$  for all  $\mu \in M$ , and let  $\bar{G} = (M_i, u_i)_{i \in I}$  denote the mixed extension of  $G$ , where  $M = \prod_{i \in I} M_i$ .

The definitions of recursive diagonal transfer continuity, etc. given in the previous sections, apply in the obvious ways to the mixed extension  $\bar{G}$ .

We now present the mixed strategy implications of Theorem 3.1.

**THEOREM 5.1** *Suppose that  $G = (M_i, u_i)_{i \in I}$  is a compact, Hausdorff game. Then  $G$  possesses a mixed strategy Nash equilibrium if and only if its mixed extension,  $\bar{G}$ , is recursively diagonal transfer continuous.*

This theorem strictly generalizes all the existence results on the mixed strategy equilibrium in the literature such as those in Nash (1950), Glicksberg (1952), Mas-Colell (1984), Dasgupta and Maskin (1986), Robson (1994), Simon (1987), Reny (1999), Monteiro and Page (2007), and Nessah and Tian (2008). Any sufficient conditions imposed in the existing theorems on the existence of mixed strategy Nash equilibrium imply the recursive diagonal transfer continuity of  $\bar{U}(\cdot)$ . To illustrate this, we present here the results of Monteiro and Page (2007), and Nessah and Tian (2008) as corollaries of Theorem 5.1. We first state some definitions introduced by them.

Monteiro and Page (2007) introduce the concept of uniform payoff security for games with compact separable metric strategy spaces and payoffs bounded and measurable in players' strategies. They show that if a game is compact and uniformly payoff secure, then its mixed extension  $\bar{G}$  is payoff secure.

**DEFINITION 5.1** The game  $G$  is uniformly payoff secure if for every player  $i \in I$ , every  $x_i \in X_i$ , and every  $\epsilon > 0$ , there is a strategy  $\bar{x}_i \in X_i$  such that for every  $y_{-i} \in X_{-i}$ , there exists a neighborhood  $\mathcal{V}_{y_{-i}}$  of  $y_{-i}$  such that  $u_i(\bar{x}_i, x'_{-i}) \geq u_i(x_i, y_{-i}) - \epsilon$ , for all  $x'_{-i} \in \mathcal{V}_{y_{-i}}$ .

**COROLLARY 5.1** [Monteiro and Page (2007)] *If a game  $G = (X_i, u_i)_{i \in I}$  is compact, bounded, separable metric, uniformly payoff secure, and has reciprocally upper semicontinuous payoffs, then it possesses a mixed strategy Nash equilibrium.*

Nessah and Tian (2008) introduce the concept of uniform transfer continuity, and show that if a game  $G = (X_i, u_i)_{i \in I}$  is uniformly transfer continuous, then the mixed extension  $\bar{G}$  is weakly transfer continuous.

**DEFINITION 5.2** The game  $G$  is said to be *uniformly transfer continuous* if for every player  $i \in I$ , every  $x_i \in X_i$ , and every  $\epsilon > 0$ , there is a strategy  $\bar{x}_i \in X_i$  such that for every  $y_{-i} \in X_{-i}$ , there exists a neighborhood  $\mathcal{V}_{(x_i, y_{-i})}$  of  $(x_i, y_{-i})$  such that

$$u_i(\bar{x}_i, x'_{-i}) + \epsilon \geq u_i(x_i, y_{-i}) \geq u_i(z) - \epsilon, \text{ for all } z \in \mathcal{V}_{(x_i, y_{-i})}.$$

**COROLLARY 5.2** [Nessah and Tian (2008)] *If a game  $G = (X_i, u_i)_{i \in I}$  is compact, bounded, Hausdorff, and uniformly transfer continuous, then it possesses a mixed strategy Nash equilibrium.*

We now provide a full characterization on the existence of symmetric mixed strategy Nash equilibrium for quasi-symmetric games. For the following result only, let  $M_0$  denote the common set of mixed strategies for each player  $i$ .

Define an extended quasi-symmetric function  $\bar{\psi} : M_0 \times M_0 \rightarrow \mathbb{R}$  by

$$\bar{\psi}(\nu, \mu) = u_i(\mu, \dots, \nu, \dots, \mu). \tag{5}$$

Since  $\bar{G}$  is quasi-symmetric,  $\mu^*$  is a symmetric mixed strategy Nash equilibrium if and only if  $\bar{\psi}(\nu, \mu^*) \leq \bar{\psi}(\mu^*, \mu^*)$  for all  $\nu \in M_0$ .

**THEOREM 5.2** *Suppose that  $G = (M_i, u_i)_{i \in I}$  is a compact, quasi-symmetric, and Hausdorff game. Then  $G$  possesses a symmetric mixed strategy Nash equilibrium if and only if its mixed extension payoff  $\bar{\psi}(\nu, \mu)$  defined by (4) is recursively diagonal transfer continuous on  $M_0$ .*

This result covers Corollary 5.3 of Reny (1999) as a special case.

## 6 Full Characterization of Bayesian Nash Equilibrium

In this section, we provide a full characterization of Bayesian Nash equilibrium in an ex ante formulation of a Bayesian game, in which each player's beliefs are common prior. The existence of Bayesian Nash equilibrium in this formulation has been studied by Radner and Rosenthal (1982), Milgrom and Weber (1985), Vives (1990), and Zandt and Vives (2007). The full characterization

of Bayesian Nash equilibrium in an interim or incomplete-information formulation of a Bayesian game studied by Van Zandt (2007) can be similarly investigated.

Let the strategy spaces be compact subsets of topological spaces and  $T_i$  the set of types of player  $i$ , a non-empty complete separable metric space. Denote by  $T$  the Cartesian product of the sets of types of the players,  $T = \prod_{i \in I} T_i$ . The common beliefs of the players are represented by  $\mu$ , a probability measure on the Borel subsets of  $T$ . The measure  $\mu_i$  will represent the marginal on  $T$ . The payoff to player  $i$  is given by  $u_i : X \times T \rightarrow \mathbb{R}$ , Borel measurable and bounded. A (pure) strategy for player  $i$  is a (Borel measurable) map  $\alpha_i : T_i \rightarrow X_i$  that assigns an action to every possible type of the player. Let  $\Sigma_i(\mu_i)$  denote the strategy space of player  $i$  when we identify strategies  $\sigma_i$  and  $u_i$  if they are equally  $\mu_i$ -almost surely (a.s.)

Let

$$\Pi_i(\sigma) = \int_T u_i(\sigma_1(t_1), \dots, \sigma_n(t_n), t_i) d\mu(dt) \quad (6)$$

be the expected payoff to player  $i$  when agent  $j$  uses strategy  $\sigma_j$ ,  $j \in I$ .

A strategy  $\sigma^*$  is a *Bayesian Nash equilibrium* of a game if

$$\Pi_i(\sigma^*) \geq \Pi_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Sigma_i.$$

There are several results available in the literature on the existence of pure strategy equilibria in Bayesian games [e.g. Radner and Rosenthal (1982), Milgrom and Weber (1985), and Vives (1990)].

As a direct corollary of Theorem 3.1, the following result strictly generalizes all the existing results on the existence of Bayesian Nash equilibrium.

**THEOREM 6.1** *Suppose a Bayesian-Nash game  $\Gamma = (\Sigma_i, \Pi_i)_{i \in I}$  is compact. Then it possesses a Bayesian Nash equilibrium if and only if  $\Pi_i$  is recursively diagonal transfer continuous on  $X$ .*

This theorem strictly generalizes the existing results such as those in Ray and Rosenthal (1982), Milgrom and Weber (1985), Vives (1990), Athey (2001), Reny (2006), Van Zandt (2007), and Zandt and Vives (2008) as special cases.

## 7 Economic Applications

The approach and main result developed in the paper can also allow us to ascertain the existence of equilibria in important classes of economic games. As an application, in this section, we show how they can be employed to fully characterize the existence of competitive (or Walrasian) equilibrium for a certain class of economies.

One of the great achievements of economic theory in the last sixty years is the general equilibrium theory. The proof of the existence of a competitive equilibrium is generally considered one of the most important and robust results of economic theory. There are many different ways of establishing the existence of competitive equilibria, including the “excess demand approach” by showing that there is a price at which excess demand can be non-positive.

The significance of such an approach lies partly in the fact that demand and/or supply may not be continuous or even not be necessarily derived from profit maximizing behavior of price taking firms, but is determined by prices in completely different ways. It is well known that Walrasian equilibrium precludes the existence of an equilibrium in the presence of increasing returns to scale and assumes price-taking and profit-maximizing behavior. Some other alternative pricing rules then have been proposed such as loss-free, average cost, marginal cost, voluntary trading, and quantity-taking pricing rules in the presence of increasing returns to scale or more general types of non-convexities— cf. Beato (1982), Bonnisseau and Cornet (1990), Quinzii(1992), Tian (2009) and the references therein. There is a large literature on the existence results using the excess demand approach, such as those in Gale (1955), Nikaido (1956, 1968, 1970), Debreu (1970, 1974, 1982), Sonnenschein (1972, 1973), Hildenbrand (1974), Hildenbrand and Kirman (1975), Grandmont (1977), Neufeind (1980), Aliprantis and Brown (1983), Hüsseinov (1999), Momi (2003), Quah (2008), etc.

We provide a complete solution to the existence of competitive equilibrium in economies with general excess demand functions,<sup>8</sup> in which commodities may be indivisible and excess demand functions may be discontinuous or do not have any structure except Walras’ law. We introduce a condition, called *recursive transfer lower semi-continuity*, which is necessary and sufficient for the existence of general equilibrium in such economies. Thus, our result strictly generalizes all the existing results on the existence of equilibrium in economies with excess demand functions.

Let  $\Delta$  be the closed  $L - 1$  dimensional unit simplex defined by

$$\Delta = \{p \in \mathbb{R}_+^L : \sum_{l=1}^L p^l = 1\}, \quad (7)$$

and let  $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L \cup \{\pm\infty\}$  denote the excess demand function of some economy. A very important property of excess demand function is Walras’ law, which can take either of the following three forms. The strong form of Walras’ law is given by

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \Delta,$$

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<sup>8</sup>In the case of strictly convex preferences and production sets, we obtain excess demand functions rather than correspondences.

the weak form of Walras' law is given by

$$p \cdot \hat{z}(p) \leq 0 \quad \text{for all } p \in \Delta,$$

and the interior form of Walras' law is given by:

$$p \cdot \hat{z}(p) = 0 \quad \text{for all } p \in \text{int } \Delta,$$

A price vector  $p^*$  is a *competitive or Walrasian equilibrium* if  $\hat{z}(p^*) \leq 0$ .

The equilibrium price problem is to find a price vector  $p$  which clears the markets for all commodities (i.e., the excess demand functions  $\hat{z}(p) \leq 0$  for the free disposal equilibrium price or  $\hat{z}(p) = 0$ ) under the assumption of Walras' law.

We say that price  $p$  *upsets* price  $q$  if  $p$  gives a higher value to  $q$ 's excess demand, i.e.  $p \cdot \hat{z}(q) > q \cdot \hat{z}(q) > 0$ .

**DEFINITION 7.1** (Recursive Upset Pricing) Let  $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L$  be an excess demand function. We say that a non-equilibrium price vector  $p^0 \in \Delta$  is *recursively upset* by  $p \in \Delta$  if there exists a finite set of price vectors  $\{p^1, p^2, \dots, p\}$  such that  $p^1 \cdot \hat{z}(p^0) > 0$ ,  $p^2 \cdot \hat{z}(p^1) > 0$ ,  $\dots$ ,  $p \cdot \hat{z}(p^{m-1}) > 0$ .

In words, a non-equilibrium price vector  $p^0$  is recursively upset by  $p$  means that there exist finite upsetting price vectors  $p^1, p^2, \dots, p^m$  with  $p^m = p$  such that  $p^0$ 's excess demand is not affordable at  $p^1$ ,  $p^1$ 's excess demand is not affordable at  $p^2$ , and  $p^{m-1}$ 's excess demand is not affordable at  $p^m$ . When the strong form of Walras' law holds, this implies that  $p^0$  is upset by  $p^1$ ,  $p^1$  is upset by  $p^2$ ,  $\dots$ ,  $p^{m-1}$  is upset by  $p$ .

**DEFINITION 7.2** (Recursive Transfer Lower Semi-Continuity) An excess demand function  $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L$  is said to be *recursively transfer lower semi-continuous* on  $\Delta$  if, whenever  $q \in \Delta$  is not a competitive equilibrium price vector, there exists some price  $p^0 \in \Delta$  (possibly  $p^0 = q$ ) and a neighborhood  $\mathcal{V}_q$  such that  $p \cdot \hat{z}(\mathcal{V}_q) > 0$  for any  $p$  that recursively upsets  $p^0$ .

Roughly speaking, recursive transfer lower semi-continuity of  $\hat{z}(\cdot)$  means that, whenever  $q$  is not a competitive equilibrium price vector, there exists another non-competitive equilibrium price vector  $p^0$  such that all excess demands in some neighborhood of  $q$  are not affordable at any price vector  $p$  that recursively upsets  $p^0$ . This implies that, if a competitive equilibrium fails to exist, then there is some non-equilibrium price vector  $q$  such that for every other price vector  $p^0$  and every neighborhood of  $q$ , excess demand of some price vector  $q'$  in the neighborhood becomes affordable at price vector  $p$  that recursively upsets  $p^0$ .

**REMARK 7.1** While continuity does not imply nor is implied by recursive diagonal transfer continuity, recursive transfer lower semi-continuity is weaker than lower semi-continuity. Indeed, when  $\hat{z}(\cdot)$  is lower semi-continuous,  $p \cdot \hat{z}(\cdot)$  is also lower semi-continuous for any nonnegative vector  $p$ , and thus we have  $p^m \cdot \hat{z}(q') > 0$  for all  $q' \in N(q)$  and  $p \in \Delta$ .

We then have the following theorem of competitive equilibrium in economies that have single-valued excess demand functions.

**THEOREM 7.1** *Suppose an excess demand function  $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L$  satisfies either the strong or the weak form of Walras' law. Then there exists a competitive price equilibrium  $p^* \in \Delta$  if and only if  $\hat{z}(\cdot)$  is recursively transfer lower semi-continuous on  $\Delta$ .*

**PROOF.** *Sufficiency* ( $\Leftarrow$ ). Define a function  $\phi : \Delta \times \Delta \rightarrow \mathfrak{R}$  by  $\phi(p, q) = p \cdot \hat{z}(p)$  for  $p, q \in \Delta$ . Since  $p \cdot \hat{z}(q) > 0$  and  $q \cdot \hat{z}(q) \leq 0$  for all  $p \in \Delta$  by Walras' law, we have  $\phi(p, q) > \phi(q, q)$  for all  $p, q \in \Delta$ . Then, by recursive transfer lower semi-continuity of  $\hat{z}(\cdot)$ ,  $\phi$  is recursively diagonal transfer continuous on  $\Delta$ .<sup>9</sup> Thus, by the sufficiency of Theorem 3.1, there exists  $p^* \in \Delta$  such that  $p \cdot \hat{z}(p^*) = \phi(p, p^*) \leq \phi(p^*, p^*) \leq 0$  for all  $p \in \Delta$ . Letting  $p^1 = (1, 0, \dots, 0)$ ,  $p^2 = (0, 1, 0, \dots, 0)$ , and  $p^L = (0, 0, \dots, 0, 1)$ , we have  $\hat{z}^l(p^*) \leq 0$  for  $l = 1, \dots, L$  and thus  $p^*$  is a competitive price equilibrium.

*Necessity* ( $\Rightarrow$ ). Suppose  $p^*$  is a competitive price equilibrium and  $p \cdot \hat{z}(q) > 0$  for  $q, p \in \Delta$ . Let  $p^0 = p^*$  and  $N(q)$  be a neighborhood of  $q$ . Since  $p \cdot \hat{z}(p^*) \leq 0$  for all  $p \in \Delta$ , it is impossible to find any sequence of finite price vectors  $\{p^1, p^2, \dots, p^m\}$  such that  $p^1 \cdot \hat{z}(p^0) > 0, p^2 \cdot \hat{z}(p^1) > 0, \dots, p^m \cdot \hat{z}(p^{m-1}) > 0$ . Hence, the recursive transfer lower semi-continuity holds trivially. ■

Theorem 7.1 assumes that the excess demand function is well defined for all prices in the closed unit simplex  $\Delta$ , including zero prices. However, when preferences are strictly monotone, excess demand functions are not well defined on the boundary of  $\Delta$ . Then, some boundary condition has been used to show the existence of competitive equilibrium in the case of an open set of price systems for which excess demand is defined —cf. Neufeind (1980). In this case, we of course cannot use Theorem 7.1 to fully characterize the existence of competitive equilibrium.

Nevertheless, like Theorem 3.1, Theorem 7.1 can be extended to the case of any set, especially the positive price open set, of price systems for which excess demand is defined. To do so, we introduced the following weak notion of recursive transfer lower semi-continuity.

<sup>9</sup>The reverse may not be true under the weak form of Walras' law. However, when the strong form of Walras' law holds,  $\hat{z}$  is recursively transfer lower semi-continuous if and only if  $\phi$  is recursively diagonal transfer continuous.



**DEFINITION 7.3** Let  $D$  be a subset of  $\Delta$ . An excess demand function  $\hat{z}(\cdot) : \Delta \rightarrow \mathbb{R}^L \cup \{\pm\infty\}$  is said to be *recursively transfer lower semi-continuous* on  $\text{int } \Delta$  with respect to  $D$  if, whenever  $q \in \Delta$  is not a competitive equilibrium price vector, there exists some price  $p^0 \in \Delta$  (possibly  $p^0 = q$ ) and a neighborhood  $\mathcal{V}_q$  such that for any finite subset of price vectors  $\{p^1, \dots, p^m\} \subset D$  with  $p^m = p$ , we have  $p \cdot \hat{z}(\mathcal{V}_q) > 0$ .

Now we have the following theorem that fully characterizes the existence of competitive equilibrium in economies with possibly indivisible commodity spaces and discontinuous excess demand functions.

**THEOREM 7.2** Let  $S$  be a subset of  $\Delta$ .<sup>10</sup> Suppose an excess demand function  $\hat{z}(\cdot) : S \rightarrow \mathbb{R}^L$  satisfies Walras' law:  $p \cdot \hat{z}(p) = 0$  for all  $p \in S$ . Then there is a competitive price equilibrium  $p^* \in S \subseteq \Delta$  if and only if there exists a compact subset  $D \subseteq S$  such that  $\hat{z}(\cdot)$  is recursively transfer lower semi-continuous on  $S$  with respect to  $D$ .

**PROOF.** The proof is essentially the same as the one of Theorem 3.2, and it is omitted here.

■

Theorem 7.2 then strictly generalizes all the existing results on the existence of competitive equilibrium in economies with single-valued excess demand functions, such as those in Gale (1955), Nikaido (1956), Debreu (1970, 1982), Hildenbrand (1974), Hildenbrand and Kirman (1975), Grandmont (1977), Neufeind (1980), Aliprantis and Brown (1983), Hüsseinov (1999), Momi(2003), and Quah (2008).

## 8 Conclusion

The existing results only give sufficient conditions for the existence of equilibrium, and no complete solution to the question of the existence of equilibrium in general games has been given in the literature. This paper fills this gap by providing a full characterization of equilibrium in games with arbitrary strategy spaces and payoffs. We fully characterize the existence of pure strategy Nash equilibrium in games with general topological strategy spaces that may be discrete, continuum, non-compact or non-convex and payoff functions that may be discontinuous or do not have any form of quasi-concavity. We establish a condition, called *recursive diagonal transfer continuity*, which is both necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs. As such, it strictly generalizes all the existing

<sup>10</sup>Say, the domain  $S$  is an open set that is given by  $\text{int } \Delta$ .

theorems on the existence of pure strategy Nash equilibrium. Recursive diagonal transfer continuity also permits full characterization results on the existence of symmetric pure strategy, mixed strategy Nash, and Bayesian Nash equilibria in games with general strategy spaces and payoffs. As an application of our approach and main result, we also fully characterize the existence of competitive equilibrium for economies with excess demand functions.

We end the paper by remarking that characterization results are mainly for the purpose of identifying whether or not a game has an equilibrium, but not whether it is easy to check. Recursive diagonal transfer continuity provides a way of understanding equilibrium, more than necessarily providing a way to check its existence. Even so, in the paper, we use many known economic examples to illustrate that it is useful to employ recursive diagonal transfer continuity to check the existence of equilibrium, especially the nonexistence of pure strategy Nash equilibrium. Nevertheless, Nessah and Tian (2008) develop some very weak sufficient conditions that are relatively easy to check and generalize most of the existing results for the existence of equilibrium in discontinuous games. A potential future work may be attempted to find ways of applying recursive diagonal transfer continuity as a useful tool for establishing equilibrium.

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