Principle of Symmetry for Network Topology with Applications to Some Networks

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Abstract—A number of Cayley-graph interconnection structures, such as cube-connected cycles, butterfly and biswapped networks, are known to be derivable by unified group semidirect product construction. In this paper, we extend these known group semidirect product constructions via a general algebraic construction based on group semidirect product. We show that under certain conditions, graphs based on the constructed groups are also Cayley graphs when graphs of the original groups are Cayley graphs. Thus, our results present a general mathematical framework-symmetry for synthesizing and exploring interconnection networks that offer many excellent properties such that lower node degrees, and thus smaller VLSI layout and simpler physical packaging of the same size and lower diameters, and thus lower delay of networks. Our constructions also lead to new insights, as well as new concrete results, for previously known interconnection schemes such as cube-connected cycles and biswapped networks.

Index Terms—Symmetry, Distributed system, Cayley graph, group semidirect product, Interconnection network, Network diameter, Parallel processor architecture, VLSI realization

I. INTRODUCTION

The fact that Cayley (di)graphs and coset graphs are excellent models for interconnection networks, studied in connection with parallel processing and distributed computation, is widely acknowledged [1], [2], [4], [12], [14], [15]. Many well-known interconnection networks are Cayley (di)graphs or coset graphs. For example, hypercube, butterfly, and cube-connected cycles networks are Cayley (di)graphs, while de Bruijin and shuffle-exchange networks are Cayley coset graphs [4], [11]. Unfortunately, much of the extensive body of work on interconnection networks consists of ad hoc design and evaluation: proposing a new interconnection scheme and showing it to be superior to some previously studied network(s) with respect to one or more performance or complexity attributes. Whereas Cayley (di)graphs have been used to explain and unify interconnection networks with some success, much work remains to be done.

Heydemann [4] believes that there is lack of general theorems pertaining to Cayley (di)graphs and advocates the exploitation of more group theory to study their properties.

One approach to reducing the implementation complexity of a network, and hence increasing its performance with constant cost, is systematic pruning of links by group semidirect product construction. Under certain conditions, when the original networks are Cayley graphs, so are the pruned versions [5], [9], [10], [16]. Thus, the pruned networks maintain node symmetry, while benefiting from lower node degree, sparser wiring, simpler layout, lower network diameter and so on. For example, in one type of construction based on torus (a Cayley graph of a finite commutative group G), defining a new group semiproduct product operator ⊗ on G leads to the group (G, ⊗) and, thereby, a Cayley-graph formulation of the pruned network [16]. Certain known interconnection architectures, such as cube-connected cycles, butterfly, and biswapped networks, may be constructed in this manner [4], [14], [16].

In this paper, we propose a general method for deriving new groups from given finite groups. Our method generalizes previously proposed pruning schemes and establishes a systematic and uniform framework for constructing new networks. Armed with this method, we also obtain new results on previously proposed networks such as honeycomb and biswapped networks, which have important applications in wireless and optical communication [7], [11], [14]. Before proceeding further, we introduce key definitions and notations related to interconnection networks and (di)graphs; Cayley (di)graphs, in particular. For more definitions and basic results on graphs and groups we refer the reader to [3], for instance, and for interconnection network concepts to [6], [8]. A list of key notation for this paper is presented in Table I for ease of reference. Unless noted otherwise, all graphs in this paper are undirected.

A digraph Ω = (V, E) is defined by a set V of vertices and a set E of arcs or directed edges. The set E is a subset of elements (u, v) of V×V. If the subset E is symmetric, that is, (u, v) ∈ E implies (v, u) ∈ E, we identify two opposite arcs (u, v) and (v, u) by the undirected edge (u, v). Because we deal primarily with undirected graphs in this paper, no problem arises from using the same
notation \((u, v)\) for a directed arc from \(u\) to \(v\) or an undirected edge between \(u\) and \(v\).

Let \(G\) be a finite group and \(S\) a subset of \(G\). The subset \(S\) is said to be a generating set for \(G\) if every element of \(G\) can be expressed as a finite product of the powers of its elements, called the generators of \(G\). We also say that \(G\) is generated by \(S\). The Cayley digraph of the group \(G\) and the subset \(S\), denoted by \(\text{Cay}(G, S)\), has vertices that are elements of \(G\) and arcs that are ordered pairs \((g, gs)\) for \(g \in G, s \in S\). If \(S\) is a generating set of \(G\), then we say that \(\text{Cay}(G, S)\) is the Cayley digraph of \(G\) generated by \(S\). If \(I \neq S\) (1 is the identity element of \(G\)) and \(S = S^{-1}\), then \(\text{Cay}(G, S)\) is a simple graph.

Assume that \(\Gamma\) and \(\Sigma\) are two graphs. The mapping \(\phi\) from \(\Gamma\) to \(\Sigma\) is a homomorphism from \(\Gamma\) to \(\Sigma\) if for any \((u, v) \in E(\Gamma)\) we have \((\phi(u), \phi(v)) \in E(\Sigma)\). In particular, if \(\phi\) is a bijection such that both \(\phi\) and the inverse of \(\phi\) are homomorphisms, then \(\phi\) is called an isomorphism of \(\Gamma\) to \(\Sigma\). Let \(G\) be a finite group and \(S\) a subset of \(G\). Assume that \(K\) is a subgroup of \(G\) (denoted as \(K \leq G\)). Let \(G/K\) denote the set of the right cosets of \(K\) in \(G\). The (right) coset graph of \(G\) with respect to subgroup \(K\) and subset \(S\), denoted by \(\text{Cos}(G, K, S)\), is the digraph with vertex set \(G/K\) such that there exists an arc \((Kg, Kg′)\) if and only if there exists \(s \in S\) and \(Kgs = Kg′\).

The remainder of our presentation is organized as follows. In Section 2, we introduce a number of group semidirect product networks. In Section 3, we discuss a general method for constructing a new group from given finite groups. In Section 4, we apply the aforementioned method to group semidirect product networks, thereby deriving a unified and systematic scheme for constructing new interconnection networks. In Section 5, we provide some new results on biswapped and many other networks. Section 6 contains our conclusions.

II. SOME KNOWN PRUNED NETWORKS BY GROUP SEMIDIRECT PRODUCT

We first give the definition of a special semidirect product called wreath product. Let \(G\) be a finite group and \(H\) a permutation group on a finite set \(\Omega\). The wreath product \(G \ltimes H\) of \(G\) and \(H\) is the set \(\{(f, h) \mid h \in H\}\), where \(f\) is a mapping from \(\Omega\) to \(G\) such that \((f, h)(f(\omega), h) = (g(\omega), h)\), and for \(i \in \Omega\), \(g(i) = f(i)f_2(j^h)\). We may verify that under this product operator \(G \ltimes H\) is a group. Its identity is \((f, e)\), where for any \(i \in \Omega\), \(f(i) = e\) and \(e\) is the identity of \(G\). The inverse of \((f, h)\) is \((g, h^{-1})\) such that \(g(i) = f(i^{-1})\). We have the following.

Lemma 1. Let \(H\) be a permutation group on \(\Omega\) of \(n\) elements, then

1. The wreath product \(G \ltimes H\) has a normal subgroup \(D = D_1 \times \ldots \times D_n, D_i = \{(f, e) \mid f(j) = e, j \neq i\} \cong G\).

2. \(H^* = \{(1, h) \mid h \in H, (i) = e\} \text{ for any } i \in \Omega\). \(H^* \cong H\) and \(G \ltimes H \cong DH\) is the semidirect product of \(D\) and \(H\).

3. For \(h^* = (1, h)\) we have \(D_i^{h^*} = D_i^h\) and \(G \ltimes H = |G^*| H^*|\).

The proof of Lemma 1, see Appendix.

If \(G\) is also a permutation group on a finite set \(\Gamma\), then the wreath product \(G \ltimes H\) may be expressed as a permutation group on \(\Gamma \times \Omega\) such that for any \((i, j) \in \Gamma \times \Omega\) we have \((i, j)(f, h) = (i^{f(j)}, j^h)\), where \(f \in G, h \in H\).

Assume that \(q\) is an integer. Let \(G = Z_{k_1} \times Z_{k_2} \times \ldots \times Z_{k_t}\), where \(Z_{k_i}\) is a cyclic group of order \(k_i\) and \(k_i\) is an integer for \(1 \leq i \leq q\). Thus \(G\) is a commutative group that is a direct product of the cyclic groups \(Z_{k_i}\) for \(1 \leq i \leq q\). Let \(s_i = (0^{(q-i)^2}, 1, 0^{(q-i)^2})\) be a vector of dimension \(q\) for \(1 \leq i \leq q\) and \(S = \{s_i\mid 1 \leq i \leq q\}\). Suppose that \(\Gamma = \text{Cay}(G, S)\), that is, \(\Gamma\) is a Cayley graph of the group \(G\) and the generator set \(S\). The network \(\Gamma\) just defined is a \(q\)-dimensional (or \(qD\)D torus. By pruning links of the graph \(\Gamma\) in a unified and systematic fashion we can obtain a pruned network which is still a Cayley graph [16]. We give a known example in the following.

Example 1. Cube-connected cycles \(CCC_q\) [13]. Let \(G = Z_2^q\). For any \(x \in G\), we write \(x = (x_1, x_2, \ldots, x_q, x_{q+1})^t\), where \(t\) denotes transpose. For \(x, y \in G\), define a new group operator \(\otimes\) as

\[
x \otimes y = x^t \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 & 0 \\
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 \\
\end{bmatrix} y^t
\]

where the matrix is of order \(q + 1\) and \(f\) is a function from \(G\) to \(Z_2\) such that \(f(x) = x_{q+1}\). It can be verified that \(H = (G, \otimes)\) is a new group under the group operator \(\otimes\).

Let \(S\) be as above and \(\Sigma = \text{Cay}(H, T)\), where \(T = \{s_1, s_{q+1}, s_{q+1}^{-1}\}\). Then \(\Sigma\) is known as cube-connected cycles \(CCC_q\).

We know that \(H = (G, \otimes)\) is the wreath product \(Z_{q} \ltimes Z_{q}^1\) of \(Z_{q}^2\) and \(Z_{q}^1\), and \(\Sigma\) is the pruned network obtained from the graph \(\Gamma = \text{Cay}(G, S)\), which is a \((q + 1)D\) torus [10]. Figure 1 depicts the \(CCC_4\) network with \(2^4 \times 4 = 64\) nodes.

As another example we consider the biswapped network [14]. Let \(\Omega\) be any digraph with the vertex set \(V(\Omega) = \{g_1, g_2, \ldots, g_r\}\) and the arc set \(E(\Omega)\). The biswapped interconnection network \(BW(\Omega) = \Sigma = (V(\Sigma), E(\Sigma))\) is a digraph with its vertex and edge sets specified as: \(\{0, p, g\}, \{1, p, g\} \mid p, g \in V(\Omega)\)
\[ E(\Sigma) = \{(0, p, g), (0, p, g), (1, p, g), (1, p, g), (g, g) \in E(\Omega), p \in V(\Omega) \} \cup \{(0, p, g), (1, g, p), (1, p, g), (0, p, g) \} \mid p, g \in V(\Omega) \] 

Intuitively, the definition postulates 2n clusters, each being an \( \Omega \) digraph: n clusters, with nodes numbered \( 0, \) \( \ldots, \) \( n \) \( \), forming part 0 of the bipartite graph, and \( n \) clusters constitute part 1, with associated node numbers \( 1, \) \( \ldots, \) \( n \) \( ). Each cluster \( p \) in either part of \( \Sigma \) has the same internal connectivity as \( \Omega \) (intracenter edges, forming the first set in the definition of \( E(\Sigma) \)). In addition, node \( g \) of cluster \( p \) in part 0/1 is connected to node \( p \) in cluster \( g \) of part 1/0 (intracenter or swap edges in the second set in the definition for \( E(\Sigma) \)). The name “biswapped network” (BSN) arises from two defining properties of the network just introduced: when clusters are viewed as superedges, the resulting graph of supernodes is a complete 2n-node bipartite graph, and the intercluster links connect nodes in which the cluster number and the node number within cluster are interchanged or swapped.

**Example 2.** As an example, when \( \Omega = C_4 \) (the undirected cycle of order 4) constitutes the basis graph, Fig. 2 depicts the resulting BSN \( C_4 \). Part 0 of the network is drawn at the top and part 1 at the bottom, with clusters 0-3 positioned from left to right.

We could continue our presentation with directed networks, deriving results for undirected networks as special cases. However, because parallel processing interconnection networks are usually undirected, we focus on undirected graphs in the rest of this paper. For undirected biswapped networks, the definition of the edge set is simplified to:

\[ E(\Sigma) = \{(0, p, g), (0, p, g), (1, p, g), (1, p, g), (g, g) \in E(\Omega), p \in V(\Omega) \} \cup \{(0, p, g), (1, g, p), (1, p, g), (0, p, g) \} \mid p, g \in V(\Omega) \] 

We need a few more notational conventions in what follows. For any graph \( \Gamma \), the number of its nodes is denoted as \( |\Gamma| \). The degree of a node \( g \) in \( \Gamma \) is \( \deg_G(g) \). The distance between nodes \( g_1 \) and \( g_2 \) in \( \Gamma \) is given by \( dist_G(g_1, g_2) \). The diameter of \( \Gamma \), that is, the maximum distance between any two nodes in \( \Gamma \), is \( D(\Gamma) \).

We begin by defining a swapped network \([14]\). For the sake of regularity, our definition is slightly different from that in [11].

Let \( \Omega \) be any digraph with the vertex set \( V(\Omega) = \{g_1, g_2, \ldots, g_n\} \) and the edge set \( E(\Omega) \). The swapped interconnection network based on \( \Omega \), that is, \( Sw(\Omega) = (V(\Gamma), E(\Gamma)) \), is a graph with its vertex and edge sets specified as:

\[ V(\Gamma) = \{(p, g) \mid p, g \in V(\Omega) \} \] 
\[ E(\Gamma) = \{(p, g), (p, g) \mid (g_1, g_2) \in E(\Omega), p \in V(\Omega) \} \cup \{(p, g), (g, p) \mid p, g \in V(\Omega) \} \] 

Note that the difference between this definition and that in [11] is that the case \( p = g \) is not excluded from the second set in the definition of \( E(\Gamma) \); in other words, here we postulate that the swap link associated with a node \( (p, p) \) in \( Sw(\Omega) \) is a self-loop, whereas in the original definition of [11], node \( (p, p) \) lacks a swap link and thus has a node degree that is one less than that of node \( (p, g) \) with \( p \neq g \). The swapped network based on a regular \( n \)-node, degree-d network \( \Omega \) has \( n^2 \) nodes of degree \( d + 1 \).

Because the class of Cayley graphs exhibits many desirable properties and also includes a significant fraction of all networks that have been found useful in parallel processing, we next consider biswapped networks built from basis networks that are Cayley graphs. It would indeed be quite an accomplishment if we could establish that biswapped networks thus formed are themselves Cayley graphs, because this would mean that certain desirable properties of the basis network are inherited by the composite biswapped network (we have already shown that Hamiltonicity is in fact transferred from the basis network to the biswapped network [14]).

In [14] we show that a Cayley-graph basis network does indeed lead to a biswapped network that is a Cayley graph.

Let \( H \) be a finite group and \( S \) a generator set of \( H \), with \( \Omega = Cay(H, S) \) and \( H \times H \) the direct product of the group \( H \) and itself. Let \( G = (H \times H)(\{t\}) = (\{t\})(H \times H) \) be a semidirect product of the group \( H \times H \) by the cyclic group \( (\{t\}) \), where \( t \) is an element of order 2, and \( (p, g) = (g, p) \) for any \( p, g \in H \). \( G \) is clearly the wreath product of \( H \) and \( (\{t\}) \). Let \( S^* = \{\{s\} \mid s \in S\} \subseteq H \times H \) and \( T = S^* \cup \{t\} \). Suppose that \( \Delta = Coset(G, \{t\}, T) \) is the coset graph of the group \( G \) with respect to the subgroup \( \{t\} \) and the generator set \( T \). Then we have the following result [14].

For completion we give its proof.

**Theorem 1.** The graph \( \Delta \) defined in the preceding paragraph is isomorphic to the swapped network \( Sw(\Omega) = \Gamma \).

**Proof.** The correspondence \( \phi: (t)(p, g) \rightarrow (g, p) \) is a mapping from \( \Delta \) to \( \Gamma \). Moreover, for \( g, p, h, q \in H \), \( (t)(g, p), (t)(h, g) \) is an edge of the graph \( \Delta \) if and only if either there is an element \( s \in S \) such that \( h = gs \) and \( p = q \) or \( (t)(h, g) = (g, p) \) is an edge of the graph \( \Delta \) if and only if \( (g, p), (h, q) \) is an edge of the graph \( \Gamma \). Hence, we have \( \Delta \cong \Gamma \).

**Example 3.** Let us consider a concrete example to illustrate the result of Theorem 1. Let \( \Omega = Cay(Z_n, S) \), with \( S = \{\pm 1\} \). Then, \( G = (Z_n \times Z_n)(\{t\}) = (\{t\})(Z_n \times Z_n) \), with \( S' = \{(0, \pm 1)\} \) and \( T = S' \cup \{t\} \). Let \( \Delta = Coset(G, \{t\}, T) \). Then, \( \Delta \cong Sw(\Omega) = \Gamma \) through the mapping \( \phi: (t)(p, g) \rightarrow (g, p) \).

By Theorem 1, there is a homomorphism from the biswapped network \( Sw(\Omega) \) to the swapped network \( Sw(\Omega) \). This is depicted in Fig. 3 for an example network.

Let \( \Psi = Cay(G, T) \) be the Cayley graph of the group \( G \) and the generator set \( T \). Then we proved the following result in a manner similar to Theorem 1[14]. For completion we also give its proof.

**Theorem 2.** The graph \( \Psi = Cay(G, T) \), with \( G \) and \( T \) as defined in the paragraph preceding Theorem 1, is isomorphic to the biswapped network \( Sw(\Omega) = \Sigma \).

**Proof.** The correspondence \( \phi: (t)(p, g) \rightarrow (i, (p, g)) \) is a mapping from \( \Psi \) to \( \Sigma \), where \( i = 0 \). Moreover, for \( g, p, h, q \in H \), \( (t)(p, g), (t)(h, q) \) with \( j = 0.1 \) is an edge of the
graph \( \Psi \) if and only if either there is an element \( s \in S \) such that \( i=j, p=q \) and \( h=gs \), or \( i \neq j, p=h \) and \( q=gs \). This is equivalent to saying that \( (i, p, g), (j, q, h) \) is an edge of the graph \( \Sigma \). Hence, we have \( \Psi \cong \Sigma \). ■

III. GROUP SEMIDIRECT PRODUCT CONSTRUCTION FROM GIVEN FINITE GROUPS

In Section 2, we defined a new group operator \( \otimes \) based on given finite groups \( G \) and \( H \) and obtained a new group wreath product \( G \otimes H \). The following are some typical examples.

1. Let \( Z_2 \otimes Z_2 \). It is isomorphic to the semidirect product \( Z_2 \times Z_2 \) by \( Z_2 \) where \((0, ..., 0; 1)(x_1, x_2, ..., x_n,0)(0, ..., 0; 1) = (x_2, ..., x_n, x_1,0)\) with \( x_i \in \{0\} \).

2. Let \( H \otimes Z_2 \). Let \( H \) be a finite group and \( H \times H \) the direct product of the group \( H \) and itself. Let \( G = (H \times H) \otimes t = \langle t \rangle \) be a semidirect product of the group \( H \times H \) by the cyclic group \( \langle t \rangle \), where \( t \) is an element of order 2, and \( \langle t, p, g \rangle = (t, p, g) \) for any \( p, g \in H \). Then \( G \) is clearly the wreath product of \( H \) and \( \langle t \rangle \) [14].

3. Let \( Z_m \otimes H \). Let \( Z_m \otimes H \leq S_n \). It is isomorphic to the semidirect product \( Z_m \times H \) by \( Z_m \) where \((0, ..., 0; 1)(x_1, x_2, ..., x_n,0)(0, ..., 0; 1) = (x_2, ..., x_n, x_1,0)\) with \( x_i \in \{0, ..., m-1\} \).

4. Let \( S_n \otimes Z_2 \). It is isomorphic to the semidirect product \( S_n \times Z_2 \) by \( Z_n \) where \((0, ..., 0; 1)(x_1, x_2, ..., x_n,0)(0, ..., 0; 1) = (x_2, ..., x_n, x_1,0)\) with \( x_i \in S_n \).

We may construct many interconnection networks of distinct and excellent properties by group semidirect product construction in the following.

IV. GENERAL RESULTS FOR CONSTRUCTING NEW NETWORKS BY GROUP SEMIDIRECT PRODUCT

Let \( G \) be a finite group and \( H \) a permutation group and we may construct the wreath product \( A = G \times H \). Then we may construct Cayley graph \( \text{Cay}(A,S) \) for some generator subsets \( S \) of \( A \). It is of distinct and excellent properties such as low node degree, low diameter and good fault tolerance and provides us with a systematic method for constructing large, scalable, modular, and robust parallel architectures, while maintaining many desirable attributes of the underlying basis network that comprise their clusters. It is easily proved that \( A \) is generated by \( S \cup T \) if \( G \) and \( H \) are generated by \( S \) and \( T \) respectively. Hence \( \text{Cay}(A,S \cup T) \) is a Cayley graph generated by \( S \cup T \) if \( \text{Cay}(G,S) \) and \( \text{Cay}(H,T) \) are Cayley graphs generated by \( S \) and \( T \) respectively. The following are some typical examples of interconnection networks by Cayley graph of wreath product.

1. Let \( A = Z_2 \otimes Z_2 \). We know that \( 1 \) generates \( Z_2 \) and \( \{\pm 1\} \) generate \( Z_2 \). Then \( U = \{(0, ..., 0), (0, ..., 0, \pm 1)\} \) generate \( A \). It is easily verified that \( \text{Cay}(A,U) \) is isomorphic to the cube-connected cycles \([4, 13]\). When \( T \) is any generator set of \( Z_2 \), we may construct a series of Cayley graphs of \( A \) and its generator set and obtain many new and distinct interconnection networks.

2. Let \( A = Z_2 \otimes S_n \). We know that \( 1 \) generates \( Z_2 \) and if \( T \) is any generator set of \( S_n \) then \( \{1\} \cup T \) is generating set of \( A \). Thus we may construct many new and distinct interconnection networks of Cayley graph \( \text{Cay}(A,1 \cup T) \), which have many excellent properties [15].

3. Let \( A = H \otimes Z_n \). If the subset \( S \) is a generator set of \( H \) and \( T \) is a generator set of \( Z_n \), then \( S \cup T \) is a generator set of \( A \) and we may construct Cayley graph \( \text{Cay}(A,S \cup T) \). Biswapped networks belong to these types of Cayley graphs [14].

V. NEW RESULTS FOR BISWAPPED AND OTHER NEW NETWORKS

A. Topological Properties

In the following we study the type (3) of Cayley graph. We know that \( \{1,-1\} \) is a generator set of \( Z_n \). For any generator set \( S \) of \( H \), \( T = \{1\} \cup S \) is a generator set of \( A \). We may construct Cayley graph \( \Gamma = \text{Cay}(A,T) \), and it is the biswapped network when \( n = 2 \) [14] and is the type (1) of networks when \( H = Z_n \). Let \( \Omega = \text{Cay}(H,S) \). We call the graph \( \Gamma \) as multiswapped network. Then \( \Gamma \) is a node transitive and of degree \( |S|+2 \) if \( n \geq 2 \) or \( |S|+1 \) if \( n = 2 \) [4]. The multiswapped network is a kind of common generalization of cube connected cycles and biswapped networks and has many excellent properties such as node symmetry, simple routing and embedding and fault-tolerance.

Now we consider digraphs in the subsection 5.1. Assume that \( \Gamma \) and \( \Sigma \) are two digraphs. We call the mapping \( \phi \) of \( V(\Gamma) \) to \( V(\Sigma) \) a homomorphism from \( \Gamma \) to \( \Sigma \) if for any \( (u,v) \in E(\Gamma) \) then \( (\phi(u),\phi(v)) \in E(\Sigma) \). The tensor product \( \Gamma \times \Sigma \) of \( \Gamma \) and \( \Sigma \) is the digraph with the vertex set \( V(\Gamma) \times V(\Sigma) \) and the arc set

\[
\{(x_1,y_1),(y_1,y_2)\} \in E(\Gamma) \times E(\Sigma).
\]

In this subsection we assume that the group \( A = NK \), where \( N = H^n \) is a normal subgroup of \( A \) (denoted by \( N < A \) ), \( K = Z_n \leq G \), and \( N \cap K = 1 \), that is, \( A \) is the semidirect product of \( N \) by \( K \). For the multiswapped network \( \Gamma = \text{Cay}(A,T) \), let \( \Sigma = \text{Cay}(\Delta,1) \) and \( \Delta = \text{Cos}(A,K,T) \), where \( T = \{1\} \cup S \) is a generating set of the group \( A \). Then any element \( g \) of \( A \) can be uniquely expressed as \( g = kn \) with \( n \in N, k \in K \). Define
the corresponding $\phi: kn \rightarrow (Kn, Nk)$ of $V(\Gamma) = A$ to $V(\Delta \times \Sigma)$. Then it is easily verified that $\phi$ is a bijection of them. We have the following.

**Proposition 1.** The mapping $\phi$ is a homomorphism of the digraph $\Gamma$ to the digraph $\Delta \times \Sigma$.

**Proof.** Consider the following diagram:

\[
\begin{array}{c}
\downarrow \\
k_n \rightarrow \downarrow \\
k_n \rightarrow (Kn, Nk)
\end{array}
\]

where $k_n = k, n_1, n_1 \in N, k_1 \in K, t \in T$ and $(kn, kn)$ is any arc of the digraph $\Gamma$.

Thus $\phi$ is a homomorphism of $\Gamma$ to $\Delta \times \Sigma$.

Proposition 1 has many applications in interconnection networks such as cube connected cycles [13].

**B. Simple Routing and Broadcasting Algorithm**

We know that the diameter is maximum distance between any two nodes of networks. It denotes maximum delay of networks and is an important network performance measure. For the diameters of the multiswapped network $\Gamma$ and the basis graph $\Omega$ we have the following relation.

**Theorem 3.** $D(\Gamma) \leq nD(\Omega) + n - 1 + \left\lfloor n/2 \right\rfloor$.

**Proof.** We have for $x_i \in H, y \in Z$

\[
(x_i, \ldots, x_v, y) = (x_i, \ldots, 1; 0)(1, \ldots, 1; 1)\ldots(1; 1; 1, 0; 1)(1, \ldots, 1; 1, 0; 1, \ldots, 1; 1, 0; 1)\ldots(1; 1, 0; 1; 1, 0; 1; 1, 0; 1)
\]

Since the diameter of a cycle $C_n$ of $n$ nodes is $\left\lfloor n/2 \right\rfloor$, we have $D(\Gamma) \leq nD(\Omega) + n - 1 + \left\lfloor n/2 \right\rfloor$.

For example, let $\Gamma$ be the cube-connected cycles, then $D(\Gamma) \leq 2n - 1 + \left\lfloor n/2 \right\rfloor$, which is 1 more than the diameter of $\Gamma$ when $n > 3$ [8]. If $\Gamma$ is the biswapped network then $D(\Gamma) \leq 2D(\Omega) + 2$, which is equal to the diameter of $\Gamma$ [14].

By the proof of the theorem 3 we easily get the routing algorithm of the multiswapped network $\Gamma$ which is a near shortest path algorithm. But we do not obtain the exact formula of the diameter of $\Gamma$.

Now we consider the broadcasting problem of interconnection networks [13]. Broadcasting is a communication scheme which sends a message from one processor to all other ones. The communication time $T$ to send a message from a processor to one of its neighbors is dependent to communication models. The linear and constant models are two main ones. We suppose that the communication model is the constant one, that is, the communication time between two adjacent processors equals to one time unit. On the other hand, in order to model real communications we must assume the different laws of communications. We assume that the messages are sent in store-and-forward mode, where a processor can not use the contents of a message until all bits received. Given a connected graph $G$ which models an interconnection network and a message originator $u$, the broadcast time of the vertex $u$, $b_u(G)$, is the minimum time required to complete broadcasting from the vertex $u$ under the model $M$. The broadcast time of the graph $G$ under the model $M$, $b_M(G)$, is defined to be the maximum broadcast time of any vertex $u$ in $G$, i.e., $b_M(G) = \max\{b_u(u) | u \in V(G)\}$. Let $\Gamma$ be the multiswapped network. For a communication model $M$ let $b_M(\Gamma_k)$ be the minimum time required to complete broadcasting in the vertices of $K \leq A$ from the identity element 1(message originator). Similar to Theorem 3 we have the following.

**Theorem 4.** $b_M(\Gamma) \leq nb_M(\Omega) + n - 1 + b_M(\Gamma_k)$. For example, let $\Gamma$ be the cube-connected cycles, then $b_a(\Gamma_k) = [n/2]$ for all-port model. Thus $b_M(\Gamma) \leq 2n - 1 + [n/2]$. If $\Gamma$ is the biswapped network then $b_M(\Gamma) \leq 2b_M(\Omega) + 2$ since $n = 2$. We also easily obtain near optimal broadcasting algorithms of these networks.

### a. Network Embeddings

Since the efficiency of parallel algorithms is dependent on the topology of interconnection networks, it is useful to explore the graph-embedding properties between two networks. We know that Hamiltonian cycle is a ring embedding of a network, which visits each node of the network exactly once. We proved that if the graph $\Omega$ is Hamiltonian then so is the graph(biswapped network) $\Gamma$ when $n = 2$ [14]. We also know that the multiswapped network is Hamiltonian when $\mid \Omega \mid$ is even or equal to 3 [14]. However we do not know whether the multiswapped network is Hamiltonian when $\mid \Omega \mid$ is odd.

Since $H$ may be any finite group and so $\Omega = \text{Cay}(H, S)$ may be any Cayley graph. Then the multiswapped network $\Gamma = \text{Cay}(A, T)$ may be many distinct networks and contains a number of embedded subgraphs, that is, abundant subgraphs can be embedded in the network $\Gamma$. We shall consider it in detail in the following.

### b. Node-disjoint Paths and Fault-tolerant

Node-disjoint paths between nodes of an interconnection network are a useful performance measure and it can speed up the transfer of data between nodes and provides alternatives routes in cases of node or link failures.

For the multiswapped network we have the following.

**Theorem 5.** If $\Omega$ is connected and $(x_1, \ldots, x_2 ; c) \neq (x_1, \ldots, x_2 ; d)$ are two nodes in the multiswapped network $\Gamma$ than $\Gamma$ has $\mid S \mid + 2$ node-disjoint paths when $n > 2$. 

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Proof. Let \( s \in S \cdot |S| > 2 \) node-disjoint paths in \( \Gamma \) are the following:
\[
(x_1, \ldots, x_n; c) \rightarrow (x_1, \ldots, x_n; x_n; c)
\]
\[
(x_1, \ldots, x_n; s, \ldots, x_n; c) \rightarrow (x_1, \ldots, x_n; d)
\]
\[
(x_1, \ldots, x_n; s, \ldots, x_n; c) \rightarrow (x_1, \ldots, x_n; s)
\]
\[
(x_1, \ldots, x_n; c) \rightarrow (x_1, \ldots, x_n; d)
\]
\[
(x_1, \ldots, x_n; c) \rightarrow (x_1, \ldots, x_n; c \pm 1) \rightarrow (x_1, \ldots, x_n; d).
\]

VI. CONCLUSION
In this paper, we have provided a general group semidirect product construction from finite groups. It generalizes previously proposed pruning schemes and provides a systematic and unified framework of constructing interconnection networks. As an application of our method, we also derived some new results on semidirect product networks, such as multiswapped and other new networks. Because of the generality of these results, which can be viewed as allowing the synthesis of alternative, more economical, interconnection networks by reducing the number of dimensions and/or link density of existing networks via mapping and pruning, we expect that they will find many more applications.

Many problems remain to be studied. We are currently investigating the applications of our method to problems related to fault-tolerant routing and average internode distance in certain subgraphs and coset graphs of multiswapped networks. These results, along with potential applications in the following areas will be reported in future:

- Load balancing and congestion control
- Scheduling and resource allocation
- Fault tolerance and graceful degradation

These constitute important practical problems in the design, evaluation, and efficient operation of parallel and distributed computer systems.

APPENDIX

Lemma 1. Let \( H \) be a permutation group on \( \Omega \) of \( n \) elements, then

(1) The wreath product \( G \ast H \) has a normal subgroup \( D = D_1 \times \ldots \times D_n, D_i = \{ (f, e) \mid f(j) = e, j \neq i \} \cong G \).

(2) \( H^* = \{ (i, h) \mid h \in H, \lambda(i) = e \} \) for any \( i \in \Omega \).

(3) For \( h^* = (1, h) \) we have \( D^*_i = D_i \) and \( G \ast H = G \upharpoonright H \).

Proof.

Let \( \Omega = \{ 1, 2, \ldots, n \} \), \( D = \{ (f, e) \mid f : \Omega \rightarrow G, D_1 = \{ (f, e) \mid f(j) = e, j \neq i \} \cong G \). \n
We have \( D_i \cong G, D = D_1 \times \ldots \times D_n \) is a normal subgroup.

\( \phi : (1, h) \rightarrow h \) is an isomorphism from \( H^* \) to \( H \) for \( h \) in \( H \) and \( G \ast H \cong DH \) is the semidirect product of \( D \) and \( H \).

For \( d = (g, e) \in D, h^* dl = (t, e) \), where \( t(j) = g(j^+) \).

Thus \( D^*_i = D_i \).

APPENDIX

| \( \cdot \) | Subgroup relationship |
| \( \ast \) | Set of (right) cosets |
| \( \times \) | Graph or set cross-product |
| \( \cdot^i \) | The vector “\( \ast \)" transposed |
| \( \circ \) | The symbol “\( \ast \)" repeated \( i \) times |
| \( (\ast) \) | Edge |
| \( \langle \cdot \rangle \) | Cyclic group |
| \( \rightarrow \) | Mapping |
| \( \cong \) | Isomorphic to |
| \( D(\ast) \) | Diameter of a graph |
| \( \Omega, \Gamma \) | Graphs or digraphs |
| \( \oplus \) | The identity element of a group |

- \( a, b, c, d \) Unit vectors
- \( \text{Cay}(\ast) \) Cayley graph
- \( \text{CCC}_i \) Cube-connected cycles of order \( q \)
- \( \text{Cos}(\ast) \) Coset graph
- \( \text{dist}(\ast) \) Distance function
- \( E(\ast) \) Edge set of a graph
- \( G, H \) Groups
- \( I_q \) The identity matrix of order \( q \)
- \( K, N \) Subgroups
- \( M, Q \) (Quasi-)permutation matrices
- \( S, T \) Generator sets, subsets of \( G \)
- \( V(\ast) \) Vertex set of a graph
- \( x_1, y_1, z_1 \) Node labels (bit vectors)
- \( Z_k \) Cyclic group of order \( k \)
- \( Z_k^q \) Elementary abelian group of order \( k^q \)

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REFERENCES


