Toric degenerations of Bézier patches

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Bézier Curves and Surfaces

Parametric Bézier curves and surfaces are widely used to represent geometric objects in a computer for design and manufacturing.

- While classical, they were rediscovered in 1959/62 by De Casteljau (Citroën) and Bézier (Renault) for automotive design.
- Used in animation software such as Adobe flash.
- You were able to fly here because of design, testing, and manufacture of airplane wings based on Bézier surfaces.
- You are looking at several thousand Bézier curves: Many fonts (LaTeX, and type 1 fonts) are defined in terms of cubic Bézier curves.

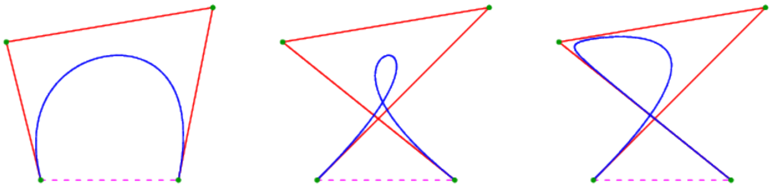
Rational Bézier Curves

For $i \in [0, d] \cap \mathbb{Z}$, and $s \in [0, d]$, we have $\beta_{i;d}(s) := s^i(1-s)^{d-i}$.

Weights $w_i > 0$ and control points $\mathbf{b}_i \in \mathbb{R}^n$ (typically, $n = 2, 3$), define a rational Bézier curve of degree d, which is the image of

$$[0, d] \ni s \longmapsto \frac{\sum_{i=0}^{d} w_i \beta_{i;d}(s) \mathbf{b}_i}{\sum_i w_i \beta_{i;d}(s)} .$$

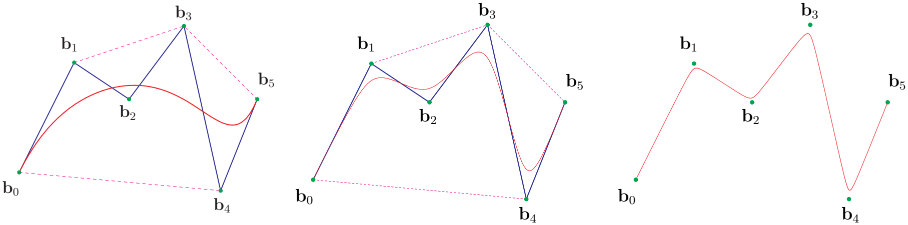
The control polygons of these rational Bézier curves connect the control points, and large weights pull the curve towards the control points.



Degenerations of Bézier Curves

Theorem (Craciun-García-S)

Let $\mathbf{b}_0, \ldots, \mathbf{b}_d \in \mathbb{R}^n$ be control points. For any $\epsilon > 0$, there are weights $w_0, \ldots, w_d \in \mathbb{R}_>$ so that the rational Bézier curve lies within a distance ϵ of the control polygon.

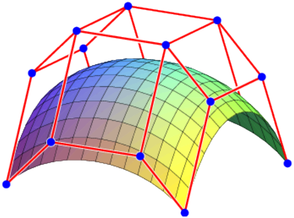


$\rightsquigarrow$ Gives meaning to the control polygon.

Rational Bézier Surfaces

Tensor product patch

is the image of $[0, c] \times [0, d]$ under



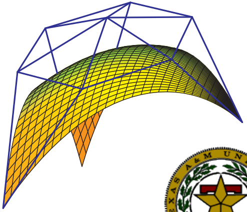
$$(s, t) \longmapsto \frac{\sum_{i=0}^{c} \sum_{j=0}^{d} w_{i,j} \beta_{i;c}(s) \beta_{j;d}(t) \mathbf{b}_{i,j}}{\sum_{i=0}^{c} \sum_{j=0}^{d} w_{i,j} \beta_{i;c}(s) \beta_{j;d}(t)} .$$

Triangular Bézier patch

For $(s, t) \in d\triangle := \{(s, t) \mid 0 \le s, t, \ s + t \le d\}$ and $(i, j) \in d\triangle \cap \mathbb{Z}^2$, set

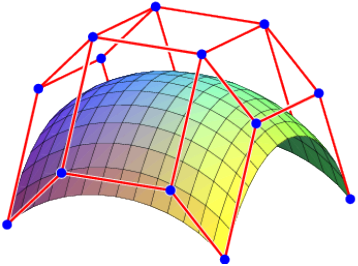
$$\beta_{i,j;d}(s, t) := s^i t^j (d - s - t)^{d-i-j} .$$

A triangular Bézier patch is the image of $\triangle$ under



$$(s, t) \longmapsto \frac{\sum_{(i,j) \in d\triangle} w_{i,j} \beta_{i,j;d}(s, t) \mathbf{b}_{i,j}}{\sum_{(i,j) \in d\triangle} w_{i,j} \beta_{i,j;d}(s, t)} .$$

Goldman and de Boor's Question



What is the significance for modeling of the control net?

Evocative Pictures

Executive Summary: Control nets encode all possible limiting positions of a patch.

The answer requires a generalization.

Toric Bézier Patches

Krasauskas's toric Bézier patches generalize the classical patches.

Let $\mathcal{A} \subset \mathbb{Z}^d$ be finite. Its convex hull $\Delta_{\mathcal{A}}$ is defined by facet inequalities

$$\Delta_{\mathcal{A}} \;:=\; \{x \in \mathbb{R}^d \mid h_{\mathcal{F}}(x) \geq 0,\ \mathcal{F} \text{ a facet of } \Delta_{\mathcal{A}}\}\,.$$

For $\mathbf{a} \in \mathcal{A}$ and $x \in \Delta_{\mathcal{A}}$, the toric Bézier function $\beta_{\mathbf{a},\mathcal{A}}(x)$ is

$$\beta_{\mathbf{a},\mathcal{A}}(x) \;:= \prod_{\mathcal{F}} h_{\mathcal{F}}(x)^{h_{\mathcal{F}}(a)}\,.$$

For weights $w \in \mathbb{R}^{\mathcal{A}}_{>}$ and control points $\mathcal{B} := \{\mathbf{b_a} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^n$, the toric Bézier patch $Y_{\mathcal{A},w,\mathcal{B}}$ is the image of $\Delta_{\mathcal{A}}$ under

$$\Delta_{\mathcal{A}} \ni x \;\longmapsto\; \frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathcal{A}}(x)\mathbf{b_a}}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathcal{A}}(x)}\;.$$

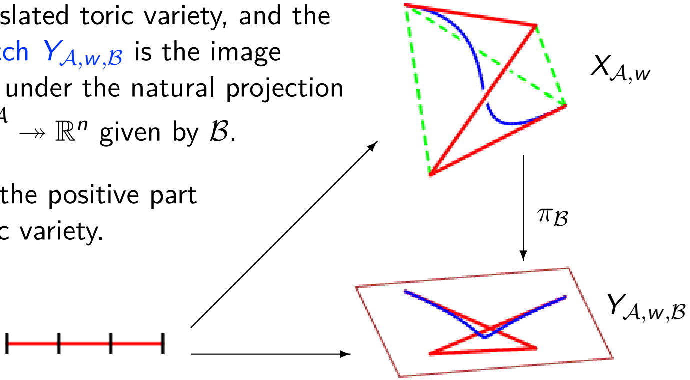
Why Toric?

Given $\mathcal{A} \subset \mathbb{Z}^d$ and $w \in \mathbb{R}^{\mathcal{A}}_{>}$ as before, let $\triangle^{\mathcal{A}}$ be the simplex with vertices indexed by $\mathcal{A}$. Then the closure $X_{\mathcal{A},w}$ of the image of the map

$$\mathbb{R}^d_{>} \ni x \longmapsto [w_{\mathbf{a}} x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \in \triangle^{\mathcal{A}},$$

is a translated toric variety, and the toric patch $Y_{\mathcal{A},w,\mathcal{B}}$ is the image of $X_{\mathcal{A},w}$ under the natural projection $\pi_{\mathcal{B}} \colon \triangle^{\mathcal{A}} \twoheadrightarrow \mathbb{R}^n$ given by $\mathcal{B}$.

$X_{\mathcal{A},w}$ is the positive part of a toric variety.

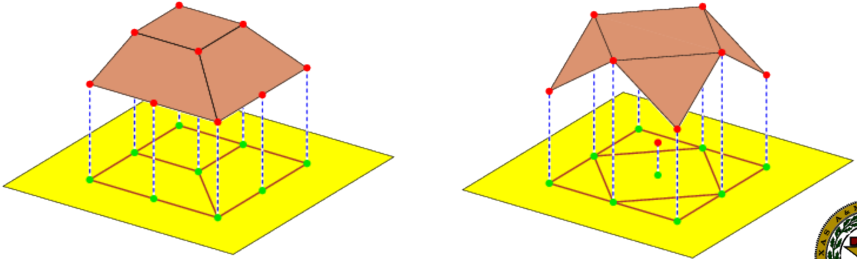


Regular Subdivisions

Given $\mathcal{A} \subset \mathbb{R}^d$ and a function $\lambda\colon \mathcal{A} \to \mathbb{R}$, form the polytope $P_\lambda := \mathrm{conv}\{(\mathbf{a}, \lambda(\mathbf{a})) \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^{d+1}$.

Upper facets of P_λ are those with positive outward pointing normals.

The system of subsets $\mathcal{F}$ of $\mathcal{A}$ consisting of lattice points lying on a face of an upper facet forms the regular subdivision $\mathcal{S}_\lambda$ of $\mathcal{A}$ induced by λ.



(The middle point on the right is not in any face $\mathcal{F}$ of $\mathcal{S}_\lambda$.)

Regular Control Surfaces

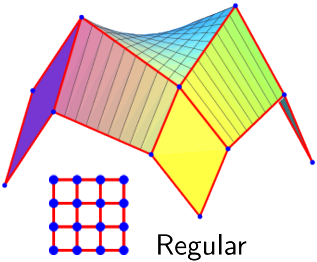
For $\mathcal{F} \subset \mathcal{A}$, the toric Bézier patch $Y_{\mathcal{F},w,\mathcal{B}} := Y_{\mathcal{F},w|_{\mathcal{F}},\mathcal{B}|_{\mathcal{F}}}$ is the image of

$$\Delta_{\mathcal{F}} \ni x \longmapsto \frac{\sum_{\mathbf{a}\in\mathcal{F}} w_{\mathbf{a}}\beta_{\mathbf{a},\mathcal{F}}(x)\mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathcal{F}} w_{\mathbf{a}}\beta_{\mathbf{a},\mathcal{F}}(x)} \,.$$

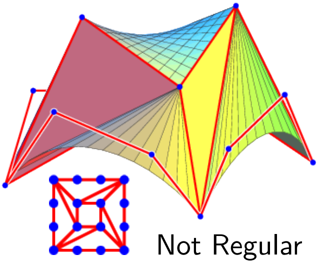
Given any system $\mathcal{S}$ of subsets $\mathcal{F}$ of $\mathcal{A}$ where the convex hulls of the $\mathcal{F}$ form a polyhedral decomposition of $\Delta_{\mathcal{A}}$, the control surface of $\mathcal{S}$ is

$$Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}) \;:=\; \bigcup_{\mathcal{F}} Y_{\mathcal{F},w,\mathcal{B}}\,.$$

It is regular if $\mathcal{S}$ is a regular subdivision.



Regular



Not Regular

Toric degenerations

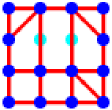
Given $\mathcal{A}, w, \mathcal{B}$ and a function $\lambda\colon \mathcal{A} \to \mathbb{R}$ inducing a regular subdivision $\mathcal{S}_\lambda$, the toric degenerations of $X_{\mathcal{A},w}$ and $Y_{\mathcal{A},w,\mathcal{B}}$ are families of varieties depending upon $t \in \mathbb{R}_>$:

Replace the weight w by $w_\lambda(t)$, where $w(t)_{\mathbf{a}} := t^{\lambda(\mathbf{a})} w_{\mathbf{a}}$, and define

$$X_{\mathcal{A},w,\lambda}(t) \;:=\; X_{\mathcal{A},w_\lambda(t)} \qquad \text{and} \qquad Y_{\mathcal{A},w,\mathcal{B}\lambda}(t) \;:=\; X_{\mathcal{A},w_\lambda(t),\mathcal{B}}$$

$$\lambda \;:=\; \begin{matrix} 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0.5 \end{matrix}$$

Regular subdivision

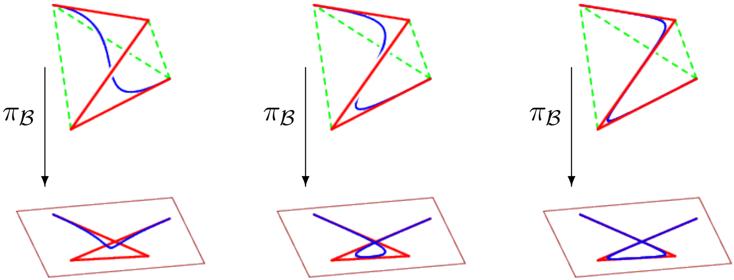


Toric Degenerations of Bézier Patches

Theorem

A regular control surface is the limit of the corresponding patch under a toric degeneration. In the Hausdorff metric,

$$\lim_{t\to\infty} Y_{\mathcal{A},w,\mathcal{B},\lambda}(t) \;=\; Y_{\mathcal{A},w,\mathcal{B}}(\mathcal{S}_\lambda)\,.$$



$t = 1$ $t = 3$ $t = 9$

Toric degenerations of a rational cubic Bézier curve

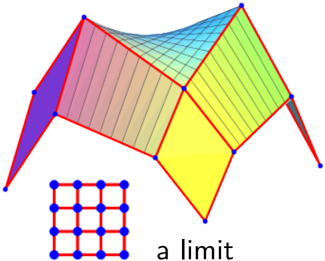
Limiting Positions are Regular

Theorem

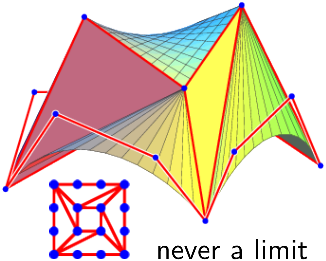
Given $\mathcal{A} \subset \mathbb{R}^d$ and control points $\mathcal{B} \subset \mathbb{R}^n$, if $Y \subset \mathbb{R}^n$ is a set for which there exists a sequence $w^1, w^2, \ldots$ of weights such that

$$Y \;=\; \lim_{i \to \infty} Y_{\mathcal{A}, w^i, \mathcal{B}}$$

in the Hausdorff metric, then Y is a regular control surface for $\mathcal{A}, \mathcal{B}$.



a limit



never a limit

One Page About the Proof

- First, consider the sequence of toric varieties $X_{\mathcal{A},w^i} \subset \triangle^{\mathcal{A}}$.
- Work of Kapranov, Sturmfels, and Zelevinsky implies that the set of all translated toric varieties is naturally compactified by the set of regular control surfaces in $\triangle^{\mathcal{A}}$, and this compactification equals the secondary polytope of $\mathcal{A}$.

M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, Quotients of toric varieties, Math. Ann. 290 (1991), no. 4, 643–655.

M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, Chow polytopes and general resultants, Duke Math. J. 67 (1992), no. 1, 189–218.

- Thus some subsequence of $\{X_{\mathcal{A},w^i}\}$ converges to a regular control surface in $\triangle^{\mathcal{A}}$, whose image must equal Y, which is therefore a regular control surface.
- We do not have a simple way to recover the lifting function λ or the weight w from the sequence $w^1, w^2, \ldots$.

Irrational Degenerations

All definitions and questions make sense for irrational patches, which are when $\mathcal{A}$ consists of real (not-necessarily rational) vectors.

In particular, regular control surfaces and toric degenerations make sense for irrational patches.

Theorem 1 (regular control surfaces are limits of toric degenerations) holds for irrational patches.

There is no version of Hilbert schemes (a tool for Kapranov, et al.) for irrational patches, nor can nice algebra be used to study them.

Nevertheless, with Postinghel and Villamizar, we can show the analog of Theorem 3 for irrational patches, using nothing but the equations and our collective wits. The result is an interpretation of the secondary polytope of any vector configuration $\mathcal{A}$ as the natural space of toric degenerations of $X_{\mathcal{A}}$ under the Hausdorff metric.

Next Step: Fibre polytopes and degenerations?

Bibliography

- G. Craciun, L. García-Puente, and F. Sottile, *Some geometrical aspects of control points for toric patches*, Mathematical Methods for Curves and Surfaces, LNCS **5862**, Springer, 2010, pp. 111–135.
- L. Garcia-Puente, F. Sottile, and C. Zhu, *Toric degenerations of Bézier patches*, to appear in ACM Transactions on Graphics.
- E. Postinghel, F. Sottile, and N. Villamizar, *Degneration of irrational toric varieties*, in progress.