

Asynchrony Induces Stability in Cellular Automata Based models

Hugues Bersini, Vincent Detours

IRIDIA - CP 194/6

Université Libre de Bruxelles

50 av. F. Roosevelt

1050 Bruxelles, Belgium

e-mail: bersini@ulb.ac.be, detours@ulb.ac.be

Abstract

Two cellular automata based computer simulations: an immune network model on one hand and the classical game of life on the other hand, despite similar algorithmic presentations, exhibit surprisingly distinct time evolution: respectively a fixed point and the complex dynamics characteristic of class IV cellular automata. At the conclusion of a complete investigation to understand better which of the algorithmic differences is responsible for this behavioural difference, we provide evidence that asynchronous rather than synchronous updating turns out to be the key factor. Experimenting and discussing in more detail this stability induction, we show that the responsibility of asynchrony for freezing game of life type of simulation can be theoretically justified in some particular cases by finding an associate Lyapunov function whose monotonous tendency proves the stability. The implications of such sensitivity to the updating mechanism for the future of cellular automata based models are reviewed.

1 Introduction

The result discussed in this paper is drawn from the comparison between two computer models, the game of life (GL) and the immune network model (INM), which although presenting very similar algorithms, display radically different dynamic regimes. GL belongs to class IV CA (Wolfram 1984), whereas INM always evolves toward fixed point attractors. These two models, both grounded in a spatially distributed type of algorithm, assume that the biological environment can be represented by a two-dimensional lattice. Each site of the lattice represents a biological entity whose state at time t can only take two values, 0 or 1, that are often interpreted as “dead or alive”. The same rules apply uniformly for all the sites in the lattice and can be stated as follows: an entity dies if its local neighbourhood (to be defined in next sections) is overcrowded or if it is empty, and it is brought to life, or it survives if already born, when the neighbouring population is in a certain range of concentration i.e. neither too high nor too low. Despite these similarities, the two models differ however in several aspects and actually one can see INM as a sophisticated version of GL.

Given the resemblance between the two models and the difference between their dynamic regimes, the question that comes to mind is: what feature present in INM but not in GL is responsible for the appearance of fixed points and thus the disappearance of complex regimes. We will see that the asynchrony of INM updating rules turns out to be the key factor in explaining this behavioural simplification. Since all distributed systems necessarily have an updating procedure, the possibility that this procedure could influence the result of simulations is of interest for all the scientists using this class of models, especially in the field of artificial life.

The results presented here could hold for a large number of distributed systems. While this paper focuses on a CA based model, Lumer and Nicolis (1994) have independently reached a similar conclusion with a coupled map lattice model. Also the dependence of Hopfield net stability on updating policy is well known (Hopfield 1982). Similar results have been reported for ising models (Choi and Huberman 1984), and for evolutionary game models (Huberman and Glance 1993; Novak, Bonhoeffer and May 1994). Thus, if all members of the CA class IV turn out to be so sensitive to the type of updating, that is, may lose their dynamical richness by increasing the degree of asynchrony, a deeper analysis on the adequacy of pure synchronous updating out of respect for biological or physical reality becomes necessary. Otherwise their appealing behaviour might just appear as pure simulation artifact leading to a strong weakening of their scientific realism and potential interest.

In section 2 are described the two models from which our result is inferred. In section 3 their dynamic behaviour is reported. Then (section 4) the effect of asynchrony will be analysed in detail: we will show that the responsibility of asynchrony for freezing GL type of simulation can in some cases be mathematically justified either by finding an associate Lyapunov function whose monotonous tendency proves stability or, as was done by McIntosh, by relying on approximate mean field theory to reveal a hidden marginally stable fixed point (McIntosh 1990). We finally discuss the implications of such sensitivity to the updating mechanism for the future of the development of CA based models.

2 Presentation of GL and INM

The game of life has been presented in a large number of publications (see for example Gardner 1970), but in order to

facilitate further comparisons we will now reintroduce its evolution rules in a more generic fashion: the state s_{ij} of a cell located at site (i, j) is determined by the sum h_{ij} of the states of its eight nearest neighbours using the following rules:

- if $h_{min} \leq h_{ij} < h_{med}$ then no change
- if $h_{med} \leq h_{ij} \leq h_{max}$ then $s_{ij} = 1$
- if $h_{ij} < h_{min}$ or $h_{ij} > h_{max}$ then $s_{ij} = 0$

with $h_{min} = 2$ and $h_{med} = h_{max} = 3$. The rules are applied synchronously for all sites.

In the following, other instances defined by different threshold values are labelled $GL_{h_{min}h_{med}h_{max}}$.

Since the biological motivations behind INM are not required to understand this article, this model will only be described from a formal point of view. Readers interested in immunological background should see Stewart and Varela 1991, where this model was first introduced. INM obeys the same transition rule as GL, however the cells are updated asynchronously (the cell to be updated is chosen at random), $h_{min} = h_{med}$, and h_{ij} is computed in a different way. Let (i', j') be the site symmetrical to (i, j) relatively to the center of the lattice (fig.1), h_{ij} is defined by:

$$h_{ij} = \sum_{k,l}^{L,L} m_{ij,kl} \cdot s_{kl}$$

where L defines the size of the lattice and $m_{ij,kl}$ is a gaussian function of the distance d between (i', j') and (k, l) .

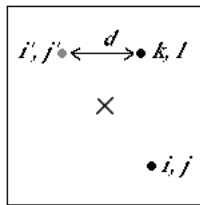


Figure 1: Computation of h_{ij} for INM.

In case of asynchrony, we say that one “time step” has elapsed when $L*L$ sites have been updated.

It is possible to construct intermediate versions between these two models in order to see which feature ($h_{min} = h_{med}$, the asynchrony, the symmetry rule, or the gaussian affinity domain) plays the most decisive role in determining at which regime the system operates.

3 The dynamics of IMN and GL

Fig.2 represents a pattern obtained when simulating INM from random initial conditions. The stripes no longer move once established. This steady state is easily explained if one notices that each stripe is coupled with a quasi-symmetric one such that both partners mutually guarantee for each other a

viable value of h_{ij} . When the lattice is covered with stripes, h_{ij} sensed by each site not yet occupied can only be either superior to the maximal threshold or inferior to the minimal one. If this is the case, no further cell can appear and no cell already present has to die. The existence of such a static equilibrium will indeed be theoretically justified in Section 4.

Complying with Wolfram’s classification, such behaviour is typical of class I CA. It is relevant in the immunological context since the stripes can memorize encounters with antigens (see Stewart and Varela 1991, Bersini 1992). Then, whatever their biological utility, either as a memory mechanism in immunology, or for the formation of geometrical patterns existing abundantly in nature, class I CAs are very promising instruments to rely on for the exploration of biological systems (see Meinhart 1982). However the key question we wanted to answer was why, despite their important resemblance in using the same type of threshold mechanism, the immune modelling just described and the game of life produce such different types of behaviour that they belong to two different classes of CA.



Figure 2: Fixed point pattern obtained with INM.

Simulations performed with the GL rules have gained increasing popularity during these last years due to the existence of a large family of fancy objects dancing on the computer screen, and known under the names of “gliders”, “blinkers”, “starships”, “guns” just to mention a few of them. They led Poundstone (1985) to say: “predictable as the game is on a cell-by-cell basis, the large scale evolution of the patterns defies intuition”. The complex dynamics caused by GL rules endows it with universal computation capacities (Berkelamp, Conway, and Wiesenfeld 1982; Langton 1991) and thus opens the door to speculations about how the laws of logic might be grounded into natural systems and even get the upper hand on the laws of physics (Langton 1991). According to several authors (Packard and Wolfram 1985, McIntosh 1990) GL appears to be an exception in its class and systematic exploration over the huge space of possible 2D CA transition rules revealed that all members of the class IV are trivial variants of the original GL. Indeed such uniqueness can be explained first by the hard time Conway spent to find a CA dynamics capable, while maintained in non-equilibrium condition, of a certain form of viability i.e. an animate self-regulatory system neither exploding nor collapsing, and secondly by the considerable care which is needed in

selecting initial conditions giving rise to interesting evolutions.

As a matter of fact, using GL in the same way as the immunological model namely initiating the simulation with a random occupation of the plane drives rapidly either to an emptying of this plane or to the survival of sparsely distributed rigid or/and oscillating objects (see fig.3 a,b for one of these simulations' final state). In order to raise interest, GL initial conditions must be selected with care (indeed this is a large part of the fun (see Poundstone 1985)) since a random selection of these conditions drives the simulation into fixed point or oscillatory configurations (as is the case for the first and the second CA classes).

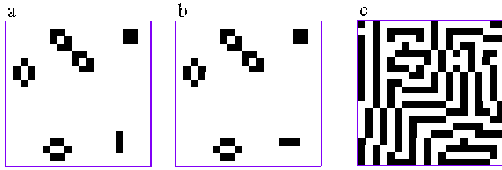


Figure 3: a,b) Snapshots of the period 2 final situation of one GL simulation started from random initial conditions (the blinker oscillating) - c) An asynchronous GL final fixed point state obtained from similar initial conditions.

4 Asynchrony freezes the game of life

Now in order to better understand why the two models operate at such different dynamic regime (at least for properly chosen initial conditions), we undertook a large number of simulations, adding to GL successively the features which differentiate it from IMN: first the symmetrical affinity, then the gaussian neighbourhood and finally the asynchrony of updating. The two first features have no influence on the regime at which the system operates. Finally we came to the conclusion that the key factor for inducing stability in INM was the asynchrony. The various dynamical behaviours obtained when varying the thresholds of the synchronous GL (with the constraint $h_{min} = h_{med}$) are listed table 1. Chaotic dynamics are obtained for 13 thresholds conditions. This number drops to 0 with asynchronous updating (table 2).

h_{max}, h_{min}	0	1	2	3	4	5	6	7	8
0	o								
1	o	c							
2	o	c	c						
3	o	c	c	n					
4	o	c	c	c	n				
5	o	c	c	c	n	n			
6	o	o	c	c	n	n	n		
7	o	o	o	o	n	n	n	n	
8	s	s	s	s	o	n	n	n	n

Table 1: Dynamics of the synchronous GL_{xy}, random initial conditions, $L = 20$. Depending on the thresholds the ca's behaviour can be oscillatory (o), chaotic (c), steady (s), or null (n, all sites equal to 0).

A similar result is observed for the regular GL and its asynchronous counterpart: the asynchronous GL (AGL) converges to a fixed point for *any* initial conditions. Noteworthy the final configurations are qualitatively different from those obtained with the synchronous GL (SGL) (fig. 3a, b) since all the available space is populated (fig. 3c). Therefore the main message of this paper is that, depending on its updating procedure, GL can belong either to the first or to the fourth CA class.

h_{max}, h_{min}	0	1	2	3	4	5	6	7	8
0	6								
1	12	*							
2	13	14	*						
3	12	16	113	22					
4	8	11	16	*	n				
5	7	13	9	150	n	n			
6	11	10	10	13	17	n	n		
7	8	7	10	5	8	n	n	n	
8	1	1	1	1	1	5	n	n	n

Table 2: Dynamics of the asynchronous GL_{xy}, random initial conditions, $L = 20$. The number of time steps before a fixed point is reached (averaged over 25 runs) is listed. Thresholds conditions marked by "*" produce dynamics that remain chaotic for more than 25000 time steps when a lattice with 20*20 site is used. However these conditions lead to fixed points within reasonable time when run on smaller lattice ($L = 7$).

Can this be justified in a more formal way? Various authors have noticed the strong similarities existing between the two popular computational models: CA and neural networks (NN) (see Garzon 1990). Their architecture is similarly organized in a set of interconnected elementary automata and their interesting behaviour emerges from the collective action of these automata. Basic differences remain however. Whereas locality and uniformity of rules are key properties of CA, they disappear in NN since neurons present a parametrized activation mechanism which depends on their "individualized" synaptic connections vector (the most sensitive NN architectural element). Moreover the range of connections is not limited to the immediate neighbourhood, and the activation mechanisms generally include slightly more complicated mathematical operations.

Among the great number of NN characterized by distinct architecture and activation mechanisms, the closest to CA ought to be the Hopfield network (Hopfield 1982) since:

- it is completely interconnected so that, despite the synaptic individuation of the neurons, it shows the greatest degree of homogeneity;
- its standard activation mechanism involves threshold functions;
- in addition, what makes it very attractive in the present context is that like for GL, asynchrony is a key factor of stability, and mainly that the formal proof of stability i.e. the existence of a monotonous Lyapunov function can be transposed in a very faithful way for proving stability of

the family of CA models (including INM and AGL) defined by the rules introduced in the previous section.

Indeed consider the CA counterpart of the Hopfield net i.e. a one-threshold CA (the update rule can boil down to: $s_{ij} = 1$ if $h_{ij} \leq h_{min}$, $s_{ij} = 0$ if $h_{ij} > h_{min}$), and associate to this rule, the Lyapunov Energy function E written below:

$$E = -\sum_{i,j} s_{ij} \left(h_{min} - \frac{h_{ij}}{2} \right)$$

In case of asynchronous transition, one cell is selected in a random or deterministic way to be updated (the selection order can be sequential or random, it is in no way a crucial point). Due to the absence of self-interaction and the symmetry of CA structure, we obtain:

$$h_{ij} \Delta s_{ij} = \sum_{kl} s_{kl} \cdot \Delta h_{kl}$$

where kl indexes the eight neighbours of ij . Thus updating E as a result of updating the selected cell (i,j) gives

$$\Delta E = -\Delta s_{ij} (h_{min} - h_{ij})$$

In consequence for each transition: $E > 0$. Given that the asynchronous form of this CA family causes E to be a monotonous decreasing function, the stability is proved: state changes will continue until a least E is reached.

In contrast, if more thresholds appear in the rules the discovery of a Lyapunov function is a non trivial problem. Let's associate with a two threshold CA of the form GLxxy the function:

$$E = \sum_{ij} s_{ij} h_{ij}^2 - \sum_{ij} \frac{s_{ij}}{2} h_{ij} h_{min} - \sum_{ij} \frac{s_{ij}}{2} h_{ij} h_{max} + \sum_{ij} s_{ij} h_{min} h_{max}$$

any transition results in:

$$\Delta E = -\Delta s_{ij} (h_{ij} - h_{min}) \cdot (h_{max} - h_{ij}) + \sum_{kl} s_{kl} \Delta h_{kl}^2$$

We easily observe that although E is always negative for a $1 \rightarrow 0$ transition, problems appear for a $0 \rightarrow 1$ transition for which the second term has a positive contribution conflicting then with the first one. Though a fixed point was always reached in our computer simulations, the existence of a monotonously decreasing Lyapunov function associated to the two-thresholds CA becomes questionable. At least it seems that its existence could be highly dependent on the thresholds values.

As a matter of fact, the existence of a Lyapunov function should imply that the fixed points of the system have large basins of attraction, which is unlikely for the threshold conditions resulting in extremely long transient time (table 2). This later remark also concerns AGL since the steady states

seem to exert no attraction. This view is consistent with the fact that a long time is often needed before a stable configuration is reached (the duration of transients varies between $5 \cdot 10^2$ and 10^5 time steps depending on initial conditions). Moreover the analysis of the damage caused by the flipping of a single site has revealed a very weak robustness of the fixed points to small changes. Actually 60% of the 200 single site perturbations performed on stabilized systems ($L=20$) led to avalanches involving all the sites. The duration of these avalanches was comparable to the duration of the transient just mentioned, i.e. the perturbation of one site had the same effect as starting from a new random initial state. In the remaining 40%, the disturbance was restricted to less than 10 sites in the neighbourhood of the perturbation.

Variants of GL with INM-like thresholds (i.e. GLxxy), were very much faster to stabilize as compared with AGL. Indeed as fig.4 shows, the final fixed point seems to really exert a strong attraction (a situation quite different than for AGL fixed points). A damage spreading analysis has demonstrated a strong robustness of the steady states to perturbations: avalanches triggered by a single site flip never spread throughout the whole lattice and remain nearly always confined to less than 10% of the cells. Also it is necessary to disturb more than 20% of the sites in order to observe significant change in the global shape of the patterns. In general the larger the thresholds window the easier to obtain and more robust the fixed points turn out to be. This could be roughly justified by observing that for the $0 \rightarrow 1$ transitions, the strength of the E expression first term increases with the enlargement of the window. We may reasonably hypothesize that in our current situation i.e in the absence of a decreasing Lyapunov function, the speed of convergence and the stability of the fixed points could be dependent on the proportion of the local increases ($E > 0$) and plateau ($E = 0$) to be encountered by the function E while showing a general decreasing tendency.

Interestingly enough, Hopfield networks present the same type of pathological behaviour as CA: in its synchronous version a lot of initial conditions cause the dynamics to be trapped into a periodic attractor (cycle of length at most 2 when the connections are symmetric). However, carrying on with the similarities, when the system evolves from more favourable initial conditions, the network settles in a fixed point following a transient shorter than in the asynchronous case. Clearly synchronous updating makes larger step in the attractive road towards fixed point than asynchronous updating, and this too rapid progression in the basin of attraction might explain both their rapid convergence when converging but also their oscillatory or even chaotic to and fro motion around the fixed points when not converging. In a similar way, it is well known that a gradient descent running with too important a learning rate can be responsible for erratic motion around the extremum.

All these common properties between CA and Hopfield networks tend to suggest that in its synchronous version the GL type of CA suffers from the same sort of instability as synchronous Hopfield networks: if the converging rate is too important, it prevents the convergence and induces an erratic wandering around the fixed point. The fact that the complex

behaviour of SGL may be attributed to a marginal stability at a fixed point was already conjectured by McIntosh (1990) relying on mean field theory. Based on a careful observation of the mean field curve for probabilistic self-consistency, it has been noticed that in general the actual transition probabilities characterizing class IV CA were located close to the theoretical solutions of the self-consistency problem and in a very stable region of the curve. This is why one may consider class IV CA behaviour as marginally stable, and why a reinforcement of the asynchrony contributes to suppress this marginality.

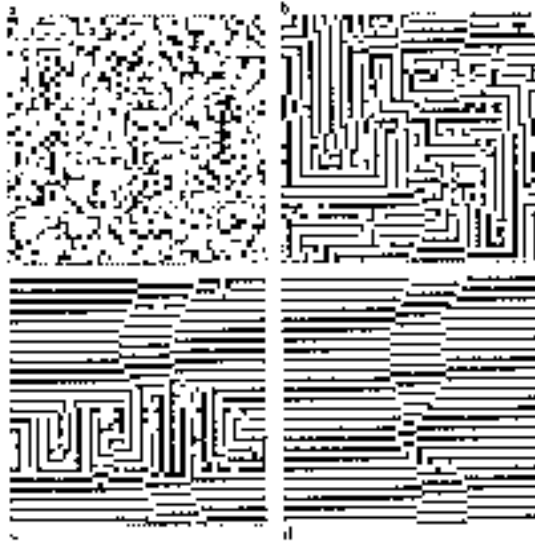


Figure 4: Time evolution of AGL223.

5 Discussion and conclusion

In a very recent work, Lumer and Nicolis (1994) discussed the induction of stability the switching from synchrony to asynchrony is responsible for in a model of coupled map lattice originally presented by Kaneko (1989) and observed to generate complex spatiotemporal dynamics when using a synchronous updating. Huberman and Glance have noticed that the same conclusion applies to simulations of evolutionary spatial games. Here we have shown that this result also holds for a whole class of threshold CA. In the last few years an important number of spatially distributed systems, generally grounded into CA models, have been largely appreciated and discussed for their capacity to evolve neither in an oscillatory way nor in a full chaotic way but somewhere in between, in a transition region called “the edge of chaos” (Langton 1991; Bak, Tang, and Wiesenfeld 1988; Chen and Bak 1989; Kauffman and Johnsen 1991). Such a critical behavioural regime is capable of storage (local quiescent state) and transmission of information (the propagating structures). Indeed manipulating these storing and propagating computational objects in a proper way can code for the elementary logical operations. SGL like Langton’s transitory stochastic CA (Langton 1991) and like Bak’s sand pile (Bak, Tang, and Wiesenfeld 1988) or trivial universe (Chen and Bak 1989) are all Turing universal.

An obvious question coming to mind is (originally raised in Huberman and Glance 1993, and Lumer and Nicolis 1994): “are the observed interesting regimes not just mere artifacts of the discretisation of the biological reality these models aim at reproducing?”. Indeed, if reality is better captured by continuous differential equations, Hopfield networks testify to the greater adequacy of asynchronous updating as compared with the synchronous one for giving the most accurate version of the differential equations solutions.

Differential equations are in principle synchronous (although some asynchrony might be introduced by adding a probabilistic term), but because of the rough discretisation that their computer simulation demands, neither a synchronous nor an asynchronous form of updating can ensure a better fidelity to the actual temporal evolution that a lack of analytical solutions may keep hidden. The ideal solution would see the synchronous and asynchronous updating giving rise to similar spatiotemporal evolution. But when this is not the case, which one is telling the truth: the fixed point or the complex regime?

In facing this delicate synchronous versus asynchronous problem which might concern numerous computer simulations behaving in a complex manner, two attitudes can be adopted but need to be clearly distinguished. A first one consists in the development of very interesting computer simulations capable of teaching us a great deal about complex systems but it must not be confused with a physical or biological scientific endeavour since no external physical or biological reality is expected to be reproduced and deeply understood in one way or another by these simulations. This attitude has much more to do with a creative or artistic effort and is more in line with a mathematical rather than a physical approach. SGL, Langton’s CA, Chen and Bak’s toy model of interacting ballistic particles in no way try to match any particular physical or biological system. Needless to say, these computer simulations share the same appeal (and indeed gliders, guns, starships, traffic lights, shuttles don’t lack of such appeal) for the same reasons and remain worth of scientific attention. They are perfect illustrations of how nature can be layered in multiple levels of coarseness, and how elementary laws at a low level can be responsible for complex behaviour at upper levels. They nicely show how unpredictability can still be grounded in predictable but finer mechanisms, and why keeping things simple but observed through a microscope can in some circumstances be the most appropriate strategy to confront apparently complex systems. If this is the attitude being adopted, the problem posed by the contrasting behaviour between different updating procedures simply vanishes. If not constrained by any biological or physical reality, just use the updating procedure which results in the kind of behaviour which better illustrates what in fact you aim at illustrating.

Now a second and to some extent much more demanding attitude is the physical one: There is some physical or biological reality whose structure and functionalities one wants to better grasp by means of computer simulation. It is supposed that such systems are continuous with direct consequence that any coarsening of the space and time resolution required by the computer simulation is not immune

of artifacts. It is well known for instance that the logistic map is chaotic only in its discrete time form, and that continuous systems need to possess at least three dimensions to exhibit similar chaos. Since there is no specific reason to privilege either the synchronous or asynchronous updating of the simulation for observing its evolution in time, and given that the two updating procedures generate totally different spatiotemporal trajectories, a hard problem is raised: which behaviour is the most faithful to reality. If possible, a Popperian type of validation might be the supreme judge in some cases, but nevertheless a high sensitivity to the updating procedure should call for an increased attention and prudence while running the simulation.

Some of these “edge of chaos”, “critical” or “phase transition” computer simulations do have physical or biological counterparts like Per Bak’s pile of sand (Bak, Tang, and Wiesenfeld 1988), earthquake models (Sornette and Sornette 1989), Kauffman’s interconnected fitness landscapes (Kauffman and Johnsen 1991) and others. On the other hand, nature seems full of systems showing fluctuations with dimension following a power law and showing spatial self-similarity as well, which are two key diagnostic factors for “criticality” and “phase transition” which these computer toy simulations aim at reproducing. Although the existence of these natural phenomena gives high legitimacy to the computer simulations endowed with the same characteristics, the existence of a contrasting behaviour depending on the updating procedure, and the crucial fact that this sensitivity often turns out to be the indicator of a hidden stability (and thus a potential outstanding behavioural simplification) is the main argument for advocating increased attention from the developers of these simulations.

References

- Bak, P., C. Tang, C. and K. Wiesenfeld. 1988. Self-Organized Criticality. *Phys. rev. A* 38 (1). 364-374.
- Berkelamp, E., Conway, J.H. and R. Guy. 1982. *Winning Ways for Your Mathematical Plays*. New York: Academic Press.
- Bersini, H. The Interplay Between the Dynamics and the Metadynamics of the Immune Network. IRIDIA Technical Report 92/12.
- Chen, K. and P. Bak. 1989. Is the Universe Operating at Self-Organized Critical State. *Physics Letters A* . Vol. 140. no 6. 299 - 302.
- Cheung, K.F., Atlas, L.E. and R.J. Marks II. 1987. Synchronous vs Asynchronous Behaviour of Hopfield’s CAM Neural Net. *Applied Optics*. Vol. 26. no 2. 4808-4813.
- Choi, M.Y., and B.A. Huberman. 1983. Digital Dynamics and Simulation of Magnetic Systems. *Phys. Rev. B* 28. 2547-2554.
- Garzon, M. 1990. Cellular Automata and Discrete Neural Networks. *Physica D* 45. 431-440.
- Hopfield J.J. 1982: Neural Networks and Physical Systems with Emergent Collective Computational Abilities. *Proc. Nat. Acad. Sci. USA* . vol. 79. 2554 - 2558.
- Huberman, B.A. and N.S. Glance. 1993. Evolutionary Games and Computer Simulations. *Proc. Nat. Acad. Sci. USA*. vol. 90. 7716-7718.
- Kaneko, K. 1989. Pattern Dynamics in Spatiotemporal Chaos. *Physica D* 34 . 1-41.
- Kauffman, S.A. and S. Johnsen. 1991. Co-Evolution to the Edge of Chaos: Coupled Fitness Landscapes, Poised States, and Co-Evolutionary Avalanches. In *Artificial Life II*. Langton, Taylor, Farmer and Rasmussen (eds.). Addison-Wesley Publishing Company.
- Langton, C. 1991. Life at the Edge of Chaos. In *Artificial Life II - Langton, Taylor, Farmer and Rasmussen (eds.)*. Addison-Wesley Publishing Company. Redwood City, CA . 41-91.
- Lumer, E.D. and G. Nicolis 1994. Synchronous Versus Asynchronous Dynamics in Spatially Distributed Systems. *Physica D* 71. 440-452.
- McIntosh, H.V. 1990: Wolfram’s Class IV Automata and a Good Life. *Physica D* 45. 105-121.
- Meinhart, H. 1982. *Models of Biological Pattern Formation*. Academic Pres. London.
- Novak, M.A., S. Bonhoeffer and R.M. May. 1994. More Spatial Games. *International J. of Bifurcation and Chaos*. Vol 4. no 1. 33-56.
- Packard, N.H. and S. Wolfram 1985. Two-dimensional cellular automata. *J. Stat. Phys.* 38. 1-35.
- Poundstone, W. 1985. *The Recursive Universe*. Morrow. New York.
- Sornette, A. and D. Sornette 1989. Self-Organized Criticality and Earthquakes. *Europhysics Letters* 9 (3). 197-202
- Stewart, J. and F. Varela. 1991. Morphogenesis in shape-space: Elementary meta-dynamics in a model of the immune network. *J. theor. Biol.* 153. 477-498.
- Wolfram, S. 1984: Universality and Complexity in Cellular Automata . *Physica D* 10 . 1-35.