Delayed Evolutionary Game Dynamics applied to Medium Access Control

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Abstract—A major contribution of biology to competitive decision making is the development of the discipline of evolutionary games. Unlike the standard Nash equilibrium that describes robustness against deviation of a single individual, the ESS (Evolutionary Stable Strategy) equilibrium concept defined in the evolutionary game paradigm is better adapted to large populations of players as it describes robustness against deviations of a whole fraction of the population. This paradigm has much potential applications in autonomous networking where we may encounter competition situations within large populations. The second appealing element of Evolutionary games, the Replicator Dynamics, describes the evolution of strategies in time; it has been proposed by evolutionary game theorists to model the evolution of species in nature. Under suitable conditions, this dynamics, well as other dynamics proposed by researchers from other disciplines, are known to converge to the ESS, which further strengthens the relevance of ESS to networking. In this paper we study the effect of small time scale delays on the convergence of various evolutionary dynamics to the ESS. The delays may correspond to the time needed to update software or to change hardware in mobile terminals. We study the delay impact on the performance of an evolutionary game describing competition between mobile terminals over the access to a channel. We compare the stability region obtained by various dynamics and show that every strategy in the Multiple Access Game remains under constant threat of extinction in the long run when the delays are large.

Keywords: evolutionary stable strategy, delay differential equation, replicator dynamic, adaptive imitation, best reply, multiple access game.

I. INTRODUCTION

One of the most important problems in networks is resource competition like bandwidth allocating or medium access control. One of the approaches studied in this context is the non-cooperative approach of classical game theory which assumes that users are rational and maximize their own payoff. Evolutionary Game Theory (EGT), developed by biologists, is a recent development in the area of game theory and aims at predicting population dynamics as a result of many local interactions between individuals. The major task of evolutionary game is the description of the dynamics of model game theory defined by strategies, payoffs and adaptation mechanism. It differs from classical game theory by (i) its focusing on the evolution dynamics of the fraction of members of the population that use a given strategy, and (ii) in its notion of evolutionary stable strategy (ESS,[23]) which includes robustness against a deviation of a whole (possibly small) fraction of the population who may wish to deviate (This is in contrast with the standard Nash equilibrium that only incorporate robustness against deviation of a single user). Recently, evolutionary game theory has gained interest among social scientists [12] and computer sciences and networking. Some examples of applications can are ALOHA system [7] and resources competition in the Internet[28].

We focus in this paper on the impact of time delays on the dynamics. Delays play an important role in many situations in networking such as in flow control and congestion problem [16], [3], [2], [19]. We focus on delays corresponding to a time scale much slower than the physical delays. In a computer network, the evolution time scale could be of the order of months or years and could apply to the evolution of services or of protocols, where as the physical delays can be of the order of a second [16] or less. To illustrate the relevance to networking, we focus on a simple model with one population of users where each individual has a choice between two strategies. We assume as usual that the interactions between the strategies are manifested through many local interactions between pairs of users. We shall use the standard representations of the evolutionary game as a two players matrix game representing the expected fitness obtained in an interaction between two individuals; the expectation is with respect to the fraction of the population that uses each strategy. We assume that time delays are not necessary symmetric. The case of a common time delay for all strategies can be found in [1], [24]. We study the stability as a function of delays. We derive new stability conditions for these dynamics with (symmetric or not) delays and apply them to the Multiple Access Game introduced in [10].

The paper is structured as follows. We first provide in the next section the needed background on evolutionary games. We then study the evolutionary game dynamics with delays and ESS for Multiple Access Game in these dynamics. Brown-von Neumann-Nash, replicator, best reply and imitation dynamics with delays are introduced. After that, we investigate the impact of the choice of some parameters in these dynamics on the stability of the system in Section IV in the Multiple Access Game. Finally we give some numerical investigations and conclude with some remarks on the velocity of these when the delays are small.
II. BACKGROUND AND RELATED WORKS

Evolutionary game theory considers a dynamic scenario where players are constantly interacting with one another and adapting their choices based on the payoff (fitness) they receive. A strategy having higher fitness than others tends to gain ground: this is formulated through rules describing the dynamics (such as the replicator dynamics or others) of the sizes of populations (of strategies). Much literature can be found on evolutionary games including on evolutionary dynamics. An example is the extensive survey [14] by that can be found on evolutionary games including on evolutionary dynamics, and in a matrix game. There is a close relation between the rest stationary points of replicator dynamics[25]. They considered a related discrete version. Tao and Wang [24] studied the effect the dynamics (such as the replicator dynamics or others) of to gain ground: this is formulated through rules describing the expected fitness of any individual at time \( t \). The payoff matrix where \( u(\cdot, \cdot) \) is affine in \( x \). Formally, consider a large population in which players that are immune from being invaded by other small populations. The conditions to be an ESS [20], [24], [5], [14] can be related to and interpreted in terms of Nash equilibrium in a matrix game. There is a close relation between the rest points of the replicator equation (more generally in payoff monotonic dynamics [27]) and the Nash equilibria of the related (symmetric) matrix game given by the so called folk theorem of evolutionary game theory [9], [15], [27].

Although ESS has been defined in the context of biological systems, it is highly relevant to engineering as well (see [26]). Formally, consider a large population in which players that have identical sets of pure strategies and symmetric payoff. Let \( S := \{1, 2, \ldots, n\} \) the set of strategies and \( \sigma_{ij} \) the fitness of subpopulation using \( i \) against the subpopulation \( j \). The reparation of the population can be represented by some probability \( s \) in \( S \). The number \( s_i \) is the proportion of individuals in the population using the strategies \( i \). Let \( \Delta(S) := \{s \in \mathbb{R}^n \mid s_i \geq 0, \sum_{i=1}^n s_i = 1\} \) the set of mixed strategies.

Suppose that, initially, the population profile is \( s \in \Delta(S) \). The average payoff in the population is \( u(s, s) = sMs \) where \( M = (m_{ij})_{i,j=1,\ldots,n} \). Now suppose that a small group of mutants enters this population playing according to a different profile \( mut \). If we call \( \epsilon \in (0, 1) \) the size of the subpopulation of mutants after normalization, then the population profile after mutation will be \( \epsilon mut + (1-\epsilon)s \). After mutation, the average payoff of non-mutants who are randomly matched to mutants is given by \( u(s, mut) = sMmut \), and the average payoff of non-mutants will be given by \( u(s, \epsilon mut + (1-\epsilon)s) = cu(s, mut) + (1-\epsilon)u(s, s) \). Analogously, we can construct the average payoff of mutant \( u(mut, cmut + (1-\epsilon)s) \). A strategy \( s \in \Delta(S) \) is an ESS if for all \( mut \neq s \), there exists some \( \epsilon_{mut} \in (0, 1) \), which may depend on \( mut \), such that for all \( \epsilon \in (0, \epsilon_{mut}) \)

\[
u(s, \epsilon mut + (1-\epsilon)s) > u(mut, \epsilon mut + (1-\epsilon)s)
\]

(1)

That is, \( s \) is ESS if, after mutation, non-mutants are more successful than mutants, in which case mutants cannot invade and will eventually get extinct. The number \( \epsilon_{mut} \) is called invasion barrier [27]. It is the maximum rate of mutants against which \( s \) is resistant. If \( s \) is an ESS then \( s \) is a Nash equilibrium. Equivalently \( s \) is an ESS if and only if it meets best reply conditions:

\[
u(mut, s) \leq u(s, s), \forall mut,
\]

\[
u(mut, s) = u(s, s) \Rightarrow u(mut, mut) < u(s, mut) \forall mut \neq s
\]

Why Evolutionary Game approach in Medium Access Problem? The classical game theoretical approach has been used in a number of fields, including Medium Access Control [18], [10] pricing, bandwidth allocating, congestion control and routing. With the increasing numbers of users, evolution of services and the apparition of new protocols like version of TCP, it is necessary that the dynamics include these evolutions and varieties. Evolutionary Game theory describes the evolution with some dynamic process and involve strategic interaction over time in large populations of users.

III. EVOLUTIONARY GAMES WITH DELAYED DYNAMICS

We consider a large population of users. We assume that there are \( n \) pure strategies. A strategy of an individual is a probability distribution over the pure strategies. An equivalent interpretation of strategies is obtained by assuming that individuals choose pure strategies and then the probability distribution represents the fraction of individuals in the population that choose each strategy. We denote by \( A = (K_{kj})_{k,j=1,\ldots,n} \) the payoff matrix where \( K \) is a positive parameter (this parameter doesn’t change the equilibrium set because the equilibrium set is invariant under affine transformation). The individuals compete through a large number of random pairwise interactions and the expected fitness of any individual at time \( t \) is given as the expected payoff value of this individual at time \( t - \tau_k \) where \( \tau_k \) is the time delay of the strategy \( k \). Let \( f_j(x(t - \tau_j)) \) be the expected payoff value of the strategy \( j \) at time \( t - \tau_j \) when the composition of population was \( x(t - \tau_j) \). The fitness of the individual using the strategy \( j \) is \( F_j(x(t - \tau_j)) := R + f_j(x(t - \tau_j)) \) where \( R \) is the individual’s fitness when there is no game (at the initial time). The payoff function \( f = (f_j)_i \) is given by \( f_j(x) = e_j A x \). Note that \( F \) is affine in \( x \).
A. Replicator Dynamics with delay

The replicator dynamics has been used for describing the evolution of road traffic congestion in which the fitness is determined by the strategies chosen by all drivers [21]. It has also been studied in the context of the association problem in wireless networks in [22]. We introduce the replicator dynamics which describes the evolution in the population of the various strategies. Replicator dynamics is one of the most studied dynamics in evolutionary game theory. In the replicator dynamics, the share of a strategy in the population grows at a rate equal to the difference between the delayed payoff of that strategy and the average delayed payoff of the population. More precisely, consider strategies. Let \( x(t) \) be the \( n \) dimensional vector whose \( j \)th element \( x_j(t) \) is the population share of strategy \( j \) at time \( t \). Thus we have \( \sum_{j=1}^{n} x_j(t) = 1 \) and \( x_j(t) \geq 0 \). In the classical replicator dynamics, the fitness of strategy \( j \) at time \( t \) has an instantaneous impact on the rate of growth of the population size that uses it. An alternative more realistic model for replicator dynamic would have some delay: the fitness acquired at time \( t \) will impact the rate of growth \( \tau_j \) time later. Below we denote by \( F_j(x(t-\tau_j)) \) the fitness for a user using strategy \( j \) when it encounters a user with strategy \( x(t-\tau_j) \) at time \( t-\tau_j \) i.e the composition of the population was \( x(t-\tau_j) \) at time \( t-\tau_j \).

Suppose that each individual of the population only uses a pure strategy \( j = 1, \ldots, n \). Then the dynamics of \( x_j(t) \) is given by
\[
\dot{x}_j(t) = Kx_j(t) \left[ F_j(x(t-\tau_j)) - \sum_{k=1}^{n} x_k(t) F_k(x(t-\tau_k)) \right],
\]
\( j = 1, \ldots, n \). Note that \( \sum_{k=1}^{n} x_k(t) F_k(x(t-\tau_k)) \) is the fitness of the population at time \( t \).

B. Imitate the better

Strategies can be transmitted within the population by imitation. We consider the model of imitation dynamics given in [14], [15] in which we introduce the delays.
\[
\dot{x}_k(t) = x_k(t) \left( \sum_{j=1}^{n} x_j(t) [\rho_{jk}(x(t-\tau_k) - \rho_{kj}(x(t-\tau_j))] \right)
\]
where \( \rho_{jk}(x) = \phi(f_k(x), f_j(x)) \), \( \phi(a, b) = 0 \) if \( a < b \) and \( \phi(a, b) = 1 \) if \( a > b \). A strategy increases if and only if its fitness is larger than the median of the fitness \( f_j(x) \), \( j = 1, \ldots, n \).

C. Best Response dynamics with delay

A strategy \( Y \in \Delta(S) \) is a best reply to \( x(t) \) at time \( t \) if
\[
\sum_{j=1}^{n} Y_j f_j(x(t-\tau_j)) \geq \sum_{j=1}^{n} x_j f_j(x(t-\tau_j)), \forall v \in \Delta(S).
\]
We use by \( BR(x(t)) \) the best reply strategies set to the strategy \( x(t) \) i.e
\[
BR(x(t)) = \left\{ Y \in \Delta(S), \sum_{j=1}^{n} Y_j f_j(x(t-\tau_j)) \geq \sum_{j=1}^{n} v_j f_j(x(t-\tau_j)), \forall v \in \Delta(S) \right\}
\]
A fraction of population revise their strategy and choose the best replies \( BR(x(t)) \) to the current population state \( x(t) \) at time \( t \). The best response dynamics is given \( \dot{x} \in BR(x) - x \).

It can be shown that the logit dynamics given by
\[
\dot{x}_j(t) = \frac{e^{F_j(x(t-\tau_j))T_0}}{\sum_{j=1}^{n} e^{F_j(x(t-\tau_j))T_0}} - x_j(t)
\]
converge to the best reply dynamics if \( T_0 \to +\infty \).

D. Brown-Von Neumann-Nash Dynamics (BNN) with delay

Delay can be introduce in BNN dynamics [8] as follows
\[
\dot{g}_j(t) = \sum_{k=1}^{n} g_k(t) - x_j(t)
\]
where \( g_j(x(t)) \) is defined as
\[
\max \left\{ 0, F_j(x(t-\tau_j)) - \sum_{k=1}^{n} x_k(t) F_k(x(t-\tau_k)) \right\}
\]
The function \( g_j(x(t)) \) is the positive part of the excess payoff for strategy \( j \) when the population state is \( x(t) \) at time \( t \).

There are a close relation between BNN dynamics with Nash’s original proofs of his equilibrium theorem (see [14]). BNN dynamics [6] is interpreted as follows: Suppose there are large users in the population in which there is steady influx and outflux. New users joining the system use only strategies that are better than average, and better strategies are more likely to be adopted. On the other hand, randomly chosen users leave the game. More precisely, strategy \( s \in \Delta(S) \) is adopted with probability proportional to the excess payoff \( g_i(s) \).

E. An adaptive dynamic with euclidian metric

We consider that adaptive dynamic which a strategy increases if its fitness is larger than the arithmetic average of the fitness \( F_j(x) \), \( j = 1, \ldots, n \).
\[
\dot{x}_j(t) = F_j(x(t-\tau_j)) - \frac{1}{n} \sum_{k=1}^{n} F_k(x(t-\tau_k))
\]
This dynamic is a particular case of adaptive dynamics with Riemannian metric and have advantageous for games with only interior equilibrium point. The absence of the term \( x_j \) at the right side make that this dynamic may be does not take account the pure equilibria. Hofbauer and Sigmund proved in [15, theorem 9.6.1] that an interior ESS is asymptotically stable for this dynamic when all the delays are zeros.

IV. Evolutionary Multiple Access Game in Ad Hoc network

We consider a large population of mobile terminals in ad hoc network (see figure 1). We assume that the density of the network is low, so that if a terminal attempts transmission one can neglect the probability of interference from more than one other mobile (called "neighbor"). We assume the mobiles move frequently and they have a packet to send in...
that the pure strategies \((T, S)\) and \((S, T)\) are also optimal in Pareto\(^1\) sense. The number \(1 - \Delta\) (resp. \(\Delta\)) represents proportion of individuals which transmit (resp. stay quiet). When the two subpopulation use this strategy, they obtain the same payoff equal to zero.

The strategy \((1 - \Delta, \Delta)\) is the unique interior Nash equilibrium. It is the unique symmetric Nash equilibrium. Thus, it is the only candidate to be ESS because symmetric Nash equilibria set content ESS set. At a mixed equilibrium, the strategies \(T\) and \(S\) must have the same fitness. We check the condition of ESS given in (1). Let

\[
A = K \begin{pmatrix} -\Delta & 1 - \Delta \\ 0 & 0 \end{pmatrix}
\]

Then \(\forall \xi \neq 1 - \Delta\)

\[
(1 - \Delta - \xi, \xi - 1 + \Delta)A \begin{pmatrix} \xi \\ 1 - \xi \end{pmatrix} = K(\xi - 1 + \Delta)^2 > 0
\]

We conclude that \((1 - \Delta, \Delta)\) an ESS.

\section*{Stability condition}
Consider the following linear delay differential equation

\[
(DDE_1) \quad \dot{z}(t) = -az(t - \tau)
\]

with \(\tau, a > 0\). It is known that all solutions of \((DDE_1)\) is asymptotically stable if and only if all roots of the characteristic equation in

\[
\lambda + ae^{-\lambda \tau} = 0
\]

have negative real parts ([4]). Necessary and sufficient condition of asymptotic stability as a function of the delay is given by the following lemma. A proof can be found in [13, proposition 1.2.8]

\begin{lemma}
A necessary and sufficient condition for all roots of (7) to have negative real parts is \(2a\tau < \pi\).
\end{lemma}

We note by \(\xi(t)\) proportion of individuals using the the strategy \(T\) at time \(t\). The following proposition gives the stability of the ESS \((1 - \Delta, \Delta)\) in the delayed replicator dynamic (2).

\begin{proposition}
The mixed equilibrium \((1 - \Delta, \Delta)\) is asymptotically stable for the replicator dynamics if \(2K\Delta(1 - \Delta)\tau_T < \pi\) and not stable if \(2K\Delta(1 - \Delta)\tau_T > \pi\).
\end{proposition}

\textit{Proof:} The replicator dynamic equation with delays in the Multiple Access Game becomes

\[
\dot{\xi}(t) = -K\xi(t)(1 - \xi(t))[\xi(t - \tau_T) - 1 + \Delta] \quad (8)
\]

\(^1\)An allocation of payoffs is said Pareto-optimal if the outcome cannot be improved upon without hurting at least one user.
Note that this equation is independent of the delay $\tau_S$ because the payoff of the strategy $S$ is equal to zero. It is known ([4, pp.336],[13, pp.188]) or from the Hartman and Grobman theorem adapted to delay differential equation that the steady state $(1 - \Delta, \Delta)$ is asymptotically stable for (8) around the stationary point $1 - \Delta$ if the trivial solution of the linearized version is asymptotically stable.

Taking linearized version of (8) around the stationary point $1 - \Delta$ and using lemma 1, we obtain the results.

The imitation rule applied to the Multiple Access Game when delays are equal to $\tau$ is given by

$$
\dot{\xi}(t) = \xi(t)(1 - \xi(t))h(\xi(t - \tau))
$$

where $h : \xi \mapsto \begin{cases} 
1 & \text{if } \xi < 1 - \Delta \\
0 & \text{if } \xi = 1 - \Delta \\
-1 & \text{if } \xi > 1 - \Delta 
\end{cases}$

The best reply at the time $t$ when the population $\xi(t)$ is

$$BR(\xi(t)) = I_{[0,1-\Delta]}(\xi(t-\tau)) + [0,1]I_{(1-\Delta)}(\xi(t-\tau))$$

where $I_A(.)$ is the indicator function of the set $A$.

The BNN dynamics becomes $\dot{\xi}(t) = K(1 - \xi(t)^2)(-\xi(t - \tau_T) + 1 - \Delta)$ if $\xi(t - \tau_T) < 1 - \Delta$, $0$ if $\xi(t - \tau_T) = 1 - \Delta$ and $K\xi(t)^2(-\xi(t - \tau_T) + 1 - \Delta)$ if $\xi(t - \tau_T) > 1 - \Delta$.

**Proposition 2.** The ESS $(1 - \Delta, \Delta)$ is asymptotically stable for the adaptive dynamics (6) if $K\tau_T < \pi$ and not stable if $K\tau_T > \pi$.

**Proof:** In the Multiple Access Game, the adaptive dynamics given in (6) becomes

$$
\dot{\xi}(t) = -\frac{K}{2}[\xi(t - \tau_T) - 1 + \Delta] 
$$

(9)

We conclude by lemma 1 with $z(t) = \xi(t) - 1 + \Delta$ and $a = \frac{K}{2} > 0$.

V. NUMERICAL INVESTIGATION

**Impact of the delay** Our first numerical experiment studies the convergence of these dynamics for the case of the unit growth parameter $K$ as a function of delay: we check the speed of convergence and the stability of the dynamics as a function of the delay $\tau_T$ (all the dynamics are independent of the delay $\tau_S$: the payoff of the strategy $S$ is zero). The state $2/3$ is a stationary point for these parameters, for which $2/3$ of the population is choose to transmit. We took $\Delta = 1/3$.

The resulting trajectories of the population using the strategy $T$ is represented as a function of time. We evaluate the stability varying the delay $\tau_T$ between $0.02$ and $10$ time units in the replicator dynamic in figure 4. For $\tau_T = 0.02$, we have stability but the convergence speed is slow. The other extreme is illustrated for $\tau_T = 10$ which the trajectory oscillates rapidly and the amplitude is seen to be greater than $2/3$. The system is unstable.

Figure 5 represents the trajectories of the population using the strategy $T$ in the imitation dynamic in which the system is unstable for all $\tau_T > 0$. The amplitude of oscillation growth with the delay. The periodicity increases when the delay decreases. In figure 6, best reply dynamic is represented for respectively $\tau_T = 0.04, 1, 2$. The strategy $(1/2, 1/2)$ is chosen for the best reply to $(1 - \Delta, \Delta)$.

We evaluate the stability varying the delay $\tau_T$ between $0.04$ and $8.4823$ time units in the Brown-von Neumann-Nash in figure 7. For $\tau_T = 0.04$, we have stability but the convergence speed is slow. The system becomes when $\tau_T$ greater than $8.4823$ is unstable.

In figure 8, adaptive dynamic is represented for respectively $\tau_T = 0.5, 2, 5$ with the initial condition $0.2$. When delay is large, the oscillating solution can be outside the interval $[0, 1]$.

**Validation of stability conditions.** In these figures we observe cases of stable and of non-stable behavior. All turn out to confirm the stability conditions that we obtained above.

**Comparison between evolutionary dynamics** The impact of a small delay is represented in figure 9. BNN and the replicator
dynamic are stable but the convergence speed is slow. Best response and imitation are unstable. The parameters are $\Delta = 1/3, K = 1, \tau_T = 0.5$.

VI. CONCLUSIONS

In this paper, we considered evolutionary games with one population of users and studied delays impact on convergence to ESS for different types of evolutionary dynamics. We observed stability phenomena that are new with respect to non-delayed evolutionary game dynamics. In all the dynamics considered, delays were shown to have negative impact and ESS can be unstable when the delays are large. In the context of the access evolutionary game of mobile terminals, this suggest that updating MAC strategies in the terminals have to be done with care so as to avoid the oscillatory behavior that we observed in the non-stable regime. Our future work is trying to analyze random delays impact on the stability on stochastic evolutionary game dynamics.

REFERENCES