Halpern-type Iterations for Strong Relatively Nonexpansive Multi-valued Mappings in Banach Spaces

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Abstract
In this paper, an iterative sequence for strong relatively nonexpansive multi-valued mapping by modifying Halpern’s iterations is introduced, and then some strong convergence theorems are proved. At the end of the paper some applications are given also.

Key words
Multi-valued mapping; Strong relatively nonexpansive; Fixed point; Iterative sequence; Normalized duality mapping

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1. INTRODUCTION

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let D be a nonempty closed subset of a real Banach space E. A single-valued mapping T : D → D is called nonexpansive if \|Tx - Ty\| ≤ \|x - y\| for all x, y ∈ D. Let N(D) and CB(D) denote the family of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on CB(D) is defined by

\[ H(A_1, A_2) = \max\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\}, \]

for A_1, A_2 ∈ CB(D), where \(d(x, A_1) = \inf \{|x - y|, y \in A_1\}\). The multi-valued mapping T : D → CB(D) is called nonexpansive if \(H(T(x), T(y)) \leq |x - y|\) for all x, y ∈ D. An element p ∈ D is called a fixed point of T : D → N(D) if p ∈ T(p). The set of fixed points of T is represented by \(F(T)\).

Let E be a real Banach space with dual \(E^*\). We denote by J the normalized duality mapping from E to \(2^{E^*}\) defined by

\[ J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in E. \]

where \(\langle \cdot, \cdot \rangle\) denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if \(\frac{\|x + y\|^2}{2} < 1\) for all x, y ∈ U = \(\{z \in E : \|z\| = 1\}\) with x ≠ y. E is said to be uniformly convex if, for each \(\varepsilon \in (0, 2]\), there exists \(\delta > 0\) such that \(\frac{\|x + y\|^2}{2} < 1 - \delta\) for
all $x, y \in U$ with $\|x - y\| \geq \epsilon$. $E$ is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|tx + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. $E$ is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

**Remark 1.1** The following basic properties for Banach space $E$ and for the normalized duality mapping $J$ can be found in Cioranescu [1].

(i) If $E$ is an arbitrary Banach space, then $J$ is monotone and bounded;
(ii) If $E$ is a strictly convex Banach space, then $J$ is strictly monotone;
(iii) If $E$ is a a smooth Banach space, then $J$ is single-valued, and hemi-continuous, i.e., $J$ is continuous from the strong topology of $E$ to the weak star topology of $E$;
(iv) If $E$ is a uniformly smooth Banach space, then $J$ is uniformly continuous on each bounded subset of $E$;
(v) If $E$ is a reflexive and strictly convex Banach space with a strictly convex dual $E^*$ and $J^* : E^* \to E$ is the normalized duality mapping in $E^*$, then $J^{-1} = J^*, J J^* = I_E^*$ and $J^* J = I_E$;
(vi) If $E$ is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J$ is single-valued, one-to-one and onto;
(vii) A Banach space $E$ is uniformly smooth if and only if $E^*$ is uniformly convex. If $E$ is uniformly smooth, then it is smooth and reflexive.

Let $E$ be a smooth Banach space. In the sequel, we always use $\phi : E \times E \to \mathbb{R}^+ \cup \{0\}$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$  \hfill (1.4)

It is obvious from the definition of $\phi$ that

$$\|\|x\| - \|y\|\|^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E.$$  \hfill (1.5)

In addition, the function $\phi$ has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, Jx - Jz \rangle, \quad \forall x, y, z \in E$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda \phi(x, y) + (1 - \lambda)\phi(x, z),$$

for all $\lambda \in [0, 1]$ and $x, y, z \in E$.

Let $C$ be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space $E$. Following Alber [2], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E.$$  

Let $D$ be a nonempty subset of a smooth Banach space. A mapping $T : D \to E$ is relatively nonexpansive [3-5], if the following properties are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, T x) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in D$;
(R3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in $D$ converges weakly to $p$ and $\{x_n - T x_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

If $T$ satisfies (R1) and (R2), then $T$ is called quasi-$\phi$-nonexpansive [6].

Recently, Weerayuth Nilsrakoo [7] introduced the following iterative sequence for finding a fixed point of strongly relatively nonexpansive mapping $T : D \to E$. Given $x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n I + (1 - \alpha_n)J T x_n)$$

where $D$ is nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$, $\Pi_D$ is the generalized projection of $E$ onto $D$ and $\{\alpha_n\}$ is a sequences in $(0, 1)$. 


Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by some authors, see for instance [8-12].

Let $D$ be a nonempty closed convex subset of a smooth Banach space $E$. A mapping $T : D \to N(D)$ is relatively nonexpansive multi-valued mapping [12], if the following properties are satisfied:

(S1) $F(T) \neq \emptyset$;
(S2) $\phi(p, z) \leq \phi(p, x), \forall x \in D, z \in T(x), p \in F(T)$;
(S3) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in $D$ which weakly to $p$ and $\lim lim d(x_n, T(x_n)) = 0$, it follows that $p \in F(T)$.

Let $D$ be a nonempty closed convex subset of a smooth Banach space $E$. We define a strongly relatively nonexpansive multi-valued mapping as follows.

**Definition 1.2** A multi-valued mapping $T : D \to N(D)$ is called strongly relatively nonexpansive, if $T$ satisfies (S1), (S2), (S3) and

(S4) If whenever $\{x_n\}$ is a bounded sequence in $D$ such that $\phi(p, x_n) - \phi(p, z_n) \to 0$, for some $p \in F(T), z_n \in T(x_n)$, it follows that $\phi(z_n, x_n) \to 0$.

In this article, inspired by Weerayuth Nilsrakoo [7], we introduce the following iterative sequence for finding a fixed point of strongly relatively nonexpansive multi-valued mapping $T : D \to N(D)$. Given $u \in E, x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n) \quad (1.8)$$

where $w_n \in T_{x_n}$ for all $n \in N, D$ is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. $\Pi_D$ is the generalized projection of $E$ onto $D$ and $\{\alpha_n\}$ is sequences in $(0, 1)$. We proved the strong convergence theorems in uniformly convex and uniformly smooth Banach space $E$.

## 2. PRELIMINARIES

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \rightharpoonup x$, respectively.

First, we recall some conclusions.

**Lemma 2.1** (Cf. [13, Proposition 2]). Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$ such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.

**Remark 2.2** For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space $E$, we have

$$\phi(x_n, y_n) \to 0 \iff x_n - y_n \to 0 \iff Jx_n - Jy_n \to 0.$$

**Lemma 2.3** (Cf. [13, Propositions 4 and 5]). Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed convex subset of $E$. Then the following conclusions hold:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;
(b) If $x \in E$ and $z \in C$, then $z = \Pi_C x \iff \langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C$;
(c) For $x, y \in E, \phi(x, y) = 0$ if and only $x = y$.

**Remark 2.4.** The generalized projection mapping $\Pi_C$ above is relatively nonexpansive and $F(\Pi_C) = C$.

Let $E$ be a reflexive, strictly convex and smooth Banach space. The duality mapping $J^*$ from $E^*$ onto $E^{**} = E$ coincides with the inverse of the duality mapping $J$ from $E$ onto $E^*$, that is, $J^* = J^{-1}$. We will use the following mapping $V : E \times E \to R$ studied in [2]:

$$V(x, x^* ) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \quad (2.3)$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^* ) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$.

**Lemma 2.5** (Cf. [2] and [14, Lemma 3.2]). Let $E$ be a reflexive, strictly convex and smooth Banach space. Then

$$V(x, x^* ) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$$
for all $x \in E$ and $x^*, y^* \in E^*$.

**Lemma 2.6** (Cf. [15, Lemma 2.1]). Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0,1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy the following conditions: $\lim_{n \to \infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n \to \infty} \delta_n \leq 0$. Then $\lim a_n = 0$.

**Lemma 2.7** (Cf. [16, Lemma 3.1]). Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \leq a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \in \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_{k+1}}$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

**Lemma 2.8** (Cf. [12, Proposition 2.1]). Let $E$ be a strictly convex and smooth Banach space, and $D$ a nonempty closed convex subset of $E$. Suppose $T : D \to N(D)$ is a relatively nonexpansive multi-valued mapping. Then, $F(T)$ is closed and convex.

### 3. MAIN RESULTS

In this section, we use Halpern’s idea [17] for finding fixed point of strongly relatively nonexpansive multi-valued mappings in a uniformly convex and smooth Banach space. In the sequel, we shall need the following lemma.

**Lemma 3.1** Let $D$ be a nonempty closed convex subset of a uniformly convex and smooth Banach space $E$, $T : D \to N(D)$ be a relatively nonexpansive multi-valued mapping, $x \in E$ and $x^* = \Pi_{F(T)} x$. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences such that $\phi(z_n, x_n) \to 0$ and $\phi(z_n, y_n) \to 0$, $z_n \in T x_n$. Then

$$\limsup_{n \to \infty} \langle y_n - x^*, Jx - Jx^* \rangle \leq 0.$$

**Proof.** From the uniform convexity of $E$ and Lemma 2.1,

$$z_n - x_n \to 0 \quad \text{and} \quad y_n - x_n \to 0.$$

From property (R3) of the mapping $T$, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to y \in F(T)$ and

$$\limsup_{n \to \infty} \langle y_n - x^*, Jx - Jx^* \rangle = \limsup_{n \to \infty} \langle x_{n_i} - x^*, Jx - Jx^* \rangle = \limsup_{i \to \infty} \langle x_{n_i} - x^*, Jx - Jx^* \rangle$$

From Lemma 2.3(b), we immediately obtain that

$$\limsup_{n \to \infty} \langle y_n - x^*, Jx - Jx^* \rangle = \langle y - x^*, Jx - Jx^* \rangle \leq 0$$

**Theorem 3.2** Let $D$ be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space $E$ and let $T : D \to N(D)$ be a strongly relatively nonexpansive multi-valued mapping. Let $\{x_n\}$ be the iterative sequence defined by (1.8), $\{\alpha_n\}$ is sequence in $(0,1)$ satisfying

(C1) $\lim_{n \to \infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} \mu$. 

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Proof. Let \( y_n \equiv J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n) \). Then \( x_{n+1} \equiv \Pi_D y_n \). By Lemma 2.8, \( F(T) \) is nonempty, closed and convex, so, we can define the generalized projection \( \Pi_{F(T)} \) onto \( F(T) \). Putting \( u^* = \Pi_{F(T)} u \), we first show that \( \{ x_n \} \) is bounded. From Remark 2.4 and (1.7), we have
\[
\phi(u^*, x_{n+1}) \leq \phi(u^*, y_n) = \phi(u^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n)) \\
\leq \alpha_n \phi(u^*, u) + (1 - \alpha_n) \phi(u^*, w_n) \\
\leq \alpha_n \phi(u^*, u) + (1 - \alpha_n) \phi(u^*, x_n) \\
\leq \max\{\phi(u^*, u), \phi(u^*, x_n)\}.
\]
By induction, we have
\[
\phi(u^*, x_{n+1}) \leq \max\{\phi(u^*, u), \phi(u^*, x_1)\},
\]
for all \( n \in \mathbb{N} \). This implies that \( \{ x_n \} \) is bounded and so is the sequence \( \{ Tx_n \} \). From Condition (C1) and (1.7), we obtain
\[
\phi(w_n, y_n) = \phi(w_n, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jw_n)) \\
\leq \alpha_n \phi(w_n, u) + (1 - \alpha_n) \phi(w_n, w_n) \\
= \alpha_n \phi(w_n, u) \to 0, \quad (n \to \infty).
\]
From Remark 2.4, Lemma 2.5 and (1.7), we have
\[
\phi(u^*, x_{n+1}) \leq \phi(u^*, y_n) = \nu(u^*, Ju) \\
\leq \nu(u^*, Ju - \alpha_n(Ju - Ju^*)) - 2\langle y_n - u^*, -\alpha_n(Ju - Ju^*) \rangle \\
= \nu(u^*, \alpha_n Ju^* + (1 - \alpha_n)Jw_n) + 2\alpha_n \langle y_n - u^*, Ju - Ju^* \rangle \\
= \phi(u^*, J^{-1}(\alpha_n Ju^* + (1 - \alpha_n)Jw_n)) + 2\alpha_n \langle y_n - u^*, Ju - Ju^* \rangle \\
\leq \alpha_n \phi(u^*, u^*) + (1 - \alpha_n) \phi(u^*, w_n) + 2\alpha_n \langle y_n - u^*, Ju - Ju^* \rangle \\
\leq (1 - \alpha_n) \phi(u^*, x_n) + 2\alpha_n \langle y_n - u^*, Ju - Ju^* \rangle,
\]
for all \( n \in \mathbb{N} \).

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists \( n_0 \in N \) such that \( \{ \phi(u^*, x_n) \}_{n=n_0}^\infty \) is nonincreasing. In this situation, \( \{ \phi(u^*, x_n) \} \) is then convergent. Then
\[
\lim_{n \to \infty} (\phi(u^*, x_n) - \phi(u^*, x_{n+1})) = 0. \tag{3.3}
\]
Notice that
\[
\phi(u^*, x_{n+1}) \leq \alpha_n \phi(u^*, u) + (1 - \alpha_n) \phi(u^*, w_n).
\]
It follows from (3.3) and Condition (C1) that
\[
\phi(u^*, x_n) - \phi(u^*, w_n) = \phi(u^*, x_n) - \phi(u^*, x_{n+1}) + \phi(u^*, x_{n+1}) - \phi(u^*, w_n) \\
\leq \phi(u^*, x_n) - \phi(u^*, x_{n+1}) + \alpha_n (\phi(u^*, u) - \phi(u^*, w_n)) \to 0.
\]
Since \( T \) is strongly relatively nonexpansive multi-valued mapping,
\[
\phi(w_n, x_n) \to 0.
\]
It follows from (3.1) and Lemma 3.1 that
\[
\limsup_{n \to \infty} \langle y_n - u^*, Ju - Ju^* \rangle \leq 0. \tag{3.4}
\]
From (3.2), we have
\[
\phi(u^*, x_{n+1}) \leq (1 - \alpha_n) \phi(u^*, x_n) + 2\alpha_n \langle y_n - u^*, Ju - Ju^* \rangle. \tag{3.5}
\]
It follows from Lemma 2.6, (3.4) and (3.5) that
\[ \lim_{n \to \infty} \phi(u^*, x_n) = 0. \]

Hence the conclusion follows from Lemmas 2.1.

**Case 2.** Suppose that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that
\[ \phi(u^*, x_{n_i}) \leq \phi(u^*, x_{n_{i+1}}), \]
for all \( i \in \mathbb{N} \). Then, by Lemma 2.7, there exists a nondecreasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( m_k \to \infty \),
\[ \phi(u^*, x_{m_k}) \leq \phi(u^*, x_{m_{k+1}}) \quad \text{and} \quad \phi(u^*, x_k) \leq \phi(u^*, x_{m_{k+1}}), \]
for all \( k \in \mathbb{N} \). This together with Condition (C1) gives
\[ \phi(u^*, x_{m_k}) - \phi(u^*, w_{m_k}) \geq \phi(u^*, x_{m_k}) - \phi(u^*, x_{m_{k+1}}) + \phi(u^*, x_{m_{k+1}}) - \phi(u^*, w_{m_k}) \leq \alpha_{m_k} (\phi(u^*, u) - \phi(u^*, w_{m_k})) \to 0. \]
This implies that
\[ \phi(w_{m_k}, x_{m_k}) \to 0. \]

It now follows from (3.1) and Lemma 3.1 that
\[ \lim_{n \to \infty} \sup \{y_{m_k} - u^*, Ju - Ju^*\} = 0. \]

(3.6)

From (3.2), we have
\[ \phi(u^*, x_{m_{k+1}}) \leq (1 - \alpha_{m_k}) \phi(u^*, x_{m_k}) + 2\alpha_{m_k} \langle y_{m_k} - u^*, Ju - Ju^* \rangle. \]

(3.7)

Since \( \phi(u^*, x_{m_k}) \leq \phi(u^*, x_{m_{k+1}}) \), we have
\[ \alpha_{m_k} \phi(u^*, x_{m_k}) \leq \phi(u^*, x_{m_k}) - \phi(u^*, x_{m_{k+1}}) + 2\alpha_{m_k} \langle y_{m_k} - u^*, Ju - Ju^* \rangle \leq 2\alpha_{m_k} \langle y_{m_k} - u^*, Ju - Ju^* \rangle. \]

In particular, since \( \alpha_{m_k} > 0 \), we get
\[ \phi(u^*, x_{m_k}) \leq 2 \langle y_{m_k} - u^*, Ju - Ju^* \rangle. \]

It follows from (3.6) that \( \phi(u^*, x_{m_k}) \to 0 \). This together with (3.7) gives
\[ \phi(u^*, x_{m_{k+1}}) \to 0. \]

But \( \phi(u^*, x_k) \leq \phi(u^*, x_{m_{k+1}}) \) for all \( k \in \mathbb{N} \). We conclude that \( x_k \to u^* \).

This implies that \( \lim_{n \to \infty} x_n = u^* \) and the proof is finished.

**Remark 3.3** The result [12, Theorem 3.3] and [18, Corollary 8] is a special case of our result.

**Lemma 3.4** Let \( D \) be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space \( E \). Let \( T : D \to \mathcal{N}(D) \) be a relatively nonexpansive multi-valued mapping. Let \( U \) be the mapping defined by
\[ U = J^{-1}(AJ + (1 - \lambda)JT), \]
where \( \lambda \in (0, 1) \), then \( U : D \to \mathcal{N}(D) \) is strongly relatively nonexpansive multi-valued mapping and \( F(U) = F(T) \).

The proof is similar to the proof of [19, Lemmas 3.1 and 3.2].

Applying Theorem 3.2 and Lemma 3.4, we have the following result.
Theorem 3.5 Let $D$ be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$ and let $T : D \rightarrow N(D)$ be a relatively nonexpansive multi-valued mapping. Let $(x_n)$ be a sequence in $D$ defined by $u \in E,$ $x_1 \in D$ and

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n Ju + (1 - \alpha_n)(\lambda Jx_n + (1 - \lambda)Jz_n))$$

where $z_n \in Tx_n$ for all $n \in \mathbb{N}, \{\alpha_n\}$ is a sequence in $(0,1)$ satisfying Conditions (C1) and (C2), and $\lambda \in (0,1).$ Then $(x_n)$ converges strongly to $\Pi_{F(T)}u.$

Remark 3.6 In Theorems 3.2 and 3.5, the condition of the nonempty interior of fixed point set of $T$ is not needed.

4. APPLICATION TO ZERO POINT PROBLEM OF MAXIMAL MONOTONE MAPPINGS

Let $E$ be a smooth, strictly convex and reflexive Banach space. An operator $A : E \rightarrow 2^E$ is said to be monotone, if $(x - y, x^* - y^*) \geq 0$ whenever $x, y \in E,$ $x^* \in Ax,$ $y^* \in Ay.$ We denote the zero point set $\{x \in E : 0 \in Ax\}$ of $A$ by $A^{-1}0.$ A monotone operator $A$ is said to be maximal, if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. If $A$ is maximal monotone, then $A^{-1}0$ is closed and convex. Let $A$ be a maximal monotone operator, then for each $r > 0$ and $x \in E,$ there exists a unique $x_r \in D(A)$ such that $J(x) \in J(x_r) + rA(x_r)$ (see, for example, [2]). We define the resolvent of $A$ by $J_r = (J + rA)^{-1}J.$ In other words $J_r = (J + rA)^{-1}J,$ $\forall r > 0.$$ A$ is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator. If $A$ is maximal monotone, then $A^{-1}0$ is closed and convex. Let $A$ be a maximal monotone operator, then for each $r > 0$ and $x \in E,$ there exists a unique $x_r \in D(A)$ such that $J(x) \in J(x_r) + rA(x_r)$ (see, for example, [2]). We define the resolvent of $A$ by $J_r = (J + rA)^{-1}J.$ In other words $J_r = (J + rA)^{-1}J,$ $\forall r > 0.$ We know that $J_r$ is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(J_r),$ $\forall r > 0,$ where $F(J_r)$ is the set of fixed points of $J_r.$ We have the following

Theorem 4.1 Let $E,$ $\{\alpha_n\}$ be the same as in Theorem 3.2. Let $A : E \rightarrow 2^E$ be a maximal monotone operator and $J_r = (J + rA)^{-1}J$ for all $r > 0$ such that $A^{-1}0 \neq \emptyset.$ Let $(x_n)$ be the sequence generated by $u, x_1 \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)JJ_r x_n),$$

then $(x_n)$ converges strongly to $\Pi_{A^{-1}0}u.$

Proof. In Theorem 3.2 taking $D = E,$ $T = J_r,$ $r > 0,$ then $T : E \rightarrow E$ is a single-valued relatively nonexpansive mapping and $A^{-1}0 = F(T) = F(J_r),$ $\forall r > 0$ is a nonempty closed convex subset of $E.$ Therefore all the conditions in Theorem 3.2 are satisfied. The conclusion of Theorem 4.1 can be obtained from Theorem 3.2 immediately.

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