Abstract—This paper deals with a novel robust estimation methodology yielding the amplitudes, frequencies and phases of the components of a biased multi-sinusoidal signal in presence of a bounded disturbance on the measurement. The proposed method is based on a suitable adaptive observer in which the parameters’ adaptation law is equipped with an excitation-based switching logic. The stability analysis shows the existence of a set of tuning parameter guaranteeing that the estimator’s dynamics is input-to-state stable with respect to bounded measurement disturbances. The effectiveness of the algorithm is illustrated by some simulation examples also reporting a few comparison results.

I. INTRODUCTION

The problem of estimating the amplitude, frequency and phase (AFP) of a sinusoidal signal arises in a variety of practical applications. In particular, the frequency estimation is a fundamental issue receiving extensive research efforts in many engineering fields such as power monitoring, vibration control and periodic disturbance rejection. While the Fast Fourier Transform (FFT) is usually preferred when discrete-time samples are available over finite-length intervals (under the assumption of bounded and stationary frequency content within this time-window), several other methods have been conceived to track time-varying amplitude/frequencies. Among them, it is worth to recall the adaptive notch-filtering method (see [1], [2]) and the classical Phase-Locked-Loop (PLL) technique (see [3], [4], [5] and [6] for their simple practical implementation). Nevertheless, the switching algorithm has to be reset if the nominal frequency changes. Moreover, estimators incorporating multiple PLL-based techniques in parallel with a de-correlation component have been conceived to be able to discriminate two nearby frequencies of two arbitrary sinusoids taking place simultaneously (see [7], [8]).

Recently, increasing attention has been devoted to techniques involving adaptive observers for the sinusoidal estimation problem (see [9] and [10]). By means of an adaptive observer, global or semi-global (in case of a noisy measurement) convergence results are obtained (see [11], [12], [13]). It is worth noting that the adaptive observer technique makes it possible to achieve multi-sinusoidal estimation by expanding the dynamic model with proper system transformation. However, due to re-parametrization, the estimated frequencies are usually not directly adapted. Instead, the parameter adaptation laws regard a set of parameters related in a nonlinear to the frequency such as the coefficients of the characteristic polynomial of the autonomous signal-generator system (see [14], [15], [16], [17], [18] and [19]). Among these methods, [15], [17], [18] are capable to handle a biased multi-sinusoidal signal, while [19] has been applied in a nonlinear plant for disturbance cancellation. In addition to the aforementioned adaptive observer-based techniques, an asymptotically convergent estimator for n-frequencies using contraction theory is proposed in [20].

Motivated by the adaptive observer proposed in [21] and its extension to the single sinusoidal case [13], the presented paper deals with a new methodology that is capable to offer reliable estimates of amplitudes, frequencies, phases and offset from a biased signal comprising n sinusoids. In contrast with other methods that adapt the coefficients of the characteristic polynomial, a direct adaptation law for the squares of the frequencies is provided. The stability analysis indicates that the robustness is guaranteed even if the measurements are corrupted by unstructured bounded disturbances, which is likely to appear in real-world applications. More specifically, once the tuning parameters are suitably set, ISS property with respect to the additive measurement noise is ensured.

The paper is organized as follows: the problem is formulated in Section II. In Section III, the adaptive observer-based estimator is proposed. Then, the stability analysis is dealt with in Section IV. Finally, simulation results showing the effectiveness of the algorithm dealt with in the paper as well as providing some comparisons are given in Section V.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a biased multi-sinusoidal signal

\[ y(t) = A_0 + \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i) \]

where the amplitudes verify the inequality \( A_i \geq 0 \), \( A_0 \) is an unknown constant bias, the frequencies parameters are strictly-positive and unique: \( \omega_i > 0, \omega_i \neq \omega_j \) for \( i \neq j \) and \( \phi_i \) is the unknown initial phase of each sinusoid.

The signal \( y(t) \) is assumed to be generated by the following observable autonomous marginally-stable dynamical system:

\[
\begin{align*}
\dot{x}(t) &= A_x x(t) + \sum_{i=1}^{n} A_i x(t) \theta_i^t \\
x(0) &= x_0 \\
y(t) &= C_x x(t)
\end{align*}
\]

with \( x(t) \in \mathbb{R}^{(2n+1)} \) and where \( x_0 \) represents the unknown initial condition which leads the output to match the stationary sinusoidal behavior since the very beginning. Moreover \( \theta_i^t = a_i + \Omega_i \) with \( \Omega_i = \omega_i^t, \forall i \in \{1,\ldots,n\} \) and

\[
A_x = \begin{bmatrix} S_1 & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0 \\ 0_{2\times 2} & S_2 & \cdots & 0_{2\times 2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2\times 2} & \cdots & \cdots & S_n & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad C_x^T = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_n^T \\ 1 \end{bmatrix},
\]

and

\[
S_i = \begin{bmatrix} 0 & 1 \\ a_i & 0 \end{bmatrix}, \quad c_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

\( a_1, a_2 \cdots a_n \) are non-zero constants can be selected arbitrarily with the only requirements to satisfy \( a_i \in \mathbb{R}, a_i \neq a_j \) for \( i \neq j \), and \( A_i \) is a matrix with \( (2i, 2i-1) \)th entry \(-1\) and \(0\).
for the all the others, for instance:

\[ A_1 = \begin{bmatrix}
S_0 & 0_{2 \times (2n-1)} \\
0_{(2n-1) \times 2} & 0_{(2n-1) \times (2n-1)}
\end{bmatrix}, \]
\[ A_2 = \begin{bmatrix}
0_{2 \times 2} & 0_{2 \times (2n-3)} \\
0_{(2n-3) \times 2} & S_0 & 0_{(2n-3) \times (2n-3)}
\end{bmatrix} \]

in which

\[ S_0 = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}. \]

In order consider the measurement uncertainty, let us assume that \( y(t) \) is corrupted by an additive disturbance \( d(t) \), bounded by a constant \( d > 0 : |d(t)| < d, \forall t \in \mathbb{R}_{\geq 0} \). Then, the perturbed signal that is available from the measurement can be written as

\[ \hat{y}(t) = A_0 + \sum_{i=1}^{n} A_i \sin(\omega_i t + \phi_i) + d(t). \]  

(3)

Thanks to (2), the signal \( \hat{y}(t) \) can be thought as generated by the observable system

\[
\begin{cases}
\dot{x}(t) = A_x x(t) + G_x(x(t))\theta^* \\
y(t) = C_x x(t) + d(t)
\end{cases}
\]

(4)

in which

\[ G_x(x(t)) = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
-x_1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \vdots \\
0 & 0 & \cdots & -x_{2n-1}
\end{bmatrix},
\]

and \( \theta^* \) denotes the true parameter vector \( \theta_1^* \theta_2^* \cdots \theta_n^* \).

**Remark 2.1:** The elements of \( G_x(x) \) are globally Lipschitz continuous functions, that is:

\[ |G_x(x) - G_x(x')| \leq |x' - x'|, \forall x, x' \in \mathbb{R}^{2n}. \]

Moreover, the true state \( x(t) \) is norm-bounded for any initial condition, i.e., \( |x(t)| \leq \bar{x}, \forall t \in \mathbb{R} \). The Lipschitz condition on \( G_x(x) \) and the bound \( \bar{x} \) allow to establish the following further bound:

\[ \|G_x(x(t))\| \leq \bar{x}, \forall t \in \mathbb{R}, \]

Now, assuming that the estimates \( \hat{x}(t) \) and \( \hat{\theta}(t) \) are available, then the full AEPs estimates are obtained by

\[
\begin{align*}
\hat{\Omega}_i &= \hat{\theta}_i - \alpha_i, \\
\hat{\omega}_i &= \sqrt{\hat{\theta}_i - \alpha_i} \\
\hat{A}_i &= \sqrt{\left(\hat{\Omega}_i x_i^2 + \hat{\omega}_i^2\right)}/\hat{\omega}_i, \\
\hat{\varphi}_i &= \angle(\hat{x}_{2i} + j\hat{\omega}_i \hat{x}_{2i-1}), \quad i = 1, 2, \cdots, n
\end{align*}
\]

(5)

(6)

(7)

In addition, the offset is evaluated directly by \( \hat{A}_0 = \hat{x}_{2n+1} \).

In order to proceed with the further analysis, the following assumption is needed.

**Assumption 1:** The frequencies of the sinusoids are bounded by a positive constant \( \omega \), such that \( \omega_i < \omega, \forall i \in \{1, \cdots, n\} \).

According to Assumption 1, there exists a known positive constant \( \theta^* \), such that \( |\theta^*| \leq \theta^* \). More specifically, in the rest of the paper we consider \( \theta^* = \theta^* \), where \( \theta^* \subseteq \mathbb{R}^n \) is a hypersphere of radius \( \theta^* \). The constraint on \( \theta^* \) is instrumental for proving the stability of the parameter adaptation law introduced in the next section.

### III. Filtered-Augmentation-based Adaptive Observer

In this section, we introduce the adaptive observer that is based on the model (4). In order to validate the estimation scheme, \( \hat{y}(t) \) is augmented by the output of a synthetic filter driven by the noisy measurement vector:

\[ \hat{y}_c(t) = A_c \hat{y}_c(t) + B_c \hat{y}(t) \]

(8)

where \( A_c \) and \( B_c \) are set by the designer such that \( A_c \) is Hurwitz and the pair \( (A_c, B_c) \) is reachable. \( \hat{y}_c(t) \in \mathbb{R}^{n_c} \) denotes the accessible state vector and with arbitrary initial condition \( \hat{y}_{c0} \). According to [13], the dimension \( n_c \) of the augmented dynamics must verify \( n_c \geq n - 1 \).

For the sake of the forthcoming analysis, it is convenient to split the expanded filtered output vector in two components:

\[ \hat{y}_c(t) = y_c(t) + d_c(t) \]

where \( y_c(t) \) and \( d_c(t) \) are produced by two virtual filters driven by the unperturbed output and by the measurement disturbance, respectively:

\[ \hat{y}_c(t) = A_c y_c(t) + B_c y(t) \]

(9)

\[ \hat{d}_c(t) = A_c d_c(t) + B_c d(t) \]

(10)

Consequently, in view of (4), (9) and (10), the overall augmented system dynamics with the extended perturbed output measurement equation can be written as follows:

\[
\begin{cases}
\dot{z}(t) = A_z z(t) + G_z(z(t))\theta^* \\
\eta(t) = C_z z(t) \\
\hat{\eta}(t) = \eta(t) + \hat{d}(t)
\end{cases}
\]

with

\[ z(0) = z_0 \in \mathbb{R}^{n_c}, \quad n_c = 2n + 1 + n_e, \quad \text{and} \]

\[ \dot{z}(t) \triangleq \begin{bmatrix} x(t)^	op \eta(t) \end{bmatrix} = \begin{bmatrix} x(t) & \eta(t) \end{bmatrix}, \quad \eta(t) \triangleq \begin{bmatrix} y(t) & \hat{y}_c(t) & \hat{y}(t) & \hat{d}(t) \end{bmatrix}^	op, \]

\[ A_z \triangleq \begin{bmatrix} A_x & 0_{(2n+1) \times n_c} \\
B_x & C_x & A_c
\end{bmatrix}, \quad C_z \triangleq \begin{bmatrix} C_x & 0_{l \times n_e} \end{bmatrix} \begin{bmatrix} 0_{n_e \times (2n+1)} & I_{n_e} \end{bmatrix} \]

and

\[ G_z(z(t)) \triangleq \begin{bmatrix} G_x \begin{bmatrix} T_x x(t) \end{bmatrix} \end{bmatrix} \]

with the transformation matrix given by

\[ T_{xx} \triangleq \begin{bmatrix} I_{2n+1} & 0_{(2n+1) \times n_e} \end{bmatrix} \]

It is worth noting that \( G_z(z(t)) \) is also Lipschitz with the same Lipschitz constant as \( G_x(x(t)) \), and norm-bounded by \( \bar{x} \). Moreover, the assumed norm-bound \( d \) on the output noise implies the existence of \( \eta \) such that \( \eta > 0 : |\eta(t)| \leq d, \forall t \in \mathbb{R}_{\geq 0} \).

Now, we introduce the structure of the adaptive observer, consisting of the measured output filter (8) and the dynamic components (12), (13) and (14) described below:
1) Augmented state estimator:
\[
\dot{\hat{z}}(t) = (A_z - L C_z) \hat{z}(t) + L \eta(t) + G_z(\hat{z}(t)) \dot{\hat{\theta}}(t) + \Xi(t) \dot{\hat{\theta}}(t)
\]
with \(\hat{z}(0) = \hat{z}_0\). The gain matrix \(L\) is given by
\[
L \triangleq \begin{bmatrix}
L_x & 0 \\
0 & 0
\end{bmatrix}
\]
where \(L_x\) is a suitable gain matrix such that \(A_x - L_x C_x\) is Hurwitz.

2) Parameter-affine state-dependent matrix filter:
\[
\dot{\hat{z}}(t) = (A_z - L C_z) \Xi(t) + G_z(\hat{z}(t)) \dot{\hat{\theta}}(t)
\]
with \(\Xi(0) = 0_{n \times n}\) and where \(\Xi(t) \in \mathbb{R}^{n \times n}\) is an auxiliary time-varying matrix whose elements are driven by the state-dependent parameter-affine matrix \(G_z(\hat{z}(t))\).

3) Parameters’ adaptation law:
Herein, an projection operator \(P\) is utilized to confine the estimated parameter \(\hat{\theta}\) to the predefined convex region \(\Theta^*\)
\[
\hat{\theta}(t) = \Psi(t) P (\hat{\theta}_{pre}(t)) \quad \text{if } |\hat{\theta}(t)| \leq \hat{\theta}^*, \text{otherwise}
\]
where \(\hat{\theta}_{pre}(t) \triangleq -\mu(\Xi(t)\Xi(t) + \rho^2 I)^{-1}\Xi(t) (C_z \hat{z}(t) - \eta(t))\)
with \(\mu\) is a positive constant, and \(\rho \in \mathbb{R}\) is another constant set by the designer. The parameters’ derivative projection operator is defined as:
\[
P (\hat{\theta}_{pre}(t)) \triangleq \begin{cases} 
\text{nsps}(\hat{\theta}(t)) (\text{nsps}(\hat{\theta}(t)))^\top \hat{\theta}_{pre}(t), & \text{if } |\hat{\theta}(t)| = \hat{\theta}^* \text{ and } \hat{\theta}(t) \hat{\theta}_{pre}(t) > 0 \\
\hat{\theta}_{pre}(t), & \text{otherwise}
\end{cases}
\]
in which \(\text{nsps}(\cdot)\) denotes the null-space of a row vector. In a compact form, the parameter adaptation law can be expressed as
\[
\hat{\theta}(t) = \Psi(t) \left[ \hat{\theta}_{pre}(t) - \mathcal{I}(\hat{\theta}) \frac{\hat{\theta}(t) \hat{\theta}(t)^\top + \hat{\theta}_{pre}(t) \hat{\theta}_{pre}(t)^\top}{\hat{\theta}^2} \right]
\]
where \(\mathcal{I}(\hat{\theta})\) denote the indicator function given by
\[
\mathcal{I}(\hat{\theta}) \triangleq \begin{cases} 
1, & \text{if } |\hat{\theta}(t)| = \hat{\theta}^* \text{ and } \hat{\theta}(t) \hat{\theta}_{pre}(t) > 0 \\
0, & \text{otherwise}
\end{cases}
\]
The activation/suppression of the parameter adaptation is determined by the binary switching signal \(\Psi(t)\), which possesses the following hysteretic property:
\[
\Psi(t) = \begin{cases} 
1, & \text{if } \min \text{eig}(\Phi(\Xi(t))) \geq 2\delta \Psi(t^-), & \text{if } 2\delta \leq \min \text{eig}(\Phi(\Xi(t))) < 2\delta \\
0, & \text{if } \min \text{eig}(\Phi(\Xi(t))) < 2\delta \Psi(t^-), & \text{if } 2\delta \leq \min \text{eig}(\Phi(\Xi(t))) < 2\delta 
\end{cases}
\]
where
\[
\Phi(\Xi(t)) = (\Xi(t)^\top \Xi(t) + \rho^2 I)^{-1} \Xi(t) C_z^\top C_z \Xi(t)
\]
represents the excitation matrix. The transition thresholds \(\delta_2\), \(\delta_1\) are selected by the designer such that \(0 < 2\delta_2 < 2\delta_1 < 1\).

The introduction of the hysteretic is inspired by the need to ensure a minimum finite duration between transitions.

IV. STABILITY ANALYSIS

A. Excitation Phase
Consider an arbitrary active identification phase, in which \(\Psi(t) = 1\) and \(\Phi(\Xi(t)) \geq 2\delta\). In order to address the stability of the adaptive observer, let us define the augmented state-estimation error vector: \(\hat{z}(t) \triangleq \hat{z}(t) - z(t)\), the parameter estimation error \(\hat{\theta}(t) \triangleq \hat{\theta}(t) - \theta^*\), and their linear time-varying combination \(\phi(t) \triangleq \Xi(t) \hat{\theta}(t) - \hat{z}(t)\). Then, the state-estimation error evolves according to the differential equation
\[
\dot{\hat{z}}(t) = (A_z - L C_z) \hat{z}(t) + L \dot{\eta}(t) + G_z(\hat{z}(t)) \dot{\hat{\theta}}(t) + \Xi(t) \dot{\hat{\theta}}(t) + G_z(z(t)) \hat{\theta}(t) + G_z(\hat{z}(t)) \theta^* + \Xi(t) \hat{\theta}(t)
\]
where \(G_z(z(t)) \triangleq G_z(z(t)) - G_z(\hat{z}(t))\). Meanwhile, the dynamics of the auxiliary variable \(\phi(t)\) evolves according to
\[
\dot{\phi}(t) = (A_z - L C_z) \phi(t) - L d_n(t) - G_z(\hat{z}(t)) \theta^*
\]
The upcoming analysis is carried out in order to demonstrate the benefits of using the derivative projection on the parameters’ estimates. To this end, in the following we will only focus on the scenario in which the projection operator is activated, since \(\dot{\hat{\theta}}(t) = \hat{\theta}_{pre}(t)\) for all the other conditions. For the sake of convenience, let us assume that \(\frac{\hat{\theta}(t) \phi(t)}{\hat{\theta}^*} \hat{\theta}_{pre}(t) = \sigma(t) \hat{\theta}(t)\), where \(\sigma(t)\) is a variable depending on \(\hat{\theta}_{pre}(t)\). Owing to the fact that \(\mathcal{I}_\theta = 1\) and \(|\hat{\theta}(t)| = \hat{\theta}^*\), we have that
\[
\dot{\phi}(t) = \phi(t) \frac{\hat{\theta}_{pre}(t)}{\hat{\theta}^*} \left[ \hat{\theta}_{pre}(t) - \sigma(t) \hat{\theta}(t) \right] - \left[ \phi(t) \sigma(t) - \phi^*(t) \sigma(t) \right] \] in which \(\langle \cdot, \cdot \rangle\) denotes the inner product. In virtue of
\[
\left\langle \phi(t), \sigma(t) \right\rangle = \sigma \phi^2 \geq \sigma \phi^* |\phi(t)| \cos \phi(t) \leq \phi^* \sigma(t)
\]
we can finally bound the scalar product \(\phi(t) \dot{\phi}(t)\) by:
\[
\hat{\theta}(t) \dot{\phi}(t) \leq \hat{\theta}^* \dot{\phi}(t) \hat{\theta}_{pre}(t)
\]
where \(\hat{\theta}_{pre}(t)\) is expanded as follows:
\[
\hat{\theta}_{pre}(t) = -\mu(\Xi(t)^\top \Xi(t) + \rho^2 I)^{-1} \Xi(t)^\top C_z \Xi(t) \hat{\theta}(t) + \mu(\Xi(t)^\top \Xi(t) + \rho^2 I)^{-1} \Xi(t)^\top C_z \hat{\theta}(t) + \mu(\Xi(t)^\top \Xi(t) + \rho^2 I)^{-1} \Xi(t)^\top C_z \hat{\theta}(t)
\]

Theorem 4.1 (ISS of the dynamic estimator): If Assumption 1 and the excitation condition hold, then given the sinusoidal signal \(y(t)\) defined in (1) and the perturbed measurement (3), there exist suitable choices of \(\mu\) and \(\rho\) such that the adaptive observer-based estimator given by (8), (12), (13) and (14) is ISS with respect to any bounded disturbance \(d_n\) and in turn ISS with respect to bounded measurement disturbance \(|d(t)| \leq \delta\).

Proof: Since \((A_z - L C_z)\) is Hurwitz, for any positive definite matrix \(Q\), there exist a positive definite matrix \(P\) that solves the linear Lyapunov equation
\[
(A_z - L C_z)^\top P + P(A_z - L C_z) = -2Q
\]
Hence, the proof is concluded, iff

$$V(t) \triangleq \frac{1}{2}(\ddot{z}(t)P\dddot{z}(t) + \dddot{z}(t) \ddot{\theta}(t) + 2g\dddot{z}(t) \dddot{\phi}(t))$$  \hspace{1cm} (19)$$

where \( g \) is a positive constant. By letting

$$\bar{c} \equiv \| \mathbf{C}_z \|, \bar{I} \equiv \| \mathbf{L} \|, \bar{q} \equiv \min \text{eig}(\mathbf{Q}), \bar{p} \equiv \max \text{eig}(\mathbf{P})$$

it follows that the time-derivative of the Lyapunov function can be bounded as

$$\dot{V}(t) \leq -q|\dot{z}(t)|^2 - qg|\dddot{z}(t)|^2 - \mu|\dot{\theta}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 + \frac{\mu}{2} |\dddot{\theta}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 + 3|\dddot{\phi}(t)|^2$$

Finally, the following inequality can be established:

$$\dot{V}(t) \leq -q|\dot{z}(t)|^2 - qg|\dddot{z}(t)|^2 - \mu|\dot{\theta}(t)|^2 - \frac{g}{2} |\dddot{\phi}(t)|^2 + \frac{\mu}{2} |\dddot{\theta}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 + 3|\dddot{\phi}(t)|^2 + \frac{3}{2} |\dddot{\phi}(t)|^2$$

By completing the squares we get

$$\dot{V}(t) \leq -q|\dot{z}(t)|^2 - qg|\dddot{z}(t)|^2 - \mu|\dot{\theta}(t)|^2 - \frac{g}{2} |\dddot{\phi}(t)|^2 + \frac{\mu}{2} |\dddot{\theta}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 + 3|\dddot{\phi}(t)|^2 + \frac{3}{2} |\dddot{\phi}(t)|^2 + \frac{3}{2} |\dddot{\phi}(t)|^2$$

Finally, the following inequality can be established:

$$\dot{V}(t) \leq -\beta_1 [V(t) - \sigma_1(\dot{d}_n)]$$

where

$$\beta_1 \triangleq 2 \min \left\{ \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \right\}$$  \hspace{1cm} (20)$$

and

$$\sigma_1(s) \triangleq \frac{1}{\beta_1} \left[ \frac{\bar{p}^2 (\bar{I} + \mu c^2)}{2(q + \mu c^2 - 3\bar{p}^2)} + \frac{3\mu c^2}{8\bar{p}^2} + \frac{g}{2} |\dddot{\phi}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 \right] s^2, \forall s \in \mathbb{R}_{\geq 0}$$

Hence, the proof is concluded, iff

$$\beta_1 > 0$$  \hspace{1cm} (21)$$

In view of (21), all the components involved in (20) should be positive, wherein \( \frac{q - \mu c^2 - 3\bar{p}^2}{2} > 0 \) can be immediately verified by choosing a positive \( \mu \). Now, we set the excitation threshold \( \bar{d} \) and the \( \mathbf{Q} \) matrix arbitrarily, determining \( \bar{q} \). Then, letting \( \bar{p} \leq \mu \), we determine a sufficient condition to ensure the positiveness of the first term in (20):

$$\frac{q - \mu c^2 - 3\bar{p}^2}{2} - \frac{2}{q} \left( \frac{\bar{p}^2}{3} \right) - \frac{3\bar{p}^2}{2\bar{d}} > 0.$$  

Being the Lyapunov parameter \( g > 0 \) arbitrary, let us set \( g = 1 \) for simplicity. Now, we can always determine a sufficiently small value of \( \mu \) for which the inequality holds true. Next, by suitably allocating the poles, we compute the output-injection gain \( \mathbf{L} \) that yields the needed \( \beta_1 \). Finally, to make \( \beta_1 \) strictly-positive, we choose a regularization parameter \( \rho \) such that

$$\frac{gq}{2} - \frac{3\mu c^2}{8\bar{d}^2} > 0.$$  

Remark 4.1: To avoid the increase of the worst-case sensitivity to bounded noises, instead of using a low value of \( \bar{p} \) that leads high-gain output injection through \( \mathbf{L} \), and high values of \( \bar{d} \) and \( \sigma_1 \) correspondingly, we can set \( \bar{p} = \mu \) and increase the regularization parameter \( \rho \).

B. Dis-Excitation Phase

Lemma 4.1 (Boundedness in dis-excitation phase): [21] When \( \Psi(t) = 0 \), the Lyapunov function admits a bound that depends on the noise level and on the initial value of the Lyapunov function itself before switching off:

$$V(t) \leq \beta_0 (L_0 V(t^-) + \sigma_0(\dot{d}_n(t^-)]) - V(t))$$

where

$$\beta_0 \triangleq 2 \min \left\{ \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \frac{q - \mu c^2 - 3\bar{p}^2}{2}, \right\}$$

and

$$\sigma_0(s) \triangleq \frac{1}{\beta_0} \left[ \frac{\bar{p}^2 \bar{I} + \mu c^2}{2(q + \mu c^2 - 3\bar{p}^2)} + \frac{3\mu c^2}{8\bar{p}^2} + \frac{g}{2} |\dddot{\phi}(t)|^2 + \frac{g}{2} |\dddot{\phi}(t)|^2 \right] s^2, \forall s \in \mathbb{R}_{\geq 0}$$

Proof: The present lemma can be proven by following the same line of reasoning adopted for the case \( \Psi(t) = 1 \).

C. Robustness Under Alternate Switching

At this stage, the stability of the adaptive observer under alternate switching is characterized by linking the results obtained for the two excitation phases.

Theorem 4.2: [21] Under the same assumptions of Theorem 4.1, consider the adaptive observer (8), (12), (13), (14) equipped with the excitation-based switching strategy defined in (15). Then, the discrete dynamics induced by sampling the adaptive observer in correspondence of the switching transitions has the asymptotic ISS property if the excitation phases last longer than \( \beta^{-1} \ln(L_0) \). Letting \( k \) denote a counter for the active identification phases, and \( V_k \) the value of the Lyapunov function (19) at then at the end of the \( k \)-th phase, then the following bound can be established:

$$V_k < e^{-\beta_1 \Delta_k} (\sigma_0(\dot{d}_n) - \sigma_1(\dot{d}_n) + L_0 V_{k-1}) + \sigma_1(\dot{d}_n),$$

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where $\Delta_k$ is the duration of the $k$-th phase.

The proof can be found in [21]. In view of Theorem 4.2, if an infinite number of active identification phases occurs asymptotically ($k \to \infty$), or if a single excitation phase lasts indefinitely, then the estimation error in the inter-sampling times converges to a region whose radius depends only on the assumed disturbance bound.

V. ILLUSTRATIVE EXAMPLES

In this section, the devised method is compared with the adaptive-observer-based method presented in [16] which offers estimated frequency and amplitude simultaneously. The dynamic estimators are discretized by forward-Euler method with fixed sampling period $T_s = 1 \times 10^{-4}$s. All the measured inputs are affected by a bounded noise denoted by $d(t)$ which is subject to uniform distribution in the interval $[-0.25, 0.25]$.

Example 1: Consider the following sum-of-sinusoids signal, perturbed by additive noise

$$\hat{y}(t) = 2 \sin 3t + \sin 2t + d(t).$$

For the sake of observer design, let $A_e = -2$, $B_e = 1$, while the observer poles are placed at $[-0.4, -0.5, -0.8, -1]$. The other tuning parameters are: $\mu = 10$, $\rho = 1$, $a_1 = 0$, $a_2 = -1$, $\delta = 1 \times 10^{-6}$, $\delta = 5 \times 10^{-7}$. The parameters for the other methods are chosen such that the response time of the frequency estimate is similar to the one provided by the method discussed in this paper. To this end, for method [16], we set $\gamma = 1200$, $\gamma_1 = 0.01$, $\gamma_2 = 0.005$, $l_1 = 10$, $l_2 = 50$ and the coefficients $\lambda_i$ are chosen: $\lambda_0 = 4$, $\lambda_1 = 6$, $\lambda_2 = 6$, $\lambda_3 = 4$.

It is worth noting from Fig.1 and 2 that both the estimators succeeded in detecting the frequencies and amplitudes in presence of bounded disturbance. However, the proposed method is more accurate in presence of additive noise and shows a smoother transient behaviour.

Example 2: Consider now a measurement signal composed of three sinusoids

$$\hat{y}(t) = 2 \sin 5t + \sin 4t + 3 \sin 3t + d(t).$$

The parameters of the synthetic filter are fixed as follows

$$A_e = \begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}, \quad B_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then, let us set $a_1 = 0$, $a_2 = -1$, $a_3 = -3$, $\mu = 60$, $\rho = 0.5$, $\delta = 1 \times 10^{-7}$, $\delta = 5 \times 10^{-8}$ respectively and place the poles at $[-0.2, -0.4, -0.5, -0.6, -0.8, -10]$. The initial values of the state variables of the observer have been all set to zero, while the parameter vector has been initialized to $\hat{\theta}(0) = [12 10 15]^T$.

As shown in Fig.3 and 4, all the three frequencies are identified individually, at the same time the true amplitudes are also successfully captured.

Example 3: In order to evaluate the performance of the method in presence of measurement bias, the following signal is considered in the present example

$$\hat{y}(t) = 1 + 4 \sin 3t + 2 \sin 2t + d(t).$$

The parameters of the proposed method are chosen as follows: $A_e = -2$, $B_e = 1$, $a_1 = -2$, $a_2 = -1$, $\mu = 50$, $\rho = 0.2$, $\delta = 1 \times 10^{-6}$, $\delta = 5 \times 10^{-7}$ and the poles’ location $[-0.2, -0.4, -0.5, -0.6, -0.8, -10]$. In Fig. 5, the behaviour of the excitation level and of the switching signal $\Psi(t)$ are shown to enhance the fact that the proposed methodology allows to check in real-time the excitation level thus allowing to
possibly stop the parameter-updating in case of low-exciting signals. The estimates are reported in Fig.6 and 7, where it can be observed that all the parameters including the offset are successfully estimated.

Future research efforts will be devoted to extend the class of measurement uncertainties, in particular, the one modelled by a time polynomial which is capable to deal with even higher order uncertainties such as drift in a systematic way.

REFERENCES


